Fourier Transform

Overview

- Signals as functions (1D, 2D)
  - Tools
- 1D Fourier Transform
  - Summary of definition and properties in the different cases
    - CTFT, CTFS, DTFS, DTFT
    - DFT
- 2D Fourier Transforms
  - Generalities and intuition
  - Examples
  - A bit of theory
- Discrete Fourier Transform (DFT)
- Discrete Cosine Transform (DCT)

Signals as functions

1. Continuous functions of real independent variables
   - 1D: $f=f(x)$
   - 2D: $f=f(x,y)\ x,y$
   - Real world signals (audio, ECG, images)
2. Real valued functions of discrete variables
   - 1D: $f=f[k]$
   - 2D: $f=f[i,j]$
   - *Sampled* signals
3. Discrete functions of discrete variables
   - 1D: $y=y[k]$
   - 2D: $y=y[i,j]$
   - *Sampled and quantized* signals
   - For ease of notations, we will use the same notations for 2 and 3

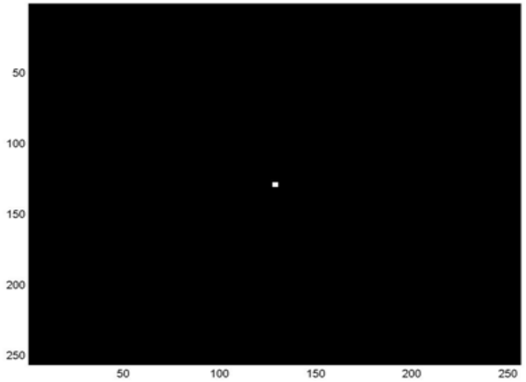
Images as functions

- Gray scale images: 2D functions
  - Domain of the functions: set of (x,y) values for which $f(x,y)$ is defined : 2D lattice $[i,j]$ defining the pixel locations
  - Set of values taken by the function : gray levels

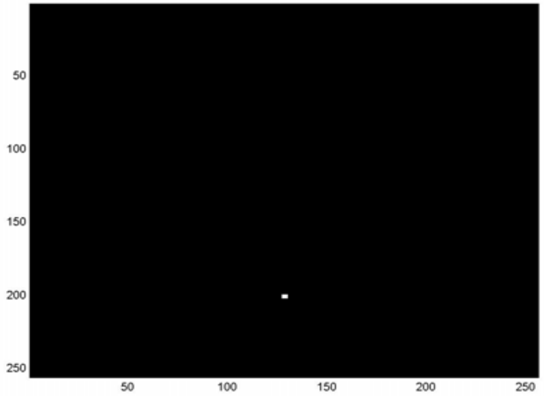
- Digital images can be seen as functions defined over a discrete domain $\{i,j: 0<i<I, 0<j<J\}$
  - I,J: number of rows (columns) of the matrix corresponding to the image
  - $f=f[i,j]$: gray level in position $[i,j]$

Example 1: δ function

$$\delta[i,j] = \begin{cases} 1 & i = j = 0 \\ 0 & i,j \neq 0; i \neq j \end{cases}$$



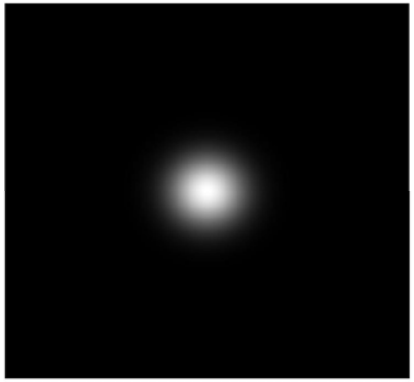
$$\delta[i,j-J] = \begin{cases} 1 & i = 0; j = J \\ 0 & otherwise \end{cases}$$



Example 2: Gaussian

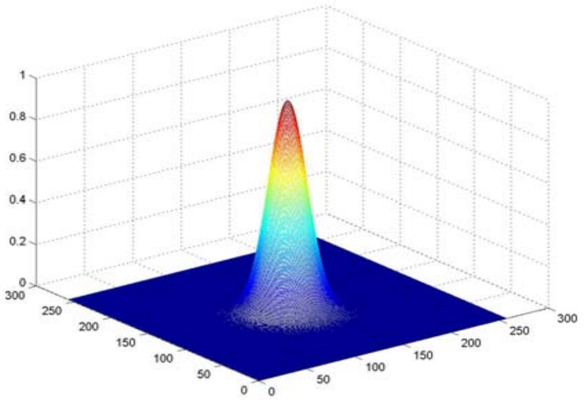
Continuous function

$$f(x, y) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{x^2+y^2}{2\sigma^2}}$$

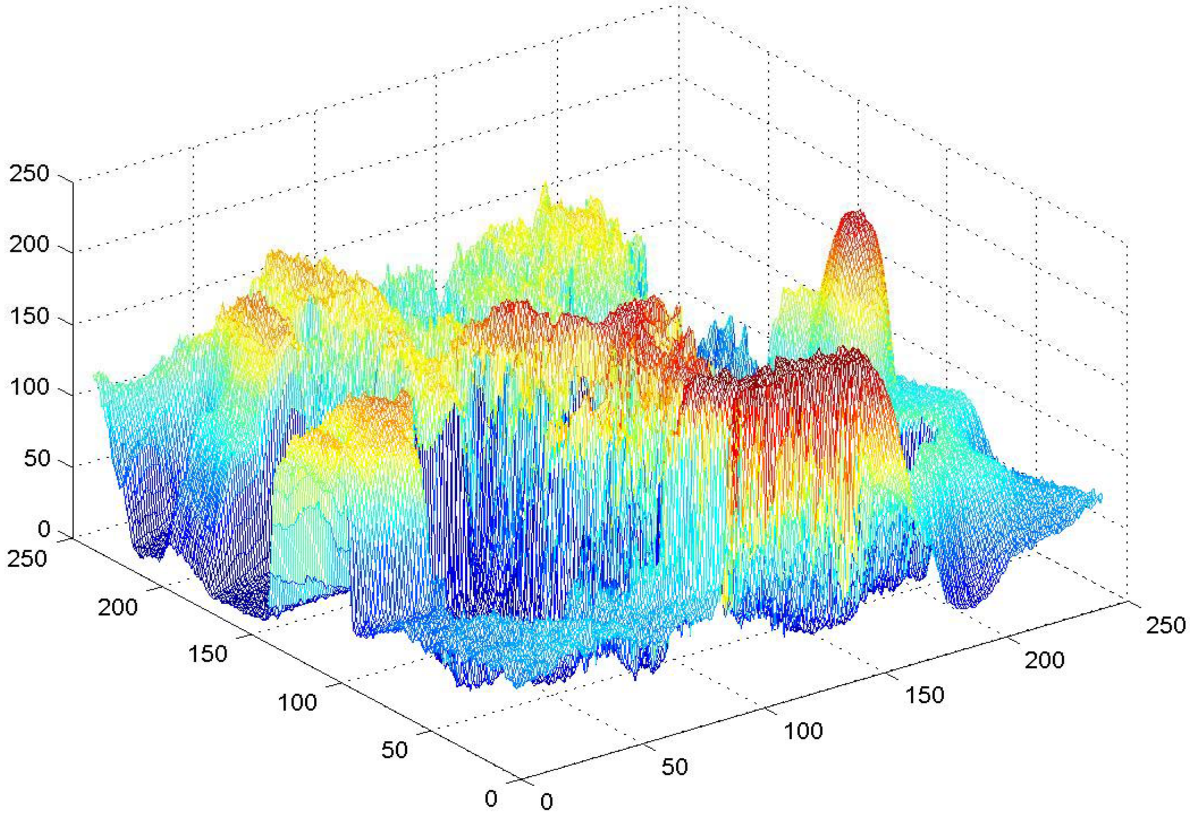


Discrete version

$$f[i, j] = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{i^2+j^2}{2\sigma^2}}$$



Example 3: Natural image



Example 3: Natural image

Fourier Transform

- Different formulations for the different classes of signals
  - Summary table: Fourier transforms with various combinations of continuous/discrete time and frequency variables.
  - Notations:
    - CTFT: continuous time FT: t is real and f real ($f=\omega$) (CT, CF)
    - DTFT: Discrete Time FT: t is discrete ($t=n$), f is real ($f=\omega$) (DT, CF)
    - CTFS: CT Fourier Series (summation synthesis): t is real AND the function is periodic, f is discrete ($f=k$), (CT, DF)
    - DTFS: DT Fourier Series (summation synthesis): $t=n$ AND the function is periodic, f discrete ($f=k$), (DT, DF)
    - P: periodical signals
    - T: sampling period
    - ω_s: sampling frequency ($\omega_s=2\pi/T$)
    - For DTFT: $T=1 \rightarrow \omega_s=2\pi$

Continuous Time Fourier Transform (CTFT)

Time is a real variable (t)

Frequency is a real variable (ω)

CTFT: Concept

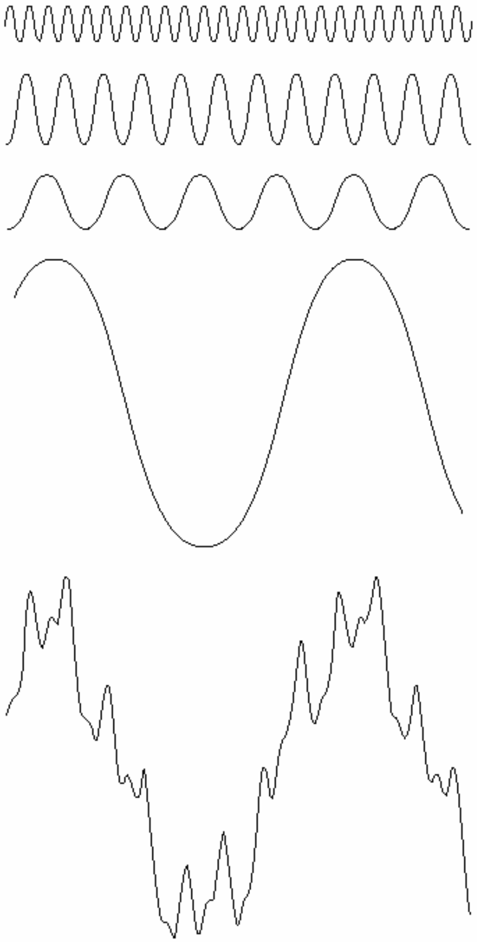


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

- A signal can be represented as a weighted sum of sinusoids.

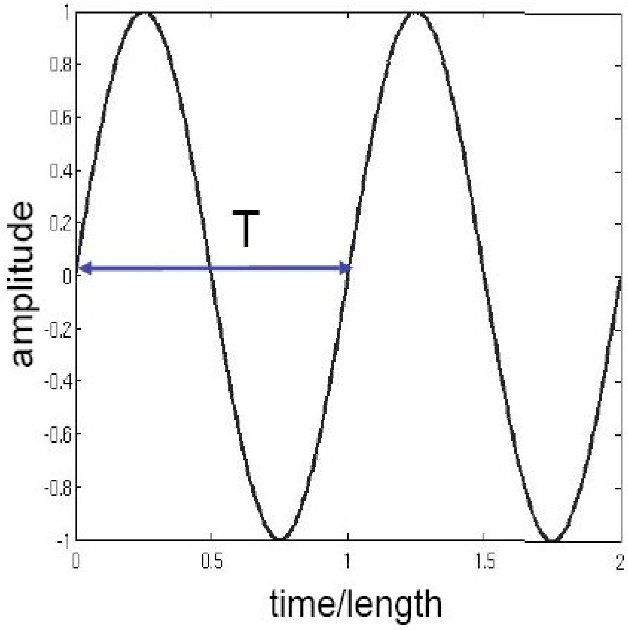
- Fourier Transform is a change of basis, where the basis functions consist of sins and cosines (complex exponentials).

[Gonzalez Chapter 4]

Continuous Time Fourier Transform (CTFT)

- Define frequency
  = 1/T
  cycles per unit time
  cycles per unit distance

- Here f = 1 T=1



Fourier Transform

- Cosine/sine signals are easy to define and interpret.
- However, it turns out that the analysis and manipulation of sinusoidal signals is greatly simplified by dealing with related signals called complex exponential signals.
- A complex number has real and imaginary parts: $z = x+j\,y$
- A complex exponential signal:

$$r\,\mathrm{e}^{j\alpha} = r\left(\cos\alpha + j\sin\alpha\right)$$

CTFT

- Continuous Time Fourier Transform
- Continuous time *a-periodic* signal
- Both time (space) and frequency are continuous variables
  - NON normalized frequency ω is used
- Fourier integral can be regarded as a Fourier series with fundamental frequency approaching zero
- Fourier spectra are continuous
  - A signal is represented as a sum of sinusoids (or exponentials) of all frequencies over a continuous frequency interval

Fourier *integral*

$$F(\omega) = \int_t f(t) e^{-j\omega t} dt$$

analysis

$$f(t) = \frac{1}{2\pi} \int_\omega F(\omega) e^{j\omega t} d\omega$$

synthesis

Then CTFT becomes

- Fourier Transform of a 1D continuous signal

$$F(u)=\int_{-\infty}^{\infty} f(x)e^{-j2\pi ux}dx$$

"Euler's formula" $e^{-j2\pi ux}=\cos\left(2\pi ux\right)-j\sin\left(2\pi ux\right)$

- Inverse Fourier Transform

$$f(x)=\int_{-\infty}^{\infty} F(u)e^{j2\pi ux}du$$

CTFT: change of notations

- Fourier Transform of a 1D continuous signal

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-j\omega x} dx$$

"Euler's formula" $e^{-j\omega x} = \cos(\omega x) - j\sin(\omega x)$

- Inverse Fourier Transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega x} d\omega$$

Change of notations:

$$\omega \to 2\pi u$$

$$\begin{cases} \omega_x \to 2\pi u \\ \omega_y \to 2\pi v \end{cases}$$

CTFT

- Replacing the variables

$$F(2\pi u) = F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx =$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} d\left(2\pi u\right) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

- More compact notations

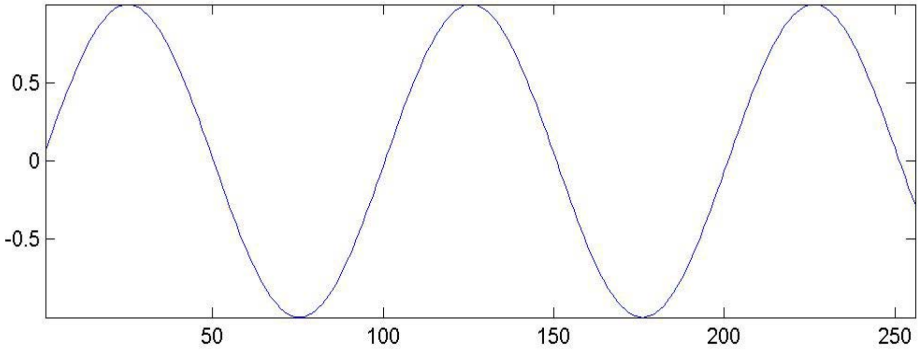
$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$$

$$f\left(x\right) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

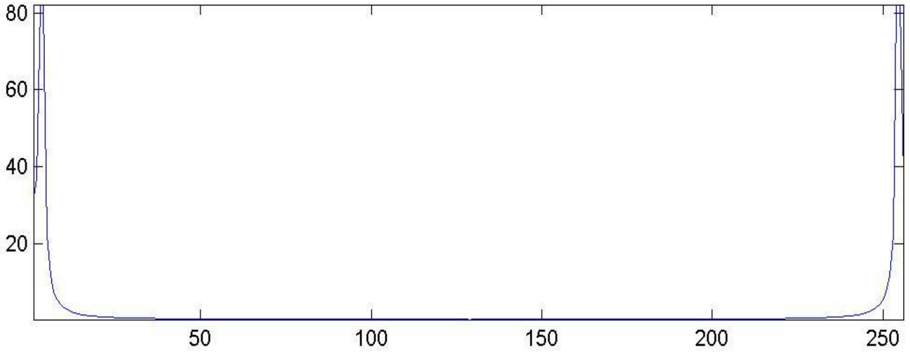
Sinusoids

- Frequency domain characterization of signals

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$



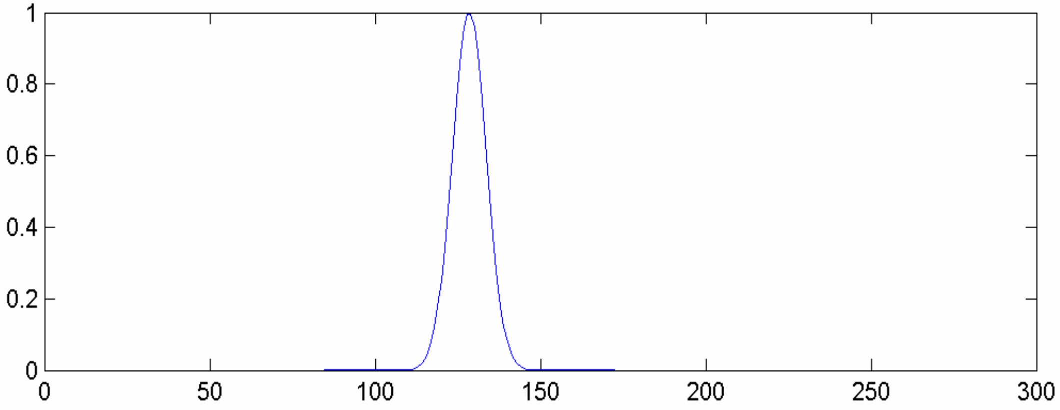
Signal domain



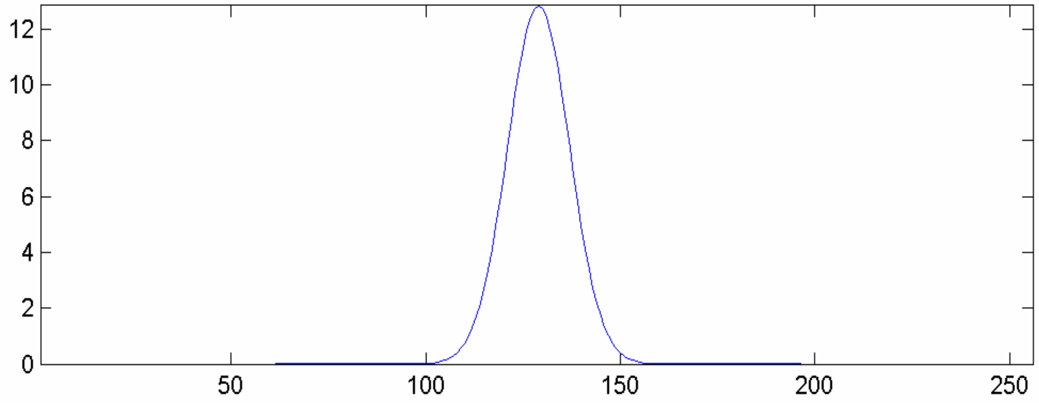
Frequency domain (spectrum, absolute value of the transform)

Gaussian

Time domain

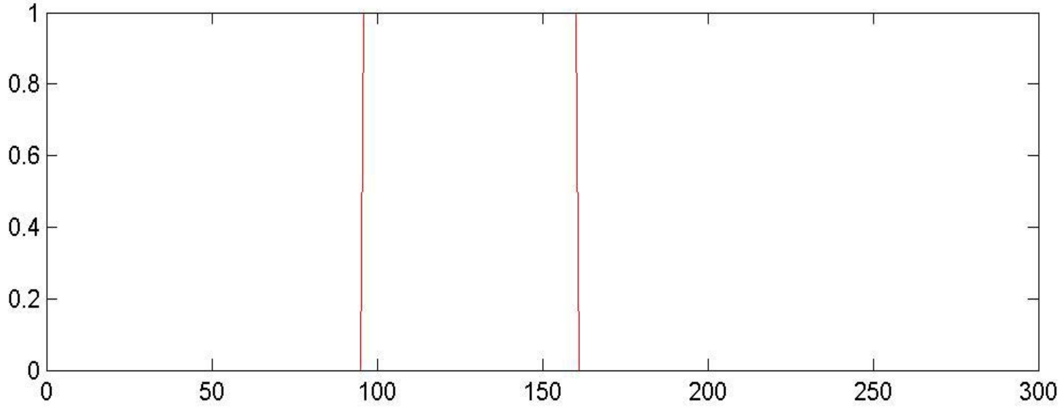


Frequency domain

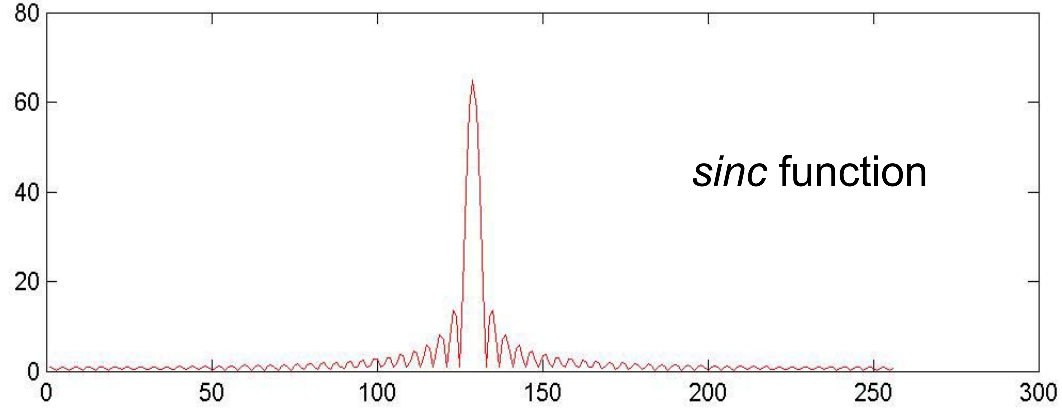


rect

Time domain

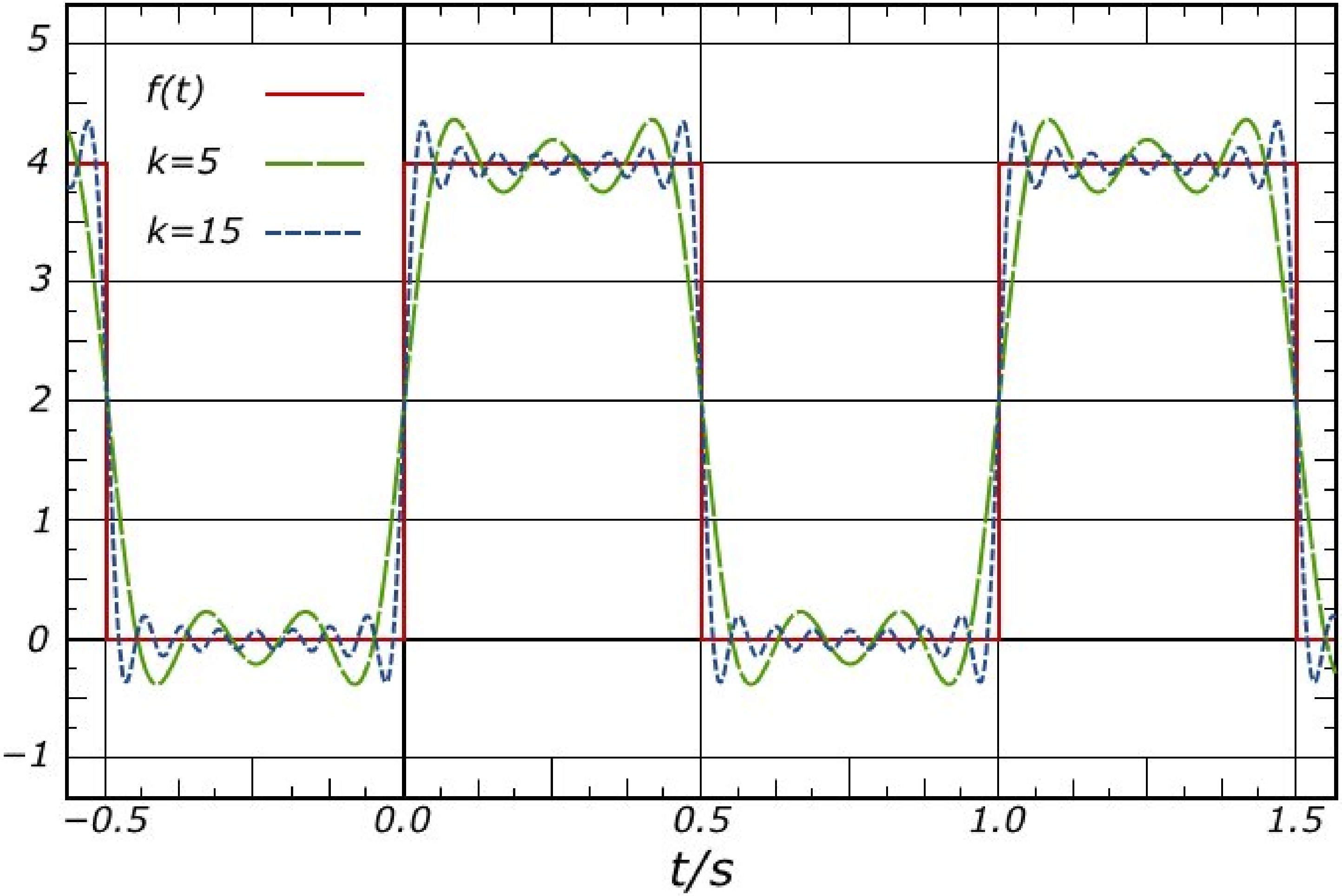


Frequency domain

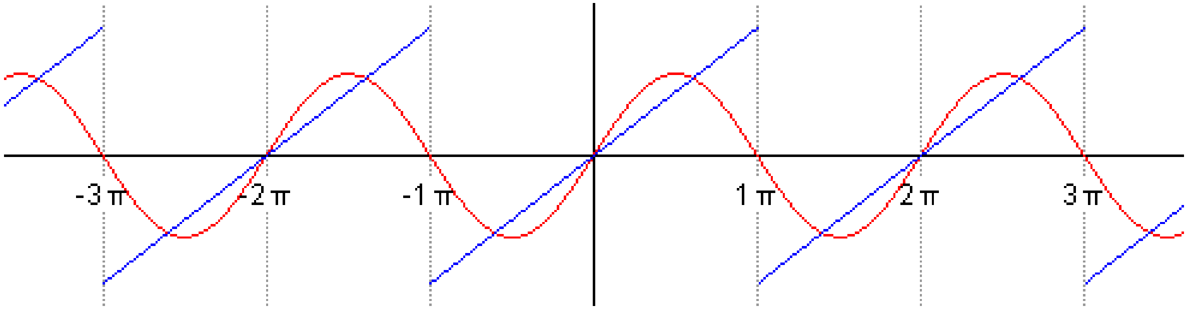


sinc function

Example



Example



Discrete Fourier Transform (DFT)

The easiest way to get to it

Time is a discrete variable (t=n)

Frequency is a discrete variable (f=k)

DFT

- The DFT can be considered as a generalization of the CTFT to discrete series

$$F[k] = \frac{1}{N} \sum_{n=0}^{N-1} f[n] e^{-j2\pi kn/N}$$

$$f[n] = \sum_{k=0}^{N-1} F[k] e^{j2\pi kn/N}$$

$$n = 0,1,\ldots, N-1$$

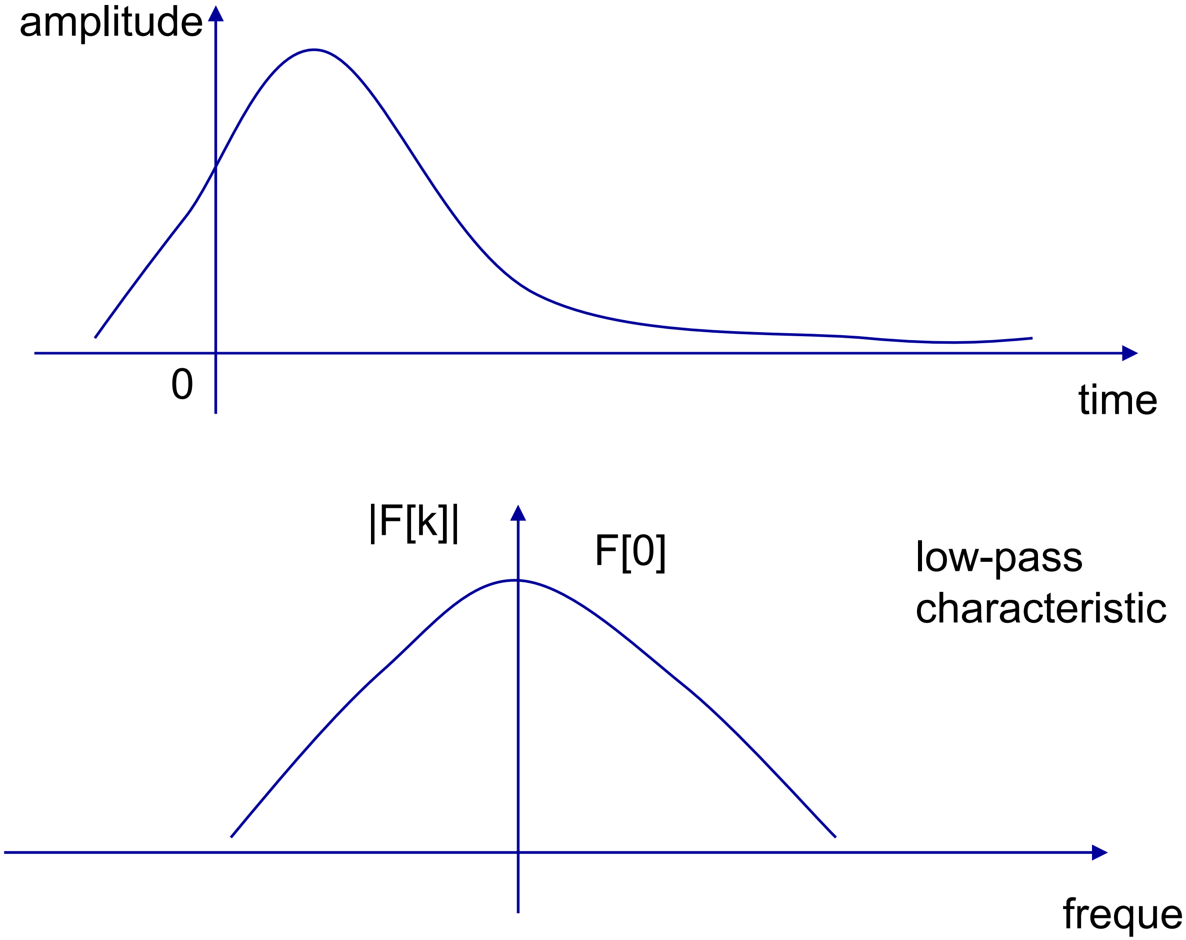
$$k = 0,1,\ldots, N-1$$

- In order to calculate the DFT we start with k=0, calculate F(0) as in the formula below, then we change to u=1 etc

$$F[0] = \frac{1}{N} \sum_{n=0}^{N-1} f[n] e^{-j2\pi 0n/N} = \frac{1}{N} \sum_{n=0}^{N-1} f[n] = \overline{f}$$

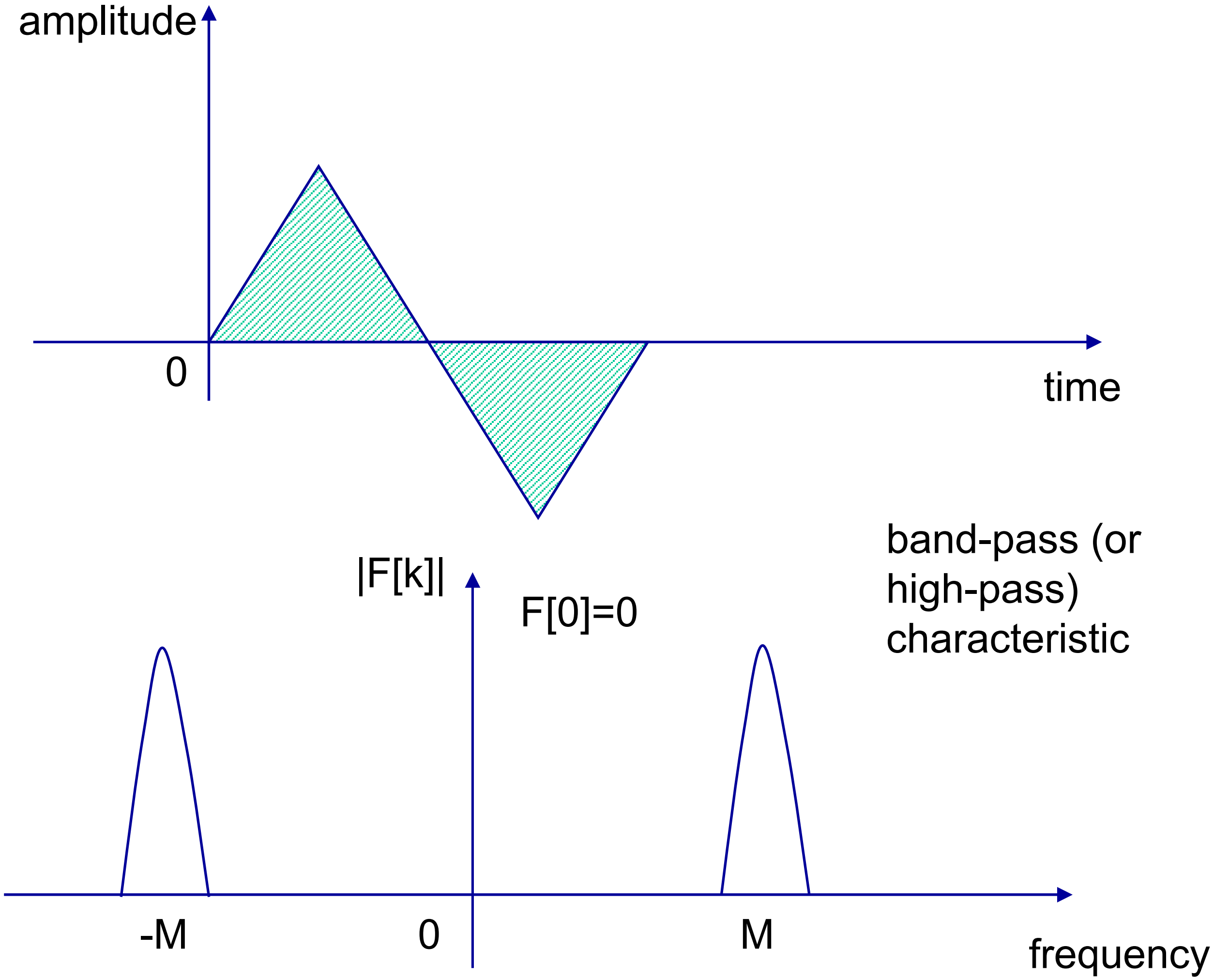
- F[0] is the average value of the function f[n] 0
  - This is also the case for the CTFT

Example 1



low-pass
characteristic

Example 2



band-pass (or
high-pass)
characteristic

DFT

- About M² multiplications are needed to calculate the DFT
- The transform F[k] has the same number of components of f[n], that is N
- The DFT always exists for signals that do not go to infinity at any point
- Using the Eulero's formula

$$e^{j\theta} = \cos\theta + j\sin\theta.$$

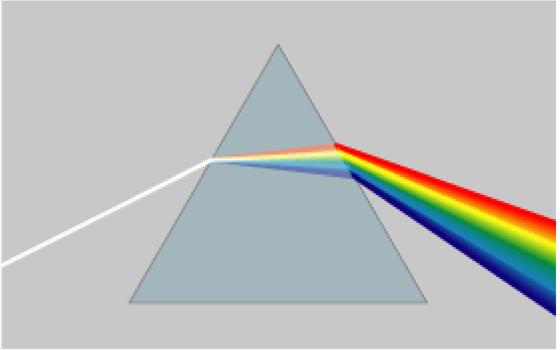
$$F[k] = \frac{1}{N}\sum_{n=0}^{N-1} f[n]e^{-j2\pi kn/N} = \frac{1}{N}\sum_{n=0}^{N-1} f[n]\left(\cos\left(j2\pi kn/N\right) - j\sin\left(j2\pi kn/N\right)\right)$$

frequency component *k*

discrete trigonometric functions

Intuition

- The FT decomposed the signal over its harmonic components and thus represents it as a sum of linearly independent complex exponential functions
- Thus, it can be interpreted as a “mathematical prism”



DFT is a complex number

- F[k] in general are complex numbers

$$F[k] = \mathrm{Re}\{F[k]\} + j\,\mathrm{Im}\{F[k]\}$$

$$F[k] = |F[k]| \exp\{ j \measuredangle F[k]\}$$

$$\left\{ \begin{array}{c} |F[k]| = \sqrt{\mathrm{Re}\{F[k]\}^2 + \mathrm{Im}\{F[k]\}^2} \\ \measuredangle F[k] = \tan^{-1}\left\{ -\dfrac{\mathrm{Im}\{F[k]\}}{\mathrm{Re}\{F[k]\}} \right\} \end{array} \right\}$$

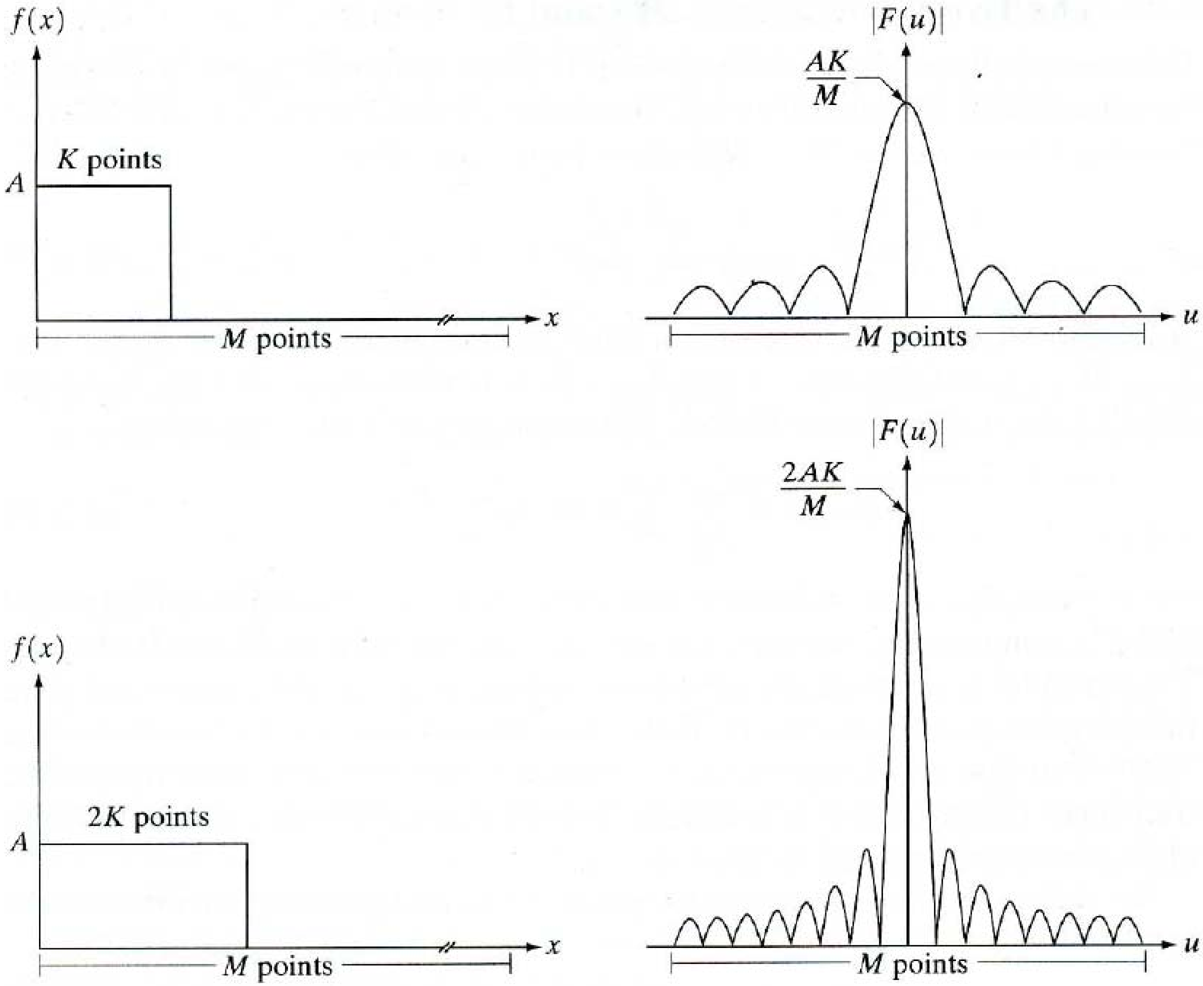
magnitude or spectrum

phase or angle

$$P[k] = |F[k]|^2$$

power spectrum

Example



a b
c d

FIGURE 4.2 (a) A discrete function of M points, and (b) its Fourier spectrum. (c) A discrete function with twice the number of nonzero points, and (d) its Fourier spectrum.

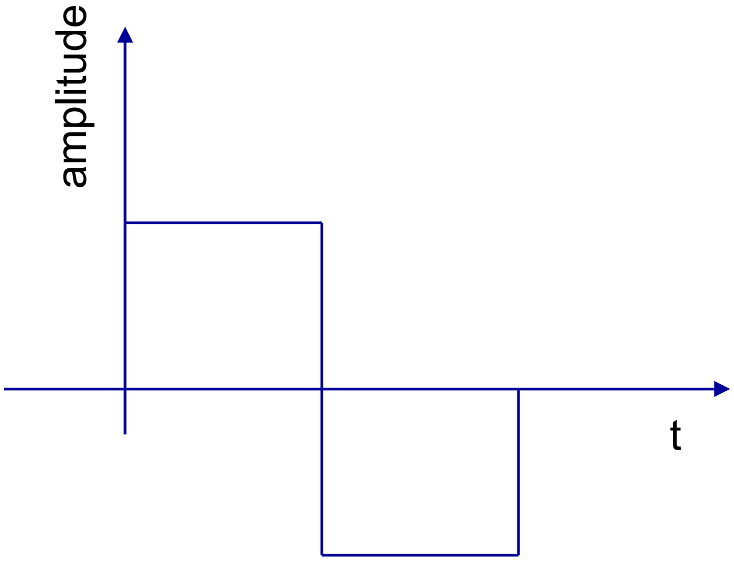
Let’s take a bit more advanced perspective…

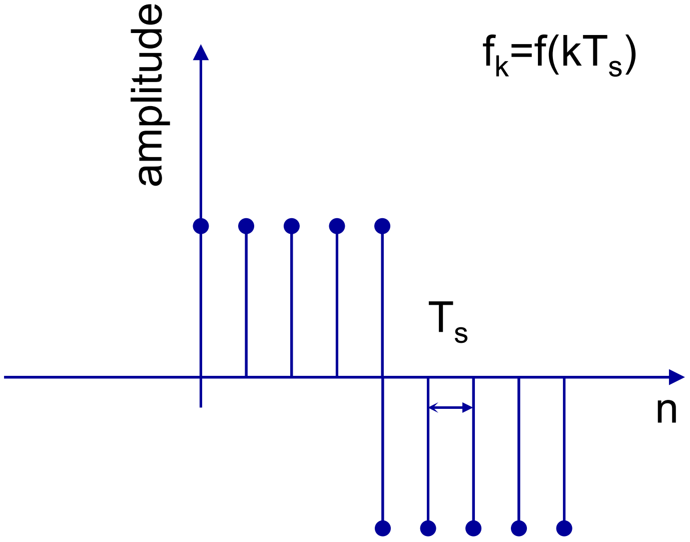
Book: Lathi, Signal Processing and Linear Systems

Overview

Transform	Time	Frequency	Analysis/Synthesis	Duality
(Continuous Time) Fourier Transform (CTFT)	C	C	$F(\omega) = \int_t f(t) e^{-j\omega t} dt$ $f(t) = \frac{1}{2\pi} \int_\omega F(\omega) e^{j\omega t} d\omega$	Self-dual
(Continuous Time) Fourier Series (CTFS)	C P	D	$F[k] = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-j2\pi kt/T} dt$ $f(t) = \sum_k F[k] e^{j2\pi kt/T}$	Dual with DTFT
Discrete Time Fourier Transform (DTFT)	D	C P	$F(\Omega) = \sum_{k=-\infty}^{+\infty} f[k] e^{-j\Omega k}$ $f[k] = \frac{1}{2\pi} \int_{2\pi} F(\Omega) e^{j\Omega k} d\Omega$	Dual with CTFS
Discrete Time Fourier Series (DTFS)	D P	D P	$F[k] = \frac{1}{N} \sum_{n=0}^{N-1} f[n] e^{-j2\pi kn/T}$ $f[n] = \sum_{n=0}^{N-1} F[k] e^{j2\pi kn/T}$	Self dual

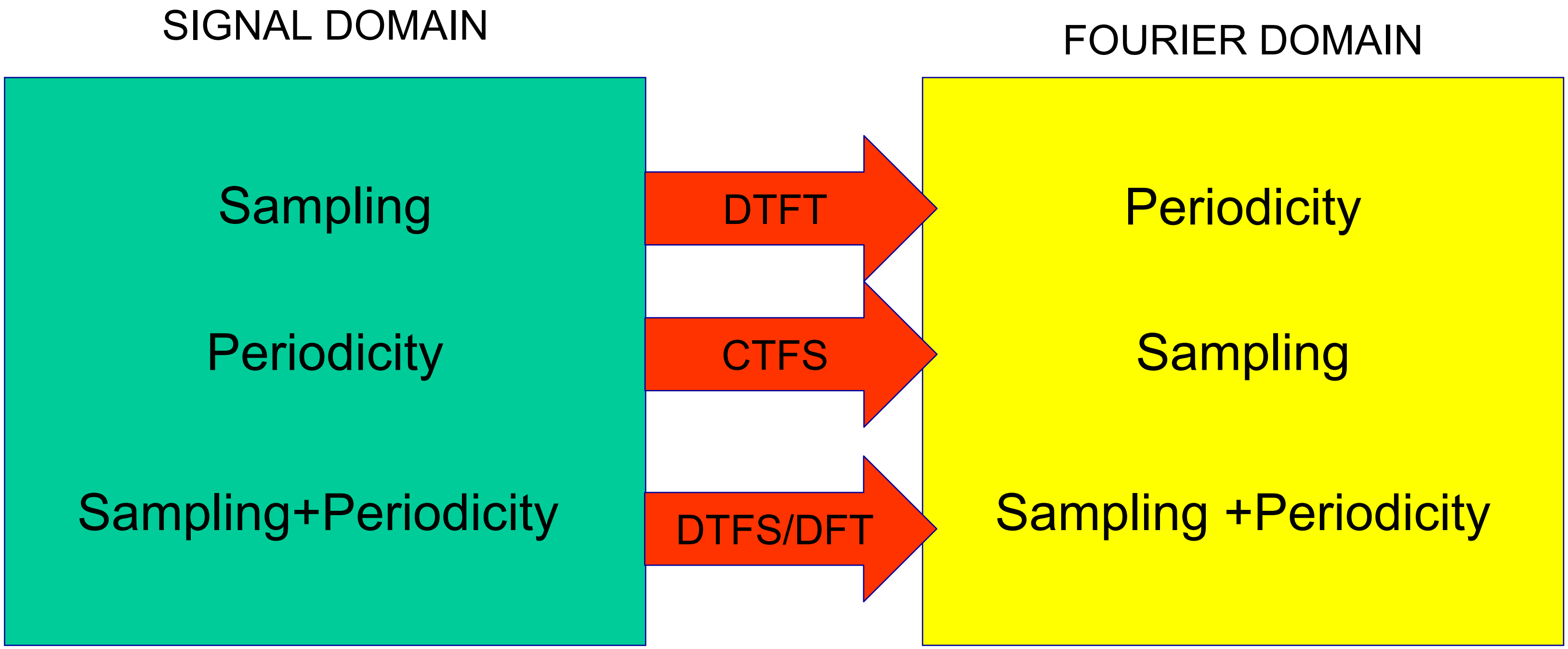
Linking continuous and discrete domains





- DT signals can be seen as *sampled* versions of CT signals
- Both CT and DT signals can be of finite duration or periodic
- There is a duality between periodicity and discretization
  - Periodic signals have discrete frequency (DF) transform (f=k) → CTFS
  - Discrete time signals have periodic transform → DTFT
  - DT periodic signals have DF periodic transforms → DTFS, DFT

Dualities



SIGNAL DOMAIN

Sampling

Periodicity

Sampling+Periodicity

DTFT

CTFS

DTFS/DFT

FOURIER DOMAIN

Periodicity

Sampling

Sampling +Periodicity

Discrete time signals

Sequences of samples

- f[k]: sample values
- Assumes a *unitary spacing* among samples (T_s=1)
- *Normalized frequency* Ω
- Transform
  - DTFT for NON periodic sequences
  - CTFS for periodic sequences
  - DFT for *periodized* sequences
- All transforms are 2π periodic

$$\Omega_s = \omega_s T_s = \frac{2\pi}{T_s} T_s = 2\pi$$

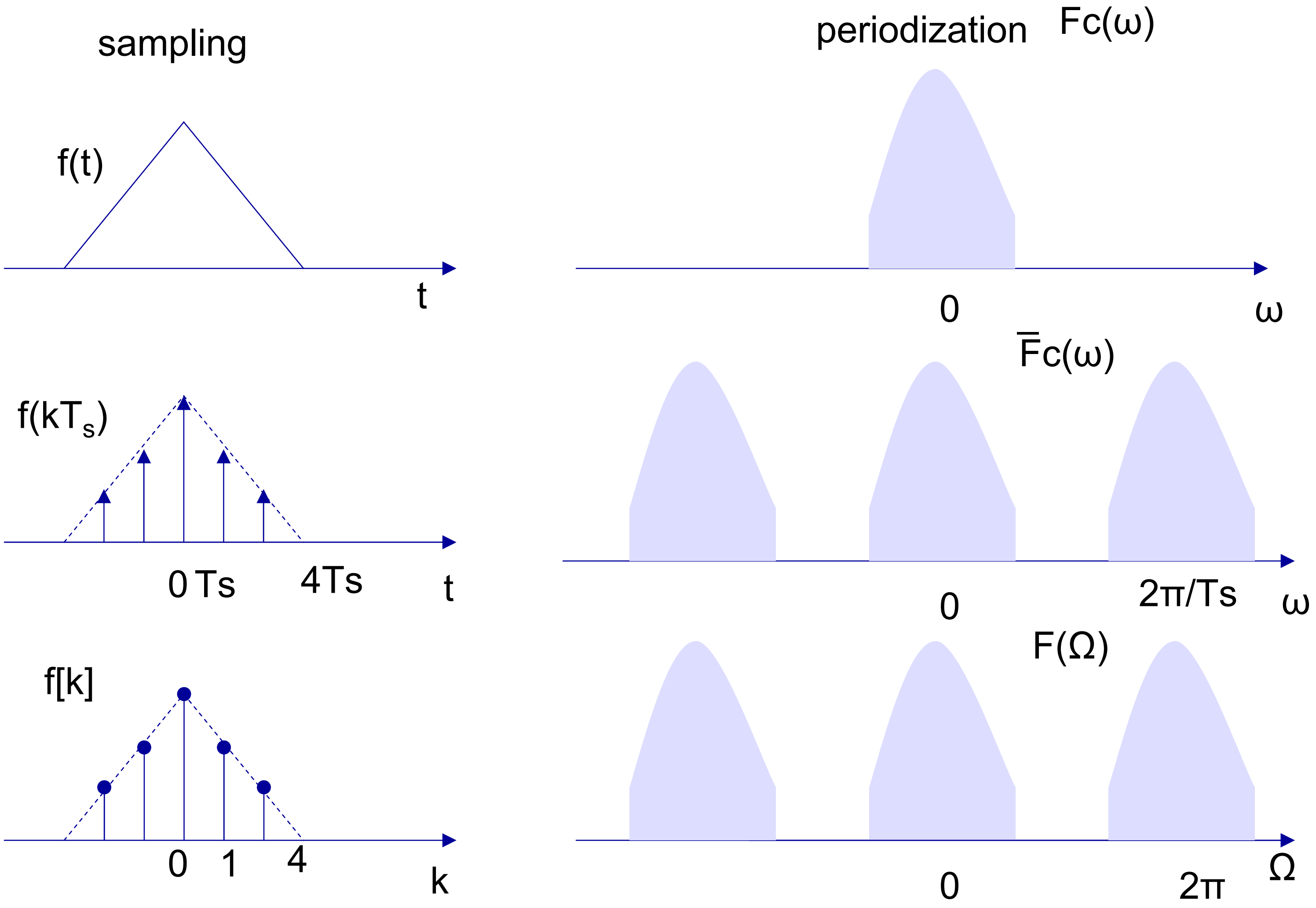
Sampled signals

- f(kT_s): sample values
- The sampling interval (or period) is T_s
- Non normalized frequency ω
- Transform
  - DTFT
  - CSTF
  - DFT
  - BUT accounting for the fact that the sequence values have been generated by sampling a real signal → f_k=f(kT_s)
- All transforms are periodic with period ω_s

$$\omega_s = \frac{2\pi}{T_s}$$

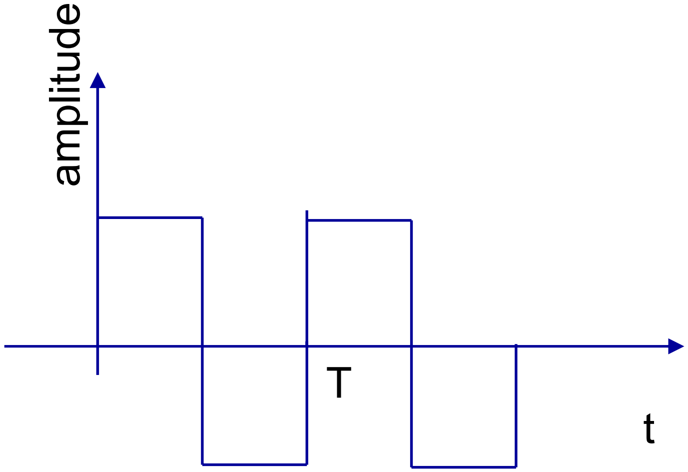
$$\boxed{\Omega = \omega T_s}$$

Connection DTFT-CTFT



CTFS

- Continuous Time Fourier Series
- Continuous time periodic signals
  - The signal is periodic with period T
  - The transform is “sampled” (it is a series)



$$F[k] = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j2\pi kt/T}dt$$

coefficients of the Fourier series

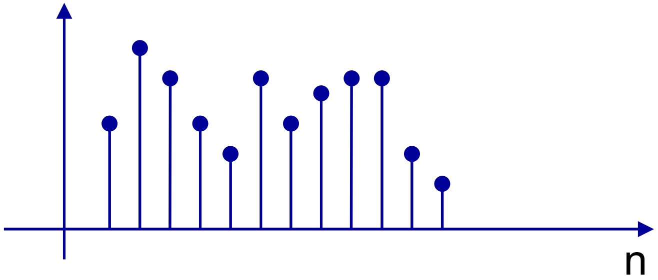
$$f(t) = \sum_{k} F[k]e^{j2\pi kt/T}$$

periodic signal

CTFS

- Representation of a continuous time signal as a sum of orthogonal components in a complete orthogonal signal space
  - The exponentials are the basis functions
- Properties
  - even symmetry → only cosinusoidal components
  - odd symmetry → only sinusoidal components

DTFT



- Discrete Time Fourier Transform
- *Discrete time a-periodic* signal
- The transform is periodic and continuous with period $\omega_s = \dfrac{2\pi}{T_s}$

non normalized frequency

$$F(\omega) = \sum_n f[n] e^{-j2\pi\omega n/\omega_s} = \sum_n f[n] e^{-j\omega n T_s}$$

$$f[n] = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} F\left(e^{j\omega t}\right) e^{j2\pi\omega n/\omega_s} d\omega = \frac{2\pi}{T_s} \int_{-\pi/T_s}^{\pi/T_s} F\left(e^{j\omega t}\right) e^{j\omega n T_s} d\omega$$

$$\omega_s = \frac{2\pi}{T_s} \rightarrow \frac{2\pi\omega}{\omega_s} = \frac{2\pi\omega}{2\pi} T_s = \omega T_s$$

$$T_s = 2\pi / \omega_s$$

sampling interval in time ↔ periodicity in frequency
the closer the samples, the farther the replicas

DTFT with normalized frequency

- Normalized frequency: change of variables

$$\boxed{\Omega = \omega T_s}$$

normalized frequency

$$\boxed{\Omega_s = \omega_s T_s = \frac{2\pi}{T_s} T_s = 2\pi}$$

$$\boxed{\Omega_s = 2\pi}$$

periodicity in the normalized frequency domain

$$F(\Omega) = \sum_{n=-\infty}^{+\infty} f[k] e^{-jn\Omega}$$

$$f[n] = \frac{1}{2\pi} \int_{2\pi} F(\Omega) e^{jk\Omega} d\Omega$$

$$F(\Omega) = \frac{1}{T_s} F_c\left(\frac{\Omega}{T_s}\right)$$

Relation to the CTFT

If Ts>1, the DTFT can be seen as a stretched periodized version of the CTFT.

DTFT with normalized frequency

- F(Ω) can be_obtained from $F_c(\omega)$ by replacing ω with Ω /T_s.
- Thus F(Ω) is identical to F(ω) frequency scaled and stratched by a factor 1/T_s, where T_s is the sampling interval in time domain
- Notations

DTFT →

$$F\left(\Omega\right)=\frac{1}{T_s}F_c\left(\frac{\Omega}{T_s}\right)$$

CTFT

$$\omega_s=\frac{2\pi}{T_s}\rightarrow T_s=\frac{2\pi}{\omega_s}$$

periodicity of the spectrum

$$\omega=\frac{\Omega}{T_s}\rightarrow\Omega=\omega T_s$$

normalized frequency (the spectrum is 2π-periodic)

$$F(\Omega)\rightarrow F\left(\omega T_s\right)=F\left(\omega 2\pi/\omega_s\right)$$

$$F(\Omega)=\sum_{n=-\infty}^{+\infty}f[n]e^{-j\Omega n}\rightarrow F\left(\omega T_s\right)=F(\omega)=\sum_{k=-\infty}^{+\infty}f[n]e^{-j2n\pi\omega/\omega_s}$$

DTFT with *unitary* frequency

$$\Omega = 2\pi u \qquad (\omega = 2\pi f)$$

$$F(\Omega) = \sum_{n=-\infty}^{\infty} f[n]e^{-j\Omega n} \rightarrow F(u) = \sum_{n=-\infty}^{\infty} f[n]e^{-j2\pi nu}$$

$$f[n] = \frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{j\Omega n}d\Omega \rightarrow f[n] = \int_{1} F(u)e^{j2\pi nu}du = \int_{-\frac{1}{2}}^{\frac{1}{2}} F(u)e^{j2\pi nu}du$$

$$\begin{cases} F(u) = \sum_{n=-\infty}^{\infty} f[k]e^{-j2\pi nu} \\ f[n] = \int_{-\frac{1}{2}}^{\frac{1}{2}} F(u)e^{j2\pi nu}du \end{cases}$$

NOTE: when T_s=1, Ω=ω and the spectrum is 2π-periodic. The unitary frequency u=2π/ Ω corresponds to the signal frequency f=2π/ω. This could give a better intuition of the transform properties.

Summary

- Sampled signals are sequences of sampels
- Looking at the sequence as to a set of samples obtained by sampling a real signal with sampling frequency ω_s we can still use the formulas for calculating the transforms as derived for the sequences by
  - Stratching the time axis (and thus squeezing the frequency axis if $T_s>1$)

normalized frequency (sample series)

$$\Omega = \omega T_s$$

frequency (sampled signal)

spectral periodicity in Ω

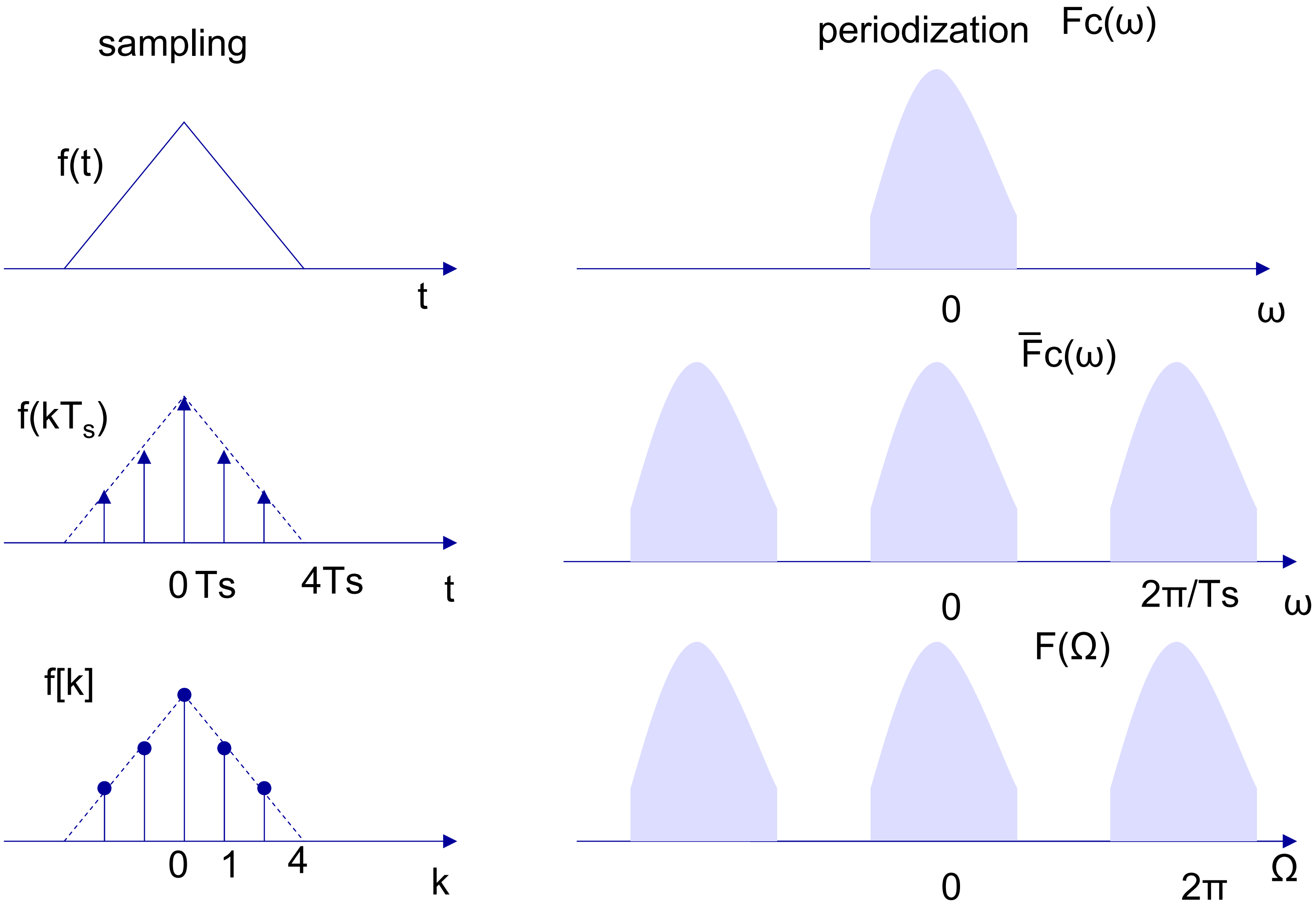
$$2\pi \rightarrow \omega_s = \frac{2\pi}{T_s}$$

spectral periodicity in ω

  - Enclosing the sampling interval T_s in the value of the sequence samples (DFT)

$$f_k = T_s f\left(kT_s\right)$$

Connection DTFT-CTFT



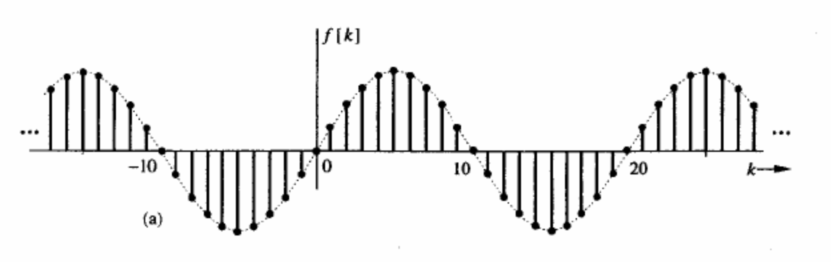
Differences DTFT-CTFT

- The DTFT is periodic with period $\Omega_s=2\pi$ (or $\omega_s=2\pi/T_s$)
- The discrete-time exponential $e^{j\Omega n}$ has a unique waveform only for values of Ω in a continuous interval of 2π
- *Numerical computations can be conveniently performed with the Discrete Fourier Transform (DFT)*

DTFS

- Discrete Time Fourier Series
- Discrete time periodic sequences of period N_0
  - Fundamental frequency

$$\Omega_0 = 2\pi / N_0$$

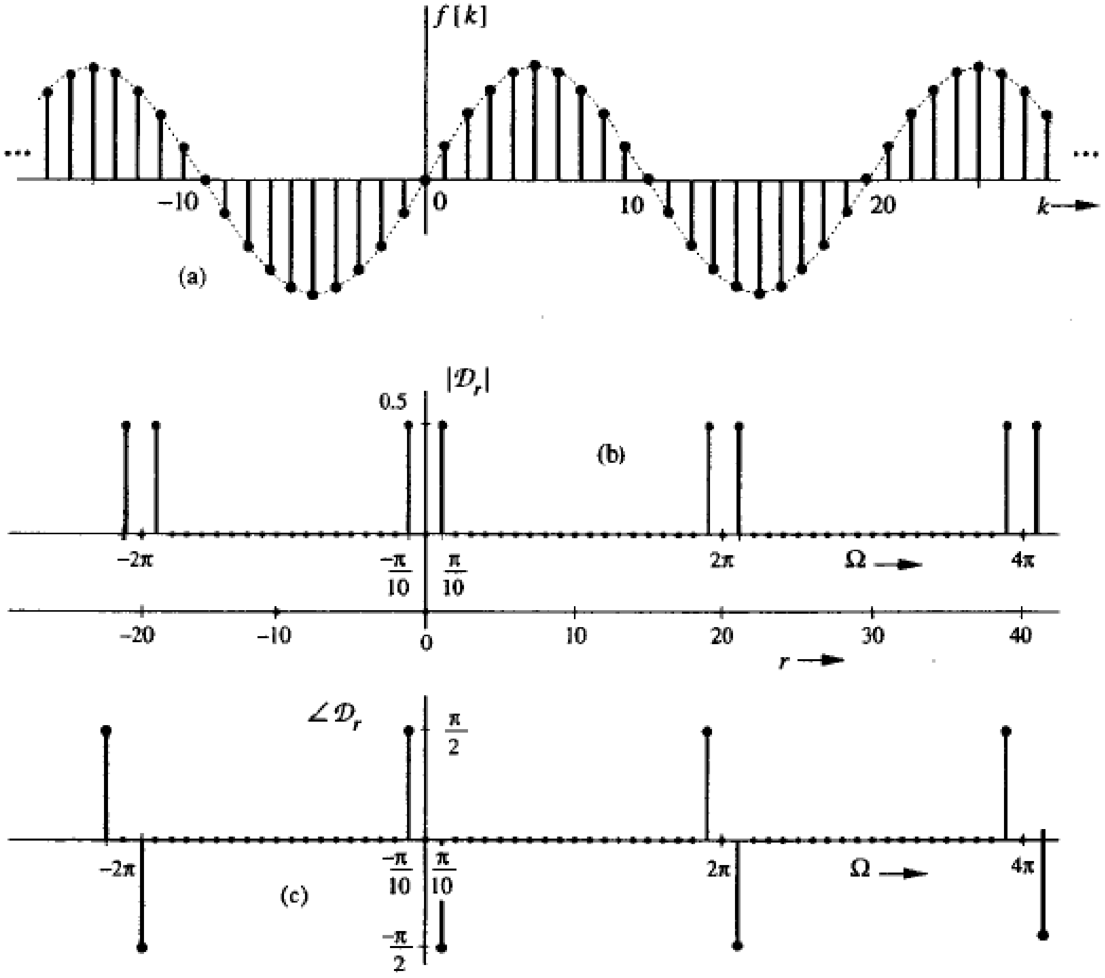


$$F[k] = \frac{1}{N_0} \sum_{n=0}^{N_0-1} f[n] e^{-jkn2\pi/N_0}$$

$$f[n] = \sum_{k=0}^{N_0-1} F[k] e^{jkn2\pi/N_0}$$

DTFS: Example

$N_0=20$
$T_s=1$
$\Omega_s=2\pi$
$\Omega_0=\pi/10$



[Lathi, pag 621]

Fig. 10.1 Discrete-time sinusoid sin $0.1\pi k$ and its Fourier spectra.

Discrete Fourier Transform (DFT)

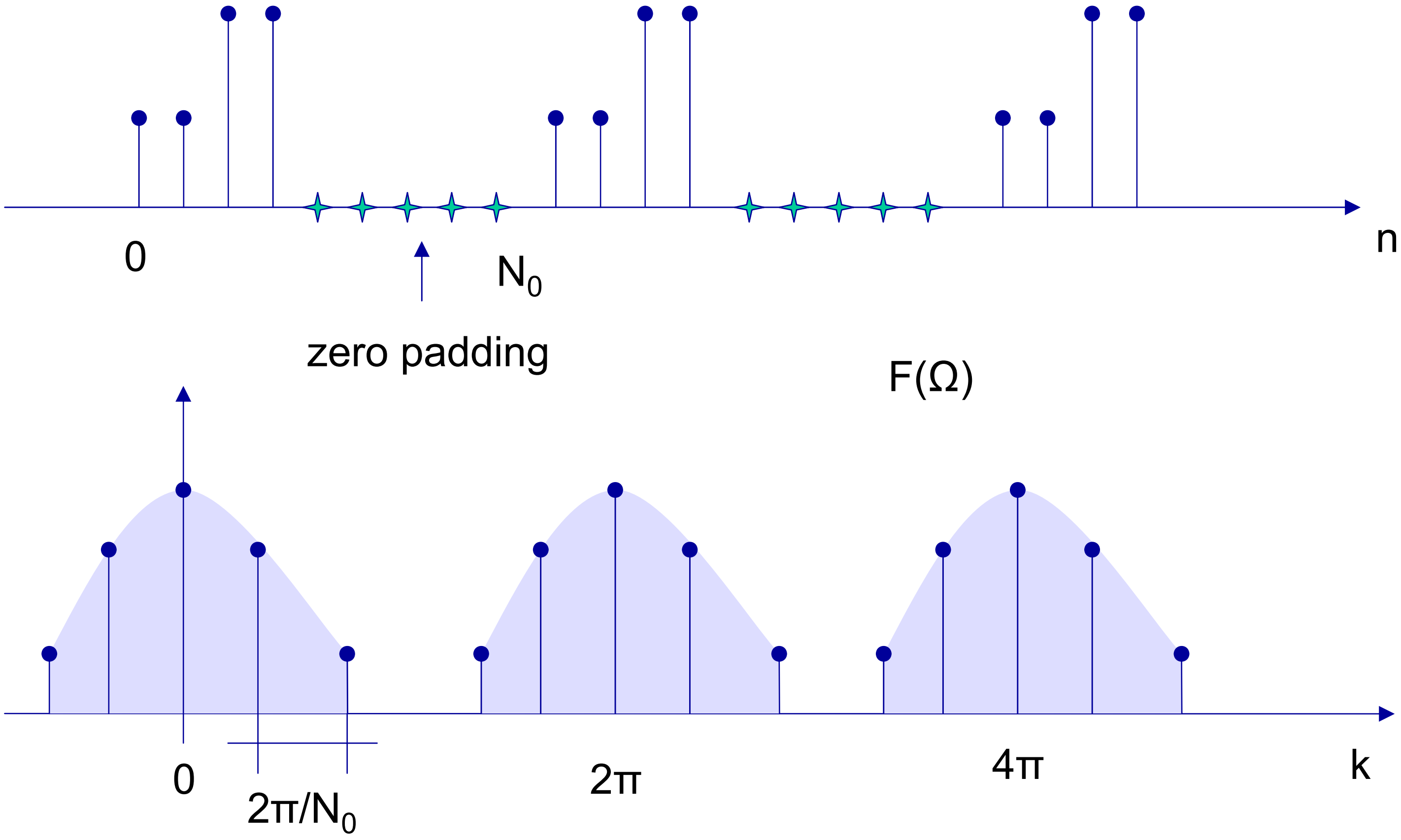
$$F[k] = \sum_{n=0}^{N_0-1} f_n e^{-jn\Omega_0 k} = \sum_{n=0}^{N_0-1} f_n e^{-j\frac{2\pi}{N_0}nk}$$

$$f[n] = \frac{1}{N_0}\sum_{k=0}^{N_0-1} F[k] e^{jn\Omega_0 k} = \frac{1}{N_0}\sum_{k=0}^{N_0-1} F[k] e^{jn\frac{2\pi}{N_0}k}$$

$$\boxed{\Omega_0 = \frac{2\pi}{N_0}}$$

- The DFT transforms N_0 samples of a discrete-time signal to the same number of discrete frequency samples
- The DFT and IDFT are a *self-contained*, one-to-one transform pair for a length-N_0 discrete-time signal (that is, the DFT is not merely an approximation to the DTFT as discussed next)
- However, the DFT is very often used as a practical approximation to the DTFT

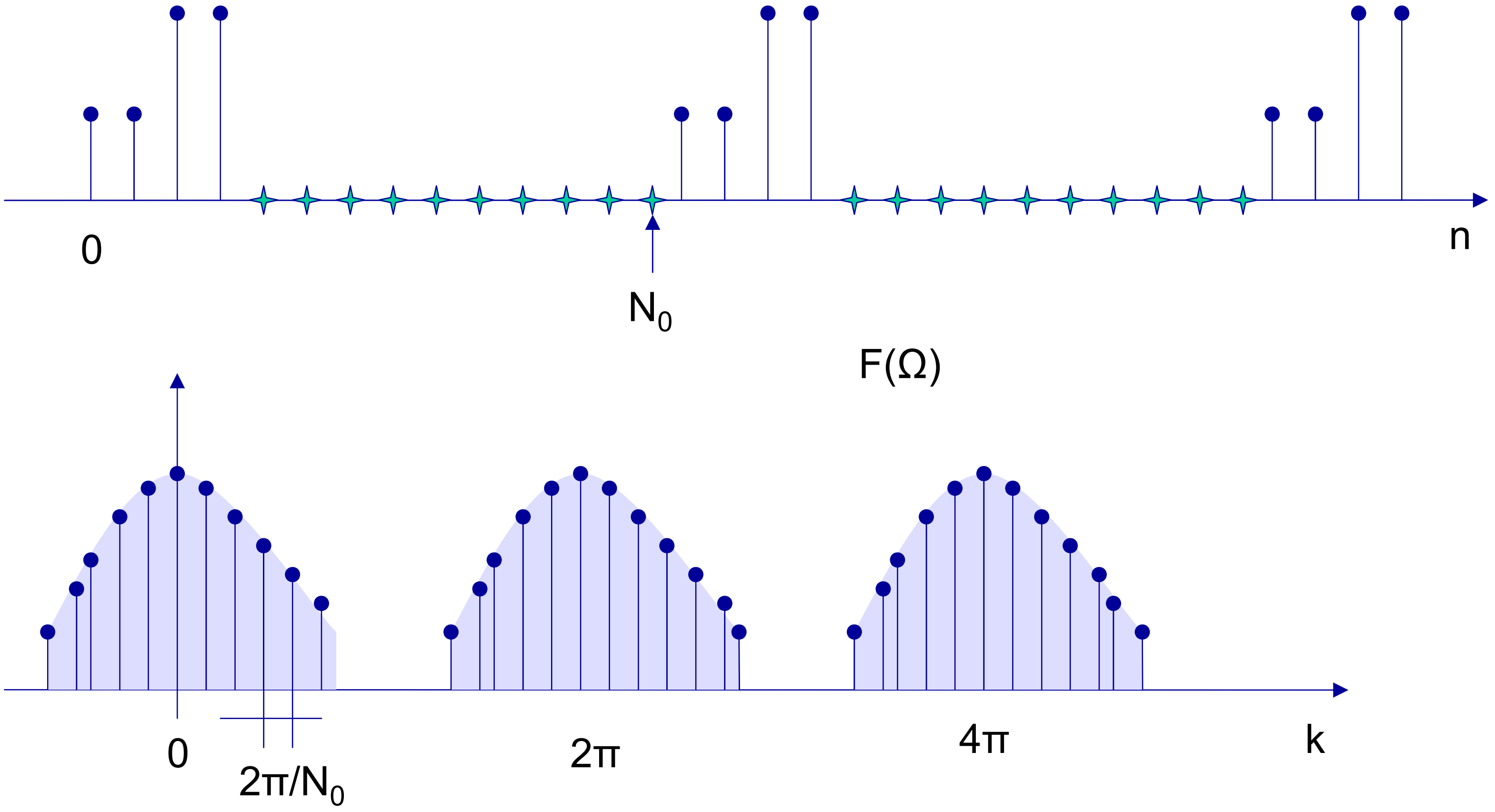
DFT



DFT

Increasing the number of zeros augments the “resolution” of the transform since the samples of the DFT gets “closer”

zero padding



Properties

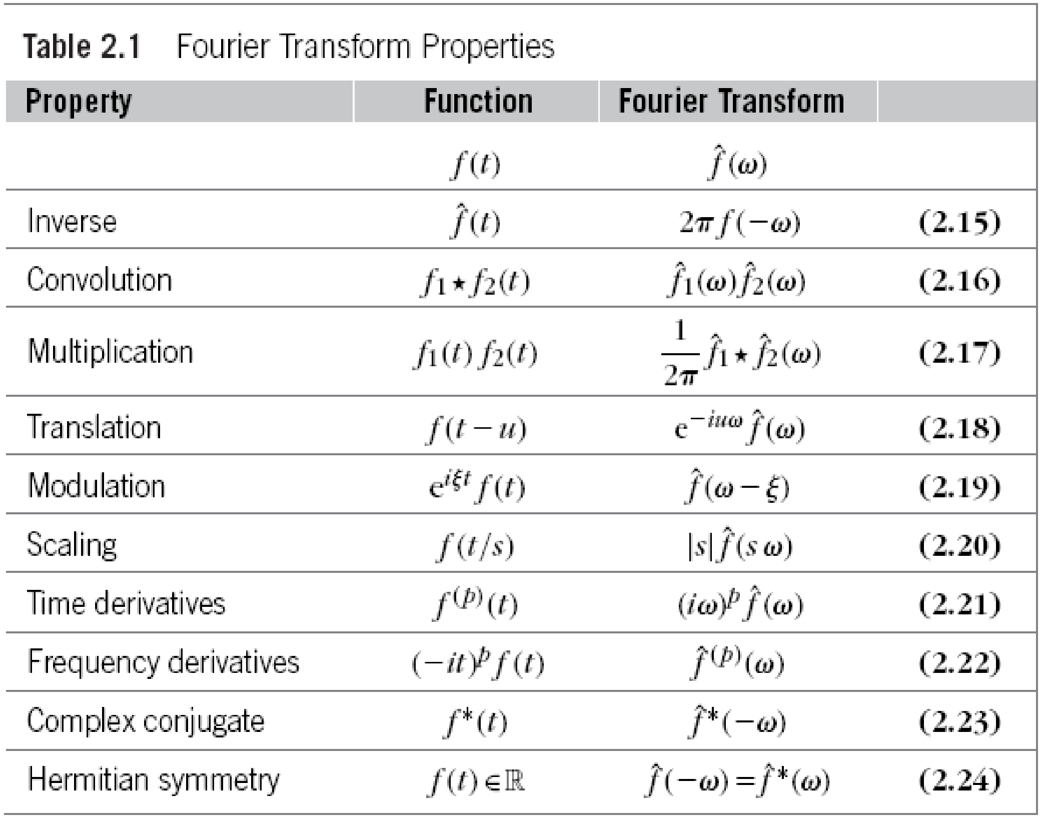


Table 2.1 Fourier Transform Properties

Property	**Function**	**Fourier Transform**	
	$f(t)$	$\hat{f}(\omega)$	
Inverse	$\hat{f}(t)$	$2\pi f(-\omega)$	**(2.15)**
Convolution	$f_1 \star f_2(t)$	$\hat{f}_1(\omega)\hat{f}_2(\omega)$	**(2.16)**
Multiplication	$f_1(t)\, f_2(t)$	$\frac{1}{2\pi}\hat{f}_1 \star \hat{f}_2(\omega)$	**(2.17)**
Translation	$f(t-u)$	$e^{-iu\omega}\hat{f}(\omega)$	**(2.18)**
Modulation	$e^{i\xi t} f(t)$	$\hat{f}(\omega-\xi)$	**(2.19)**
Scaling	$f(t/s)$	$\lvert s\rvert \hat{f}(s\,\omega)$	**(2.20)**
Time derivatives	$f^{(p)}(t)$	$(i\omega)^p \hat{f}(\omega)$	**(2.21)**
Frequency derivatives	$(-it)^p f(t)$	$\hat{f}^{(p)}(\omega)$	**(2.22)**
Complex conjugate	$f^*(t)$	$\hat{f}^*(-\omega)$	**(2.23)**
Hermitian symmetry	$f(t) \in \mathbb{R}$	$\hat{f}(-\omega) = \hat{f}^*(\omega)$	**(2.24)**

For real signals f(t)

$$f(t) \rightarrow \hat{f}(\omega)$$

$$f(-t) \rightarrow \hat{f}(-\omega) = \hat{f}^*(\omega)$$

Proof

$$\Im\{f(-t)\} = \int_{-\infty}^{+\infty} f(-t) e^{-j\omega t} dt = \int_{-\infty}^{+\infty} f(t') e^{j\omega t'} dt' = \hat{f}(-\omega)$$

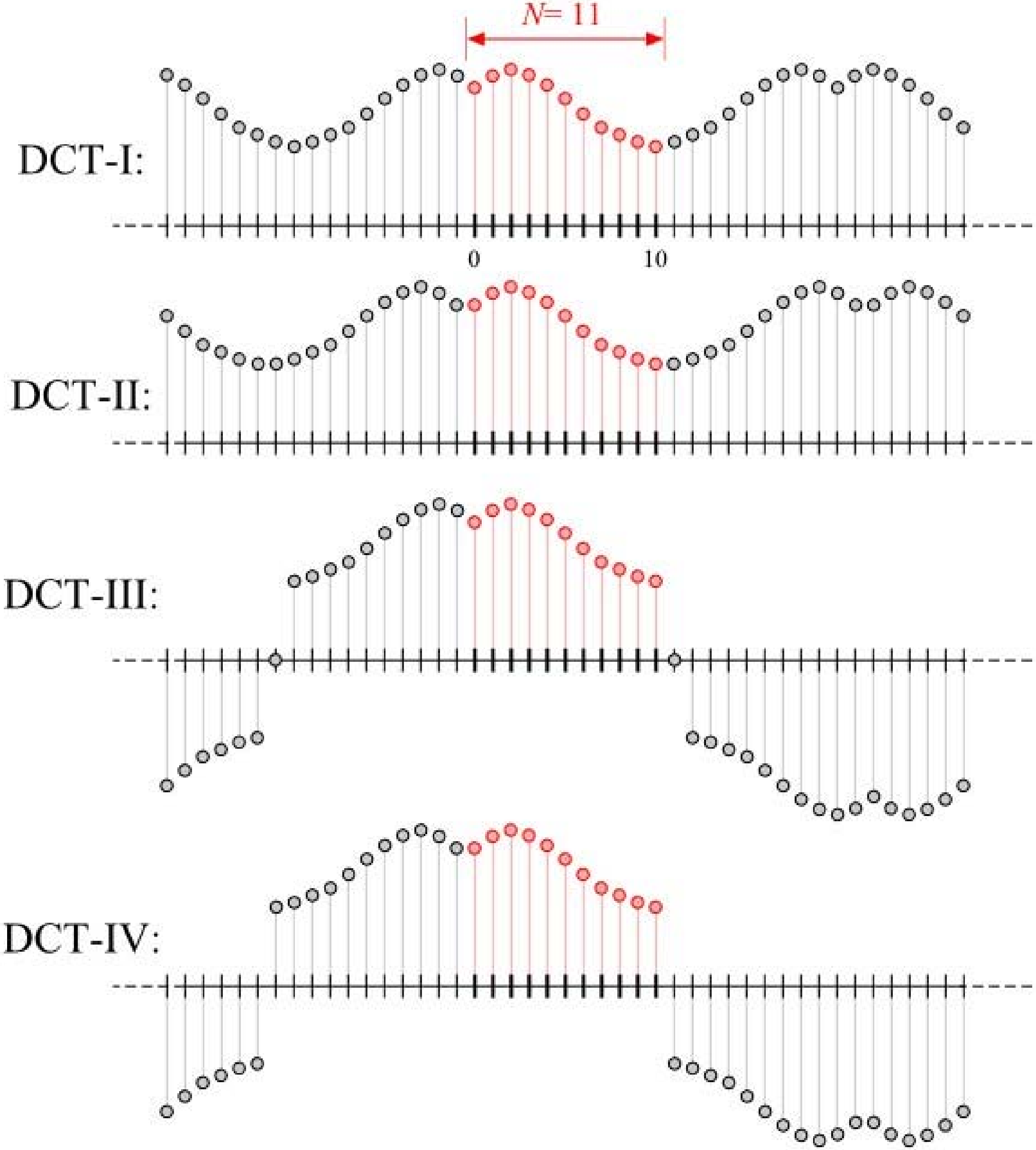
Discrete *Cosine* Transform (DCT)

- Operate on finite discrete sequences (as DFT)
- A **discrete cosine transform** (**DCT**) expresses a sequence of finitely many data points in terms of a sum of cosine functions oscillating at different frequencies
- DCT is a Fourier-related transform similar to the DFT but using only real numbers
- DCT is equivalent to DFT of roughly twice the length, operating on real data with even symmetry (since the Fourier transform of a real and even function is real and even), where in some variants the input and/or output data are shifted by half a sample
- There are eight standard DCT variants, out of which four are common
- Strong connection with the Karunen-Loeven transform
  - VERY important for signal compression

DCT

- DCT implies different boundary conditions than the DFT or other related transforms
- A DCT, like a cosine transform, implies an *even periodic* extension of the original function
- Tricky part
  - First, one has to specify whether the function is even or odd at *both* the left and right boundaries of the domain
  - Second, one has to specify around *what point* the function is even or odd
    - In particular, consider a sequence *abcd* of four equally spaced data points, and say that we specify an even *left* boundary. There are two sensible possibilities: either the data is even about the sample *a*, in which case the even extension is ***dcbabcd***, or the data is even about the point *halfway* between *a* and the previous point, in which case the even extension is ***dcbaabcd*** (*a* is repeated).

Symmetries



DCT

$$X_k = \sum_{n=0}^{N_0-1} x_n \cos\left[\frac{\pi}{N_0}\left(n+\frac{1}{2}\right)k\right] \qquad k = 0,\ldots., N_0 - 1$$

$$x_n = \frac{2}{N_0}\left\{\frac{1}{2}X_0 + \sum_{k=0}^{N_0-1} X_k \cos\left[\frac{\pi k}{N_0}\left(k+\frac{1}{2}\right)\right]\right\}$$

- *Warning*: the normalization factor in front of these transform definitions is merely a convention and differs between treatments.
  - Some authors multiply the transforms by $(2/N_0)^{1/2}$ so that the inverse does not require any additional multiplicative factor.
    - Combined with appropriate factors of √2 (see above), this can be used to make the transform matrix orthogonal.

Images vs Signals

1D

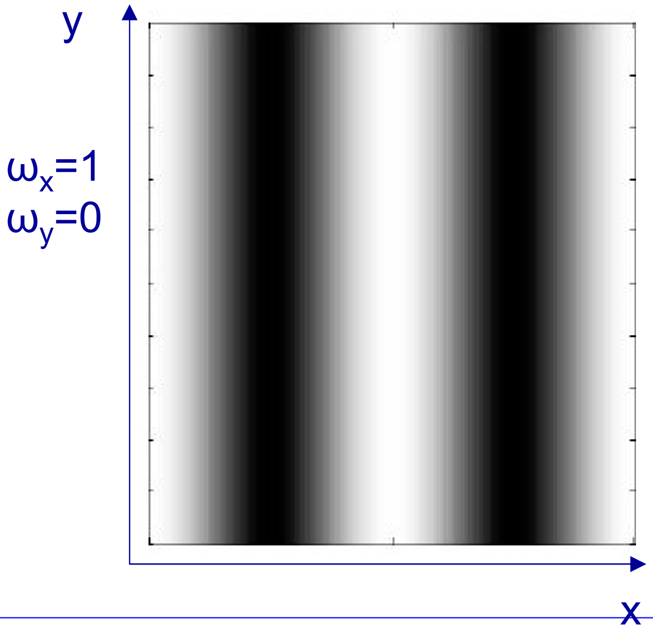
- Signals
- Frequency
  - Temporal
  - Spatial
- Time (space) frequency characterization of signals
- Reference space for
  - Filtering
  - Changing the sampling rate
  - Signal analysis
  - ….

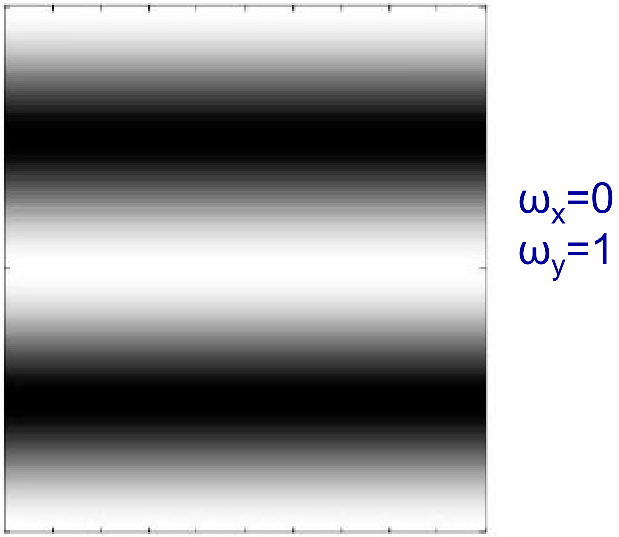
2D

- Images
- Frequency
  - Spatial
- Space/frequency characterization of 2D signals
- Reference space for
  - Filtering
  - Up/Down sampling
  - Image analysis
  - Feature extraction
  - Compression
  - ….

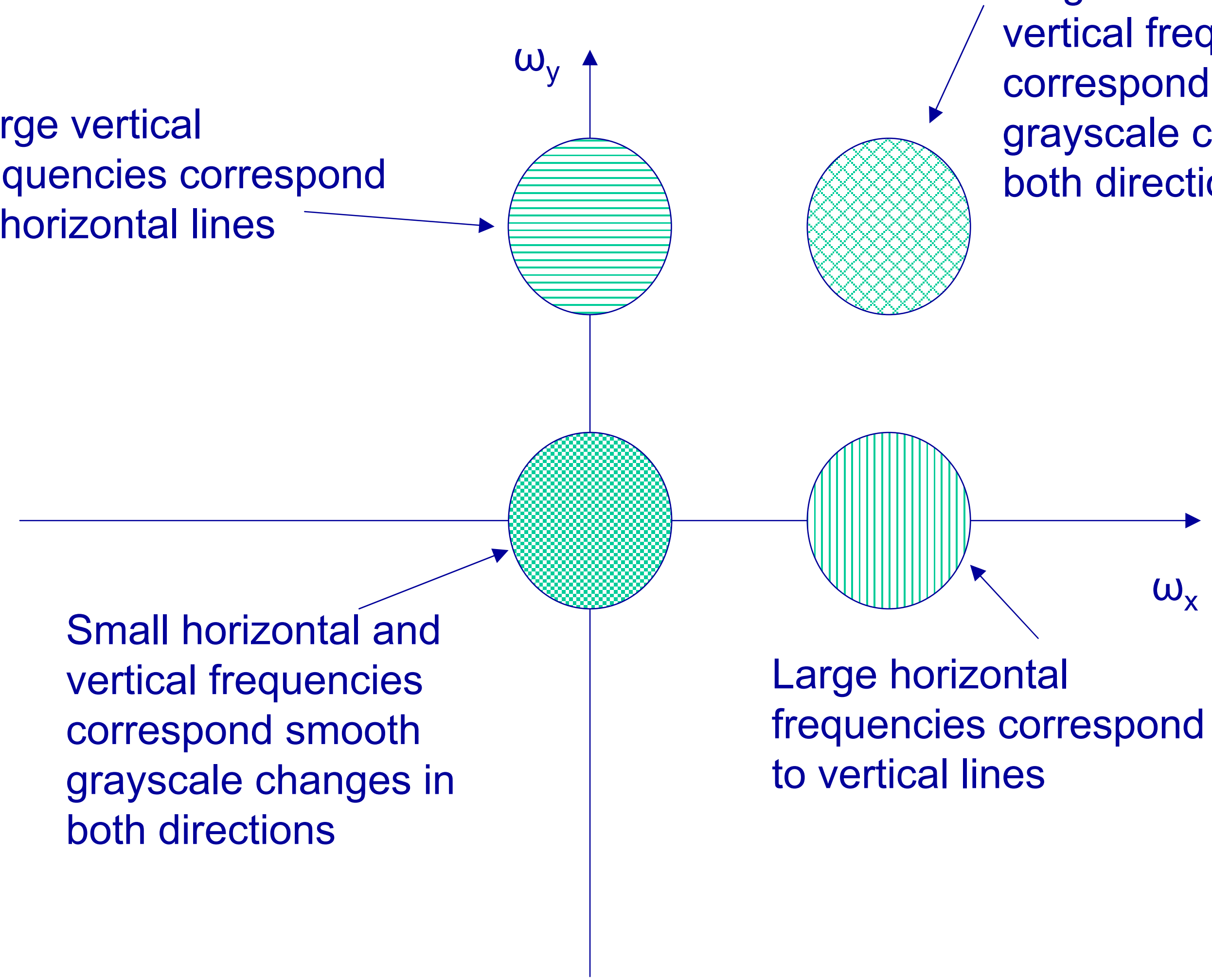
2D spatial frequencies

- 2D spatial frequencies characterize the image spatial changes in the horizontal (x) and vertical (y) directions
  - Smooth variations -> low frequencies
  - Sharp variations -> high frequencies





2D Frequency domain



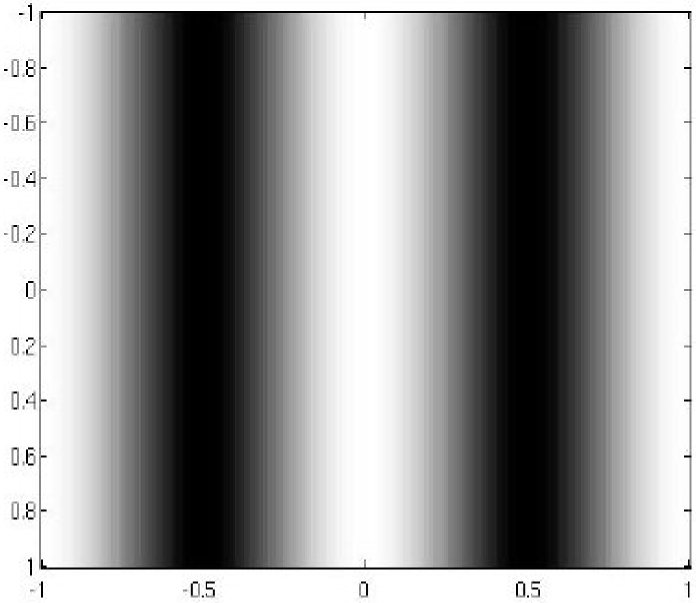
Large vertical frequencies correspond to horizontal lines

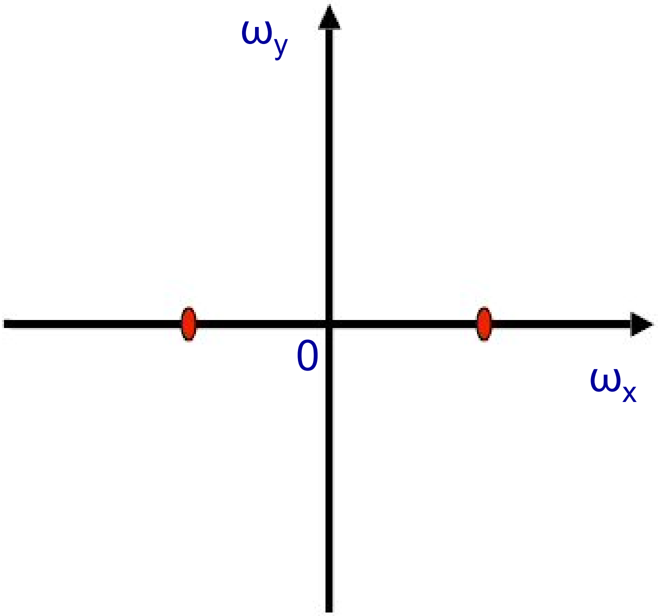
Large horizontal and vertical frequencies correspond sharp grayscale changes in both directions

Small horizontal and vertical frequencies correspond smooth grayscale changes in both directions

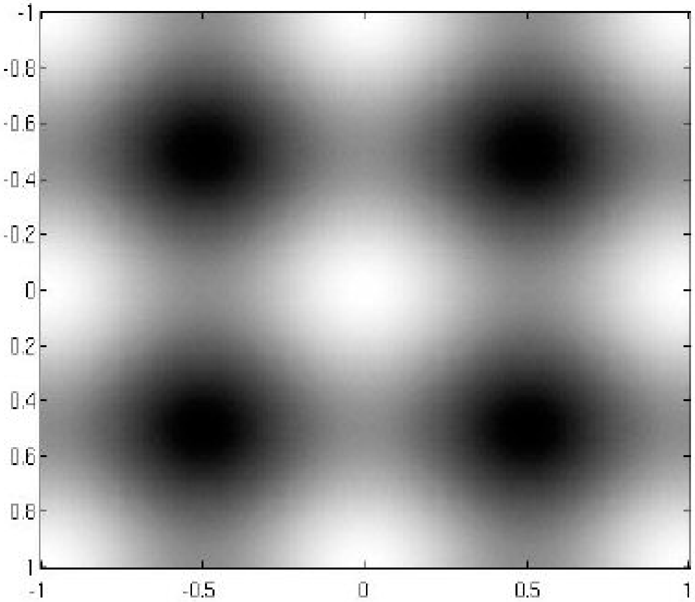
Large horizontal frequencies correspond to vertical lines

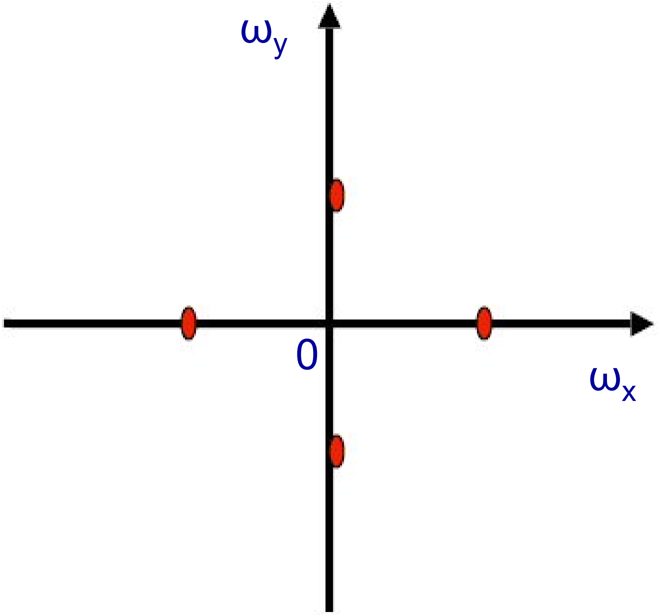
Vertical grating



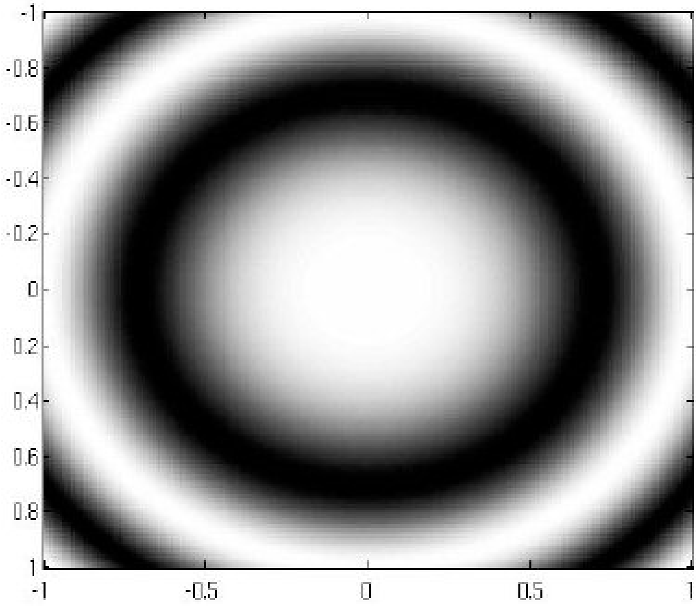


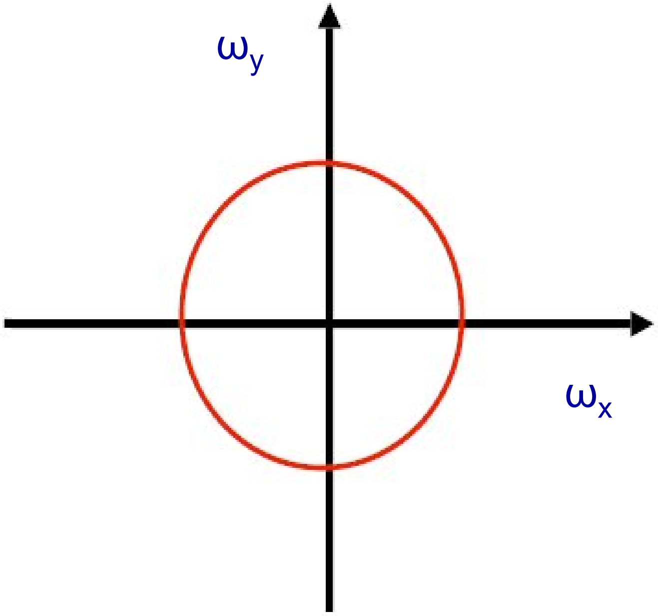
Double grating





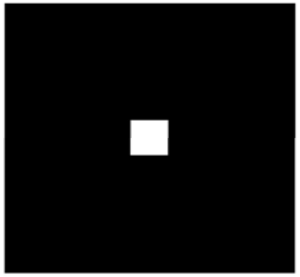
Smooth rings

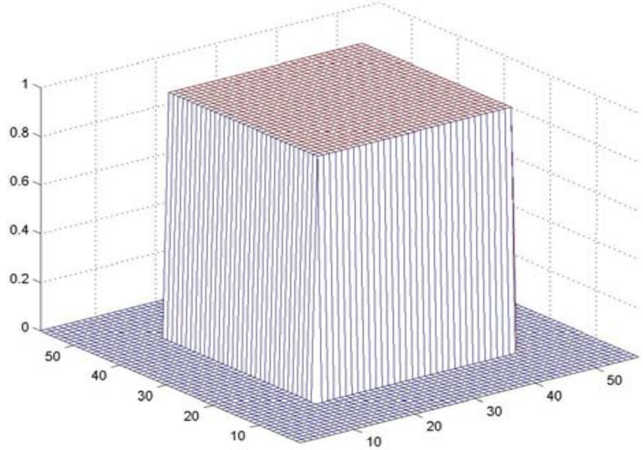


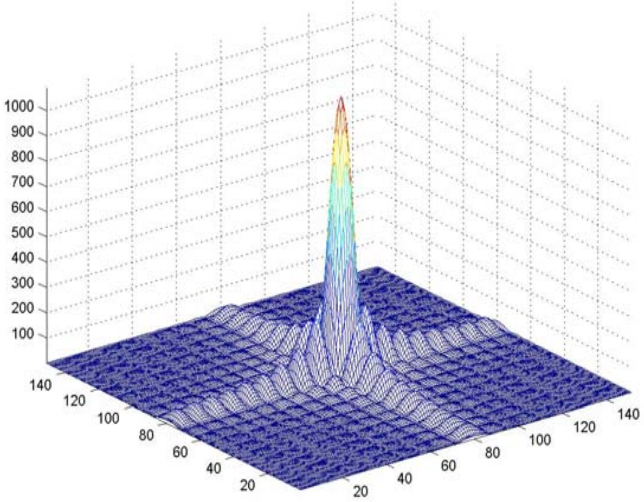


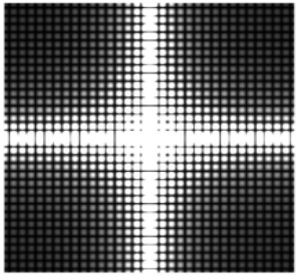
2D box

2D sinc

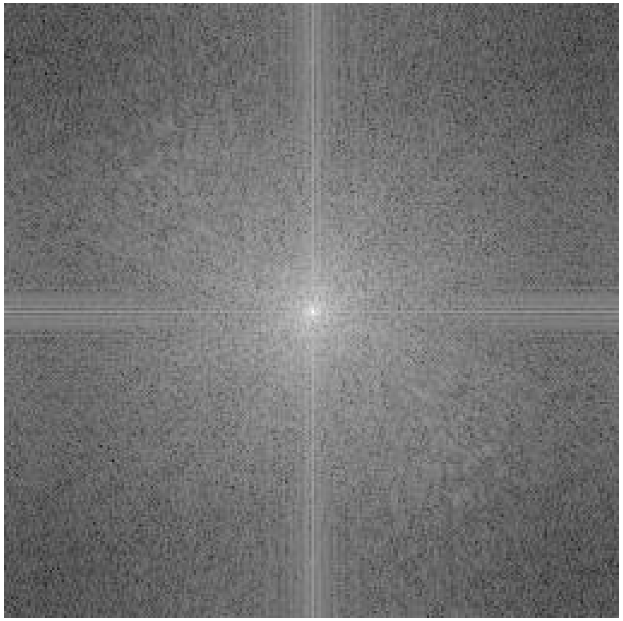






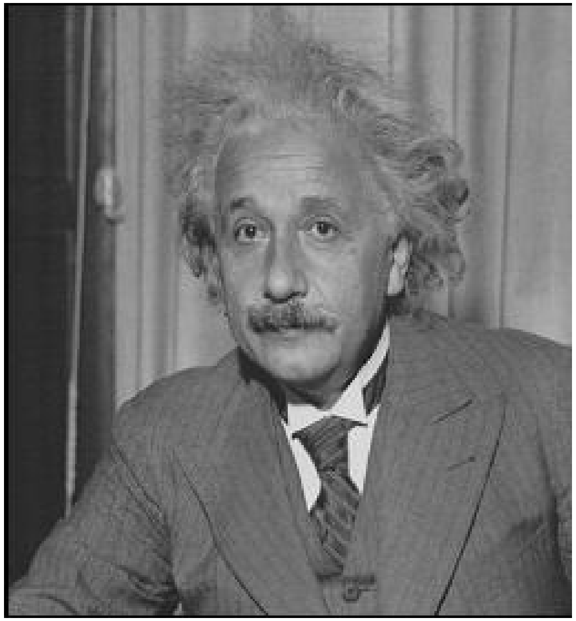


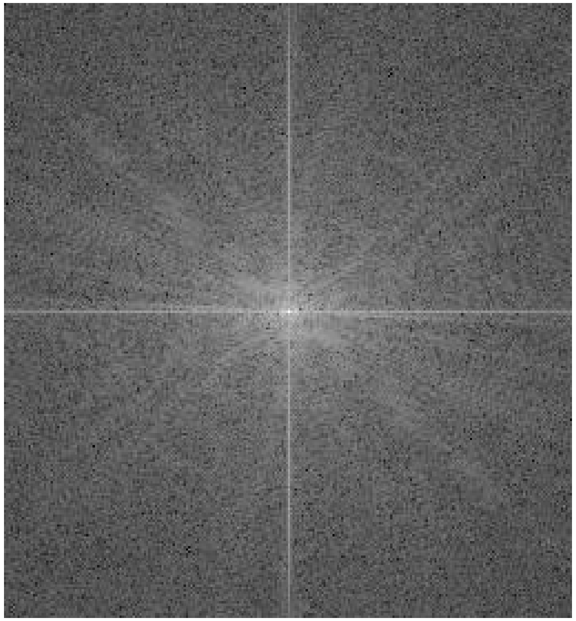
Margherita Hack



log amplitude of the spectrum

Einstein





log amplitude of the spectrum

What we are going to analyze

- 2D Fourier Transform of continuous signals (2D-CTFT)

1D $$F(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt, f(t) = \int_{-\infty}^{+\infty} F(\omega)e^{j\omega t}dt$$

- 2D Fourier Transform of discrete space signals (2D-DTFT)

1D $$F(\Omega) = \sum_{k=-\infty}^{\infty} f[k]e^{-j\Omega k}, f[k] = \frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{j\Omega k}dt$$

- 2D Discrete Fourier Transform (2D-DFT)

1D $$F_r = \sum_{k=0}^{N_0-1} f[k]e^{-jr\Omega_0 k}, f_{N_0}[k] = \frac{1}{N_0}\sum_{r=0}^{N_0-1} F_r e^{jr\Omega_0 k}, \Omega_0 = \frac{2\pi}{N_0}$$

2D Continuous Fourier Transform

- Continuous case (x and y are real) – 2D-CTFT (notation 1)

$$\hat{f}\left(\omega_x,\omega_y\right)=\int_{-\infty}^{+\infty} f\left(x,y\right)e^{-j\left(\omega_x x+\omega_y y\right)}dxdy$$

$$f\left(x,y\right)=\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\hat{f}\left(\omega_x,\omega_y\right)e^{j\left(\omega_x x+\omega_y y\right)}d\omega_x d\omega_y$$

$$\iint f\left(x,y\right)g^*\left(x,y\right)dxdy=\frac{1}{4\pi^2}\iint \hat{f}\left(\omega_x,\omega_y\right)\hat{g}^*\left(\omega_x,\omega_y\right)d\omega_x d\omega_y$$ Parseval formula

$$f=g\rightarrow\iint\left|f\left(x,y\right)\right|^2dxdy=\frac{1}{4\pi^2}\iint\left|\hat{f}\left(\omega_x,\omega_y\right)\right|^2d\omega_x d\omega_y$$ Plancherel equality

2D Continuous Fourier Transform

- Continuous case (x and y are real) – 2D-CTFT

$$\omega_x = 2\pi u$$

$$\omega_y = 2\pi v$$

$$\hat{f}(u,v) = \int_{-\infty}^{+\infty} f(x,y) e^{-j2\pi(ux+vy)} dxdy$$

$$f(x,y) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \hat{f}(u,v) e^{j2\pi(ux+vy)} (2\pi)^2 dudv =$$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \hat{f}(u,v) e^{j2\pi(ux+vy)} (2\pi)^2 dudv$$

2D Continuous Fourier Transform

- 2D Continuous Fourier Transform (notation 2)

$$\hat{f}\left(u,v\right)=\int_{-\infty}^{+\infty}f\left(x,y\right)e^{-j2\pi\left(ux+vy\right)}dxdy$$

$$f\left(x,y\right)=\int_{-\infty}^{+\infty}\hat{f}\left(u,v\right)e^{j2\pi\left(ux+vy\right)}dudv=$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|f(x,y)\right|^2 dxdy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\hat{f}(u,v)\right|^2 dudv$$

Plancherel’s equality

2D Discrete Fourier Transform

The independent variable (t,x,y) is discrete

$$F_r = \sum_{k=0}^{N_0-1} f[k] e^{-jr\Omega_0 k}$$

$$f_{N_0}[k] = \frac{1}{N_0} \sum_{r=0}^{N_0-1} F_r e^{jr\Omega_0 k}$$

$$\Omega_0 = \frac{2\pi}{N_0}$$

$$\Rightarrow$$

$$F[u,v] = \sum_{i=0}^{N_0-1} \sum_{k=0}^{N_0-1} f[i,k] e^{-j\Omega_0 (ui+vk)}$$

$$f_{N_0}[i,k] = \frac{1}{{N_0}^2} \sum_{u=0}^{N_0-1} \sum_{v=0}^{N_0-1} F[u,v] e^{j\Omega_0 (ui+vk)}$$

$$\Omega_0 = \frac{2\pi}{N_0}$$

[Lathi’s notations]

Delta

- Sampling property of the 2D-delta function (Dirac’s delta)

$$\int_{-\infty}^{\infty} \delta(x - x_0, y - y_0) f(x, y) dxdy = f(x_0, y_0)$$

- Transform of the delta function

$$F\{\delta(x, y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x, y) e^{-j2\pi(ux+vy)} dxdy = 1$$

$$F\{\delta(x - x_0, y - y_0)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - x_0, y - y_0) e^{-j2\pi(ux+vy)} dxdy = e^{-j2\pi(ux_0+vy_0)}$$

shifting property

Constant functions

- Inverse transform of the impulse function

$$F^{-1}\left\{\delta(u,v)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(u,v)e^{j2\pi(ux+vy)}dudv = e^{j2\pi(0x+v0)} = 1$$

- Fourier Transform of the constant (=1 for all x and y)

$$k(x,y) = 1 \qquad \forall x, y$$

$$F\left(u,v\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi(ux+vy)}dxdy = \delta(u,v)$$

Trigonometric functions

- Cosine function oscillating along the x axis
  - Constant along the y axis

$$s(x,y) = \cos(2\pi fx)$$

$$F\left\{\cos(2\pi fx)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos(2\pi fx) e^{-j2\pi(ux+vy)} dxdy =$$

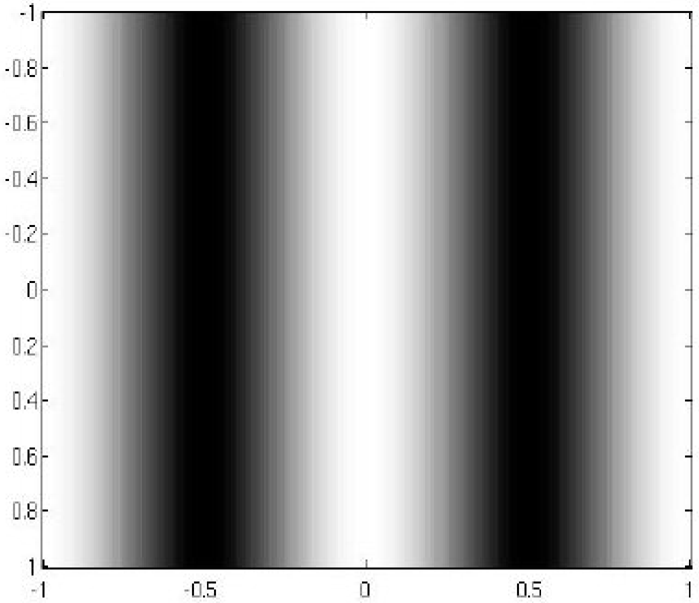
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{e^{j2\pi(fx)} + e^{-j2\pi(fx)}}{2}\right] e^{-j2\pi(ux+vy)} dxdy$$

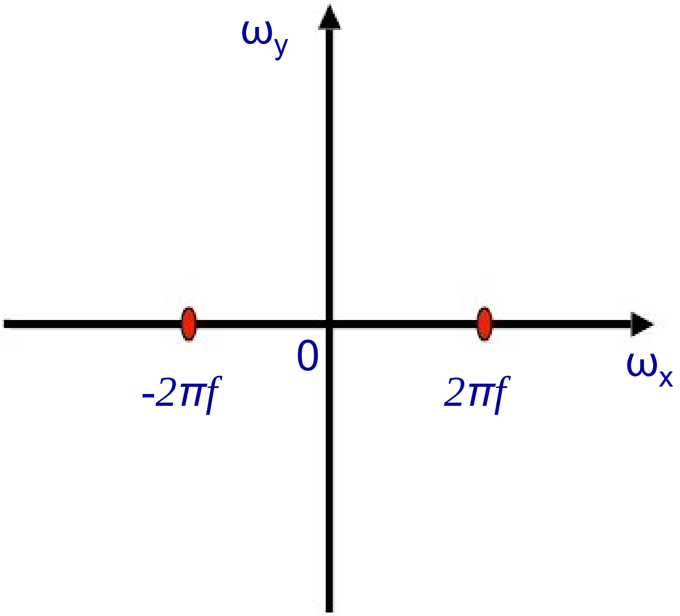
$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[e^{-j2\pi(u-f)x} + e^{-j2\pi(u+f)x}\right] e^{-j2\pi vy} dxdy =$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-j2\pi vy} dy \int_{-\infty}^{\infty} \left[e^{-j2\pi(u-f)x} + e^{-j2\pi(u+f)x}\right] dx = \frac{1}{2} 1 \int_{-\infty}^{\infty} \left[e^{-j2\pi(u-f)x} + e^{-j2\pi(u+f)x}\right] dx =$$

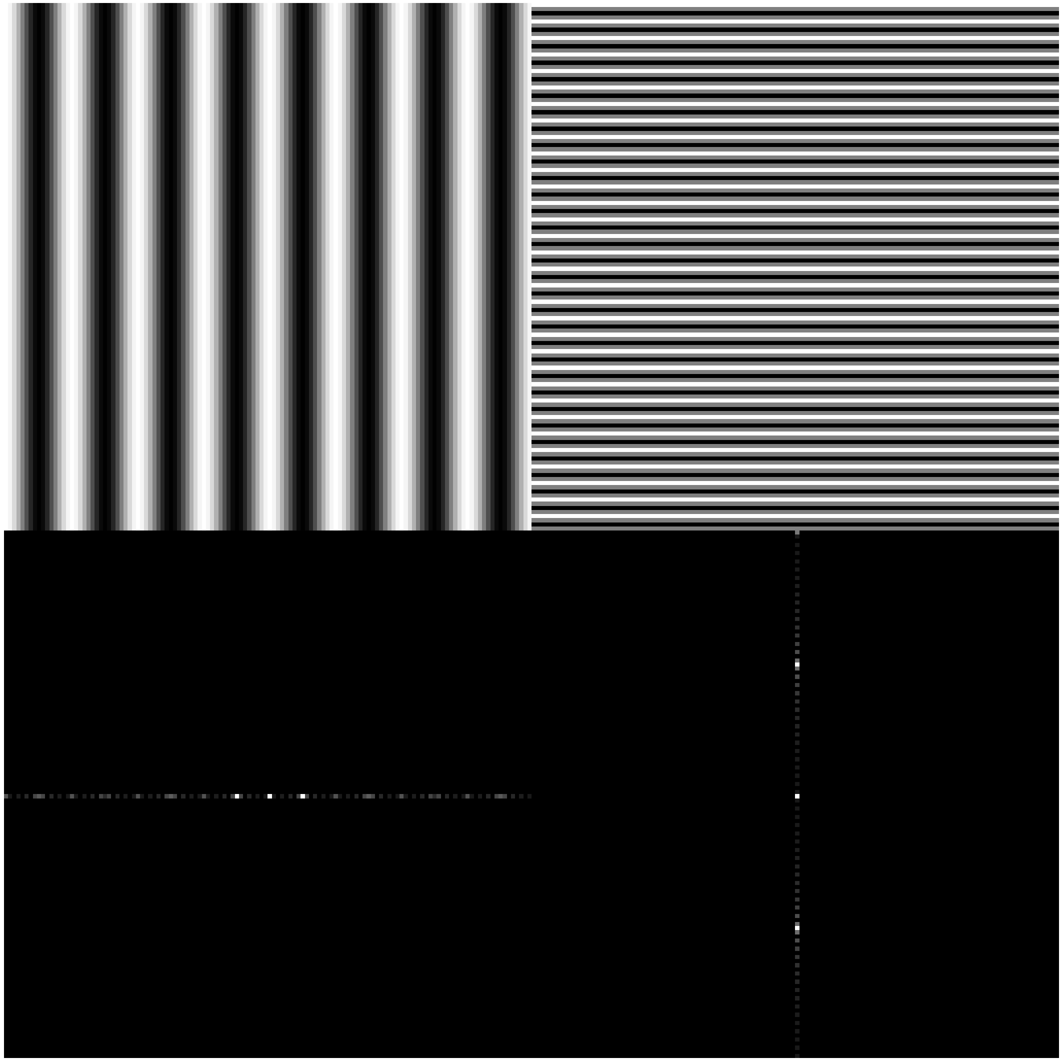
$$\frac{1}{2}\left[\delta(u-f) + \delta(u+f)\right]$$

Vertical grating

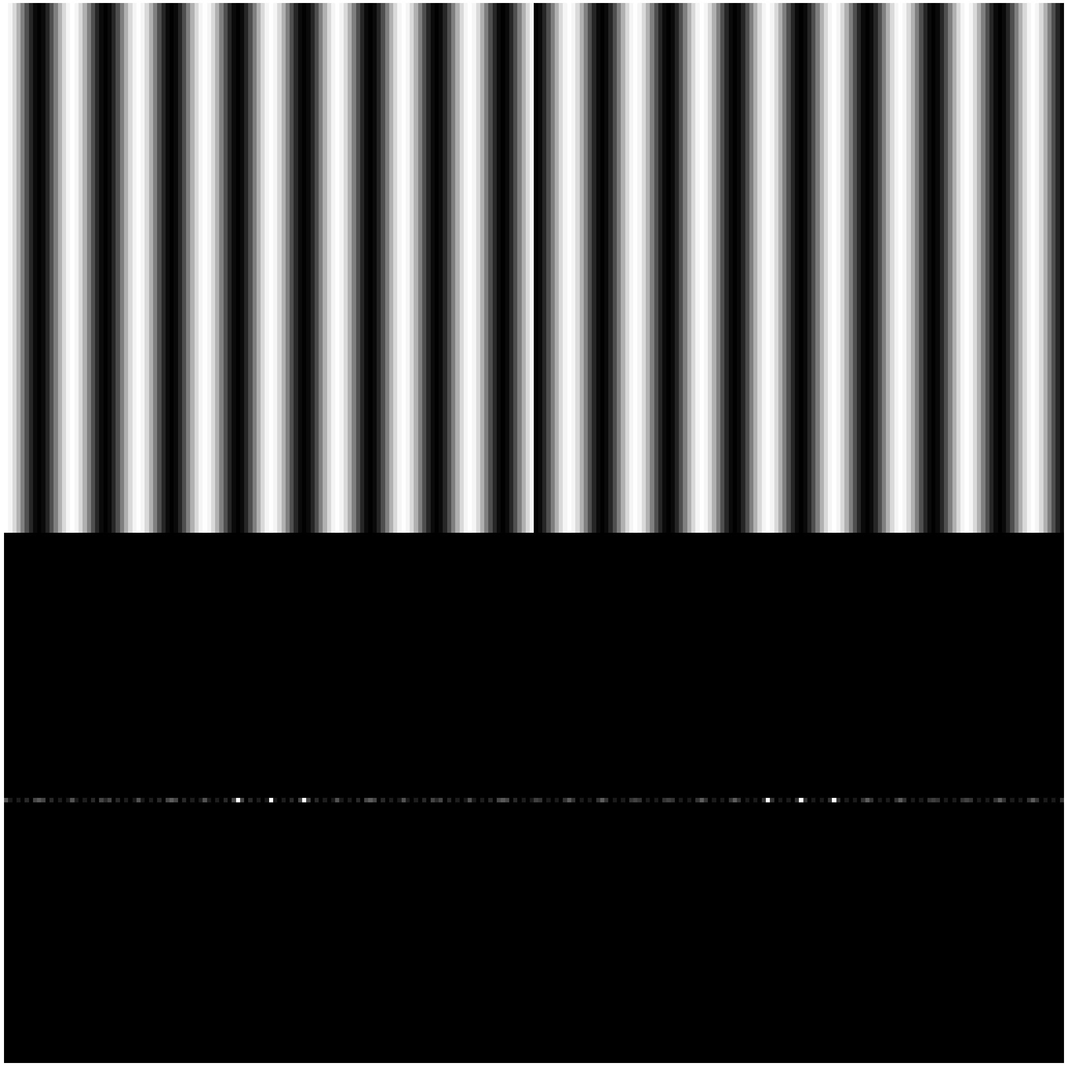




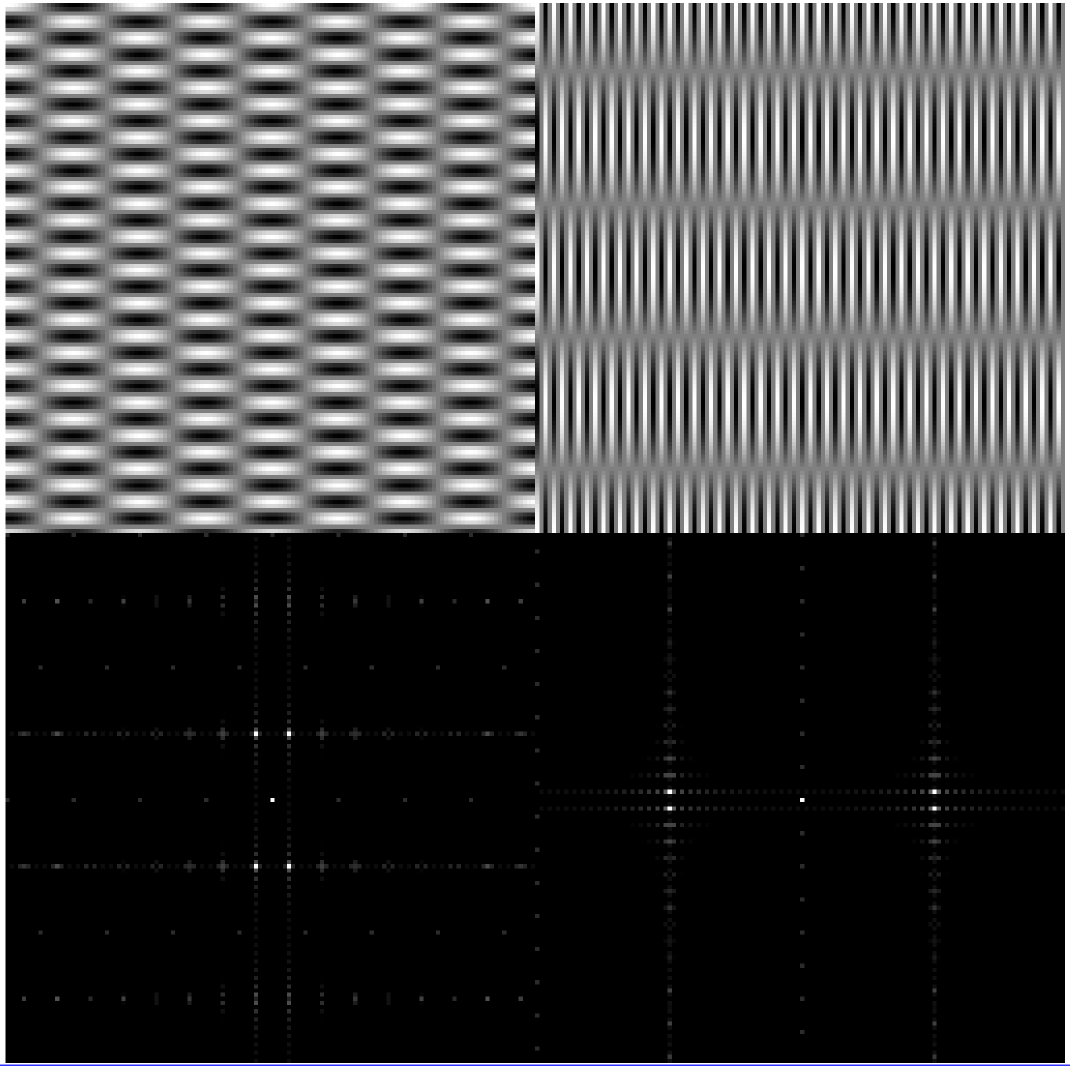
Ex. 1



Ex. 2

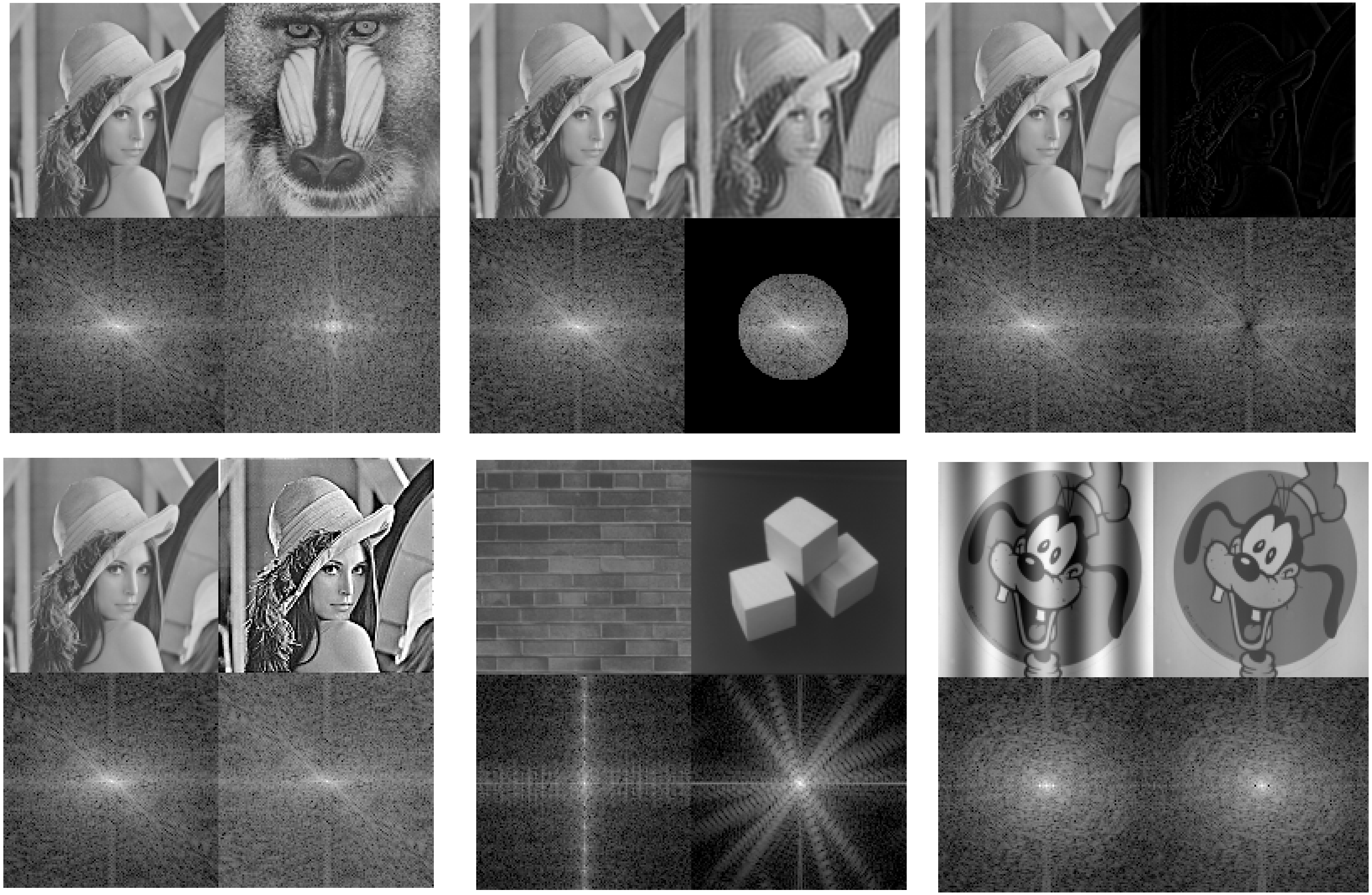


Ex. 3



Magnitudes

Examples



Properties

- Linearity $af(x,y)+bg(x,y) \Leftrightarrow aF(u,v)+bG(u,v)$
- Shifting $f(x-x_0, y-x_0) \Leftrightarrow e^{-j2\pi(ux_0+vy_0)}F(u,v)$
- Modulation $e^{j2\pi(u_0x+v_0y)}f(x,y) \Leftrightarrow F(u-u_0, v-v_0)$
- Convolution $f(x,y)*g(x,y) \Leftrightarrow F(u,v)G(u,v)$
- Multiplication $f(x,y)g(x,y) \Leftrightarrow F(u,v)*G(u,v)$
- Separability $f(x,y)=f(x)f(y) \Leftrightarrow F(u,v)=F(u)F(v)$

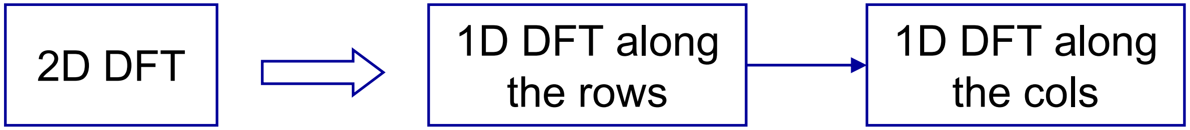
Separability

1. Separability of the 2D Fourier transform
    - 2D Fourier Transforms can be implemented as a sequence of 1D Fourier Transform operations performed *independently* along the two axis

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi ux}e^{-j2\pi vy}dxdy = \int_{-\infty}^{\infty} e^{-j2\pi vy}dy\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi ux}dx =$$

$$= \int_{-\infty}^{\infty} F(u,y)e^{-j2\pi vy}dy = F\left(u,v\right)$$



2D DFT ⇒ 1D DFT along the rows → 1D DFT along the cols

Separability

- Separable functions can be written as $f\left(x,y\right)=f\left(x\right)g\left(y\right)$

2. The FT of a separable function is the product of the FTs of the two functions

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-j2\pi(ux+vy)}dxdy=$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(x)g\left(y\right)e^{-j2\pi ux}e^{-j2\pi vy}dxdy=\int_{-\infty}^{\infty}g\left(y\right)e^{-j2\pi vy}dy\int_{-\infty}^{\infty}h(x)e^{-j2\pi ux}dx=$$

$$=H\left(u\right)G\left(v\right)$$

$$f\left(x,y\right)=h\left(x\right)g\left(y\right)\Rightarrow F\left(u,v\right)=H\left(u\right)G\left(v\right)$$

2D Fourier Transform of a Discrete function

- Fourier Transform of a 2D *a-periodic* signal defined over a 2D discrete grid
  - The grid can be thought of as a 2D brush used for sampling the continuous signal with a given spatial resolution (T_x,T_y)

1D

$$F(\Omega) = \sum_{k=-\infty}^{\infty} f[k]e^{-j\Omega k},\ f[k] = \frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{j\Omega k}dt$$

2D

$$F(\Omega_x,\Omega_y) = \sum_{k_1=-\infty}^{+\infty}\sum_{k_2=-\infty}^{+\infty} f[k_1,k_2]e^{-j\left(k_1\Omega_x+k_2\Omega_y\right)}$$

$$f[k] = \frac{1}{4\pi^2}\int_{2\pi}\int_{2\pi} F(\Omega_x,\Omega_y)e^{j\left(k_1\Omega_x+k_2\Omega_y\right)}d\Omega_x\Omega_y$$

Ω_x,Ω_y: normalized frequency

Unitary frequency notations

$$\begin{cases} \Omega_x = 2\pi u \\ \Omega_y = 2\pi v \end{cases}$$

$$F(u,v) = \sum_{k_1=-\infty}^{+\infty} \sum_{k_2=-\infty}^{+\infty} f[k_1,k_2] e^{-j2\pi(k_1 u + k_2 v)}$$

$$f[k_1,k_2] = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} F(u,v) e^{-j2\pi(k_1 u + k_2 v)} du dv$$

- The integration interval for the inverse transform has width=1 instead of 2π
  - It is quite common to choose

$$\frac{-1}{2} \leq u,v < \frac{1}{2}$$

Properties

- Periodicity: 2D Fourier Transform of a *discrete* a-periodic signal is *periodic*
  - *The period is 1 for the unitary frequency notations and 2π for normalized frequency notations*.
  - Proof (referring to the firsts case)

$$F(u+k,v+l)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f[m,n]e^{-j2\pi\left((u+k)m+(v+l)n\right)}$$

Arbitrary integers

$$=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f[m,n]e^{-j2\pi\left(um+vn\right)}\overset{1}{e^{-j2\pi km}}\,\overset{1}{e^{-j2\pi ln}}$$

$$=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f[m,n]e^{-j2\pi(um+vn)}$$

$$=F(u,v)$$

Properties

- Linearity
- shifting
- modulation
- convolution
- multiplication
- separability
- energy conservation properties also exist for the 2D Fourier Transform of discrete signals.
- NOTE: in what follows, (k_1,k_2) is replaced by (m,n)

2D DTFT Properties

- Linearity $af[m,n]+bg[m,n] \Leftrightarrow aF(u,v)+bG(u,v)$
- Shifting $f[m-m_0,n-n_0] \Leftrightarrow e^{-j2\pi(um_0+vn_0)}F(u,v)$
- Modulation $e^{j2\pi(u_0m+v_0n)}f[m,n] \Leftrightarrow F(u-u_0,v-v_0)$
- Convolution $f[m,n]*g[m,n] \Leftrightarrow F(u,v)G(u,v)$
- Multiplication $f[m,n]g[m,n] \Leftrightarrow F(u,v)*G(u,v)$
- Separable functions $f[m,n]=f[m]f[n] \Leftrightarrow F(u,v)=F(u)F(v)$
- Energy conservation $\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}|f[m,n]|^2 = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}|F(u,v)|^2\,dudv$

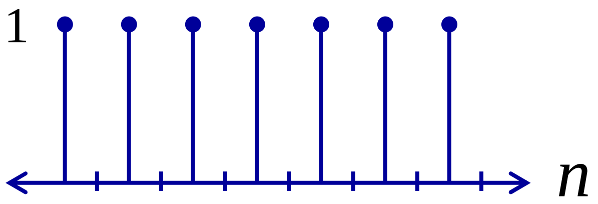
Impulse Train

- Define a *comb* function (impulse train) as follows

$$comb_{M,N}[m,n]=\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta[m-kM,n-lN]$$

where *M* and *N* are integers

$$comb_2[n]$$



Appendix

2D-DTFT: delta

- Define *Kronecker delta function*

$$\delta[m,n] = \begin{cases} 1, & \text{for } m = 0 \text{ and } n = 0 \\ 0, & \text{otherwise} \end{cases}$$

- DT Fourier Transform of the Kronecker delta function

$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left[\delta[m,n] e^{-j2\pi(um+vn)} \right] = e^{-j2\pi(u0+v0)} = 1$$

2D DT Fourier Transform: constant

- Fourier Transform of 1

$$f[k,l]=1, \forall k,l$$

$$F[u,v]=\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\left[1e^{-j2\pi(uk+vl)}\right]=$$

$$=\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta(u-k,v-l)$$

periodic with period 1
along u and v

To prove: Take the inverse Fourier Transform of the Dirac delta function and use the fact that the Fourier Transform has to be periodic with period 1.