Sampling 2D signals

Impulse Train

$$comb_{M,N}[m,n] \triangleq \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta[m-kM, n-lN]$$

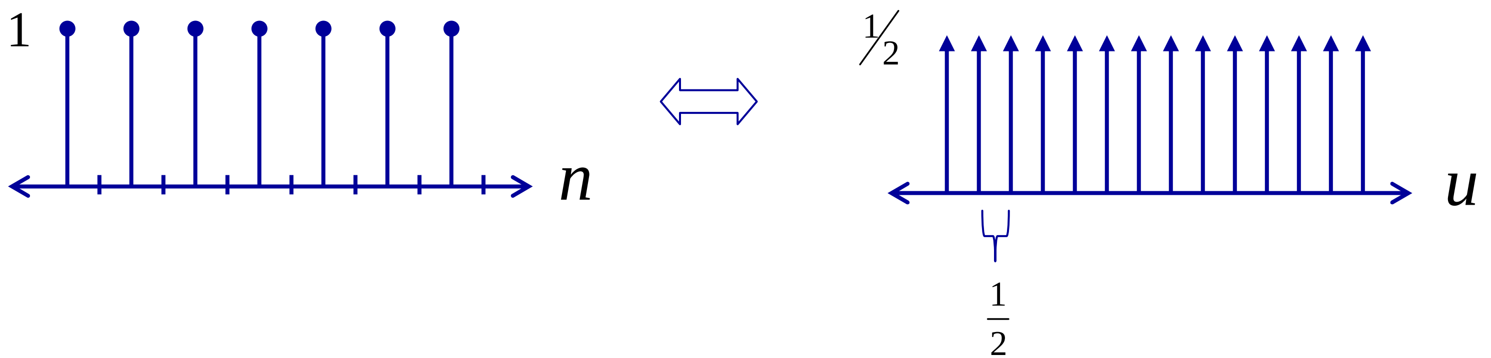
$$comb_{M,N}(x,y) \triangleq \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta\left(x-kM, y-lN\right)$$

- Fourier Transform of an impulse train is also an impulse train:

$$\underbrace{\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta\left[m-kM, n-lN\right]}_{comb_{M,N}[m,n]} \Leftrightarrow \underbrace{\frac{1}{MN} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta\left(u-\frac{k}{M}, v-\frac{l}{N}\right)}_{comb_{\frac{1}{M},\frac{1}{N}}(u,v)}$$

Impulse Train

$$comb_2[n] \qquad\qquad \frac{1}{2}comb_{\frac{1}{2}}(u)$$



Impulse Train

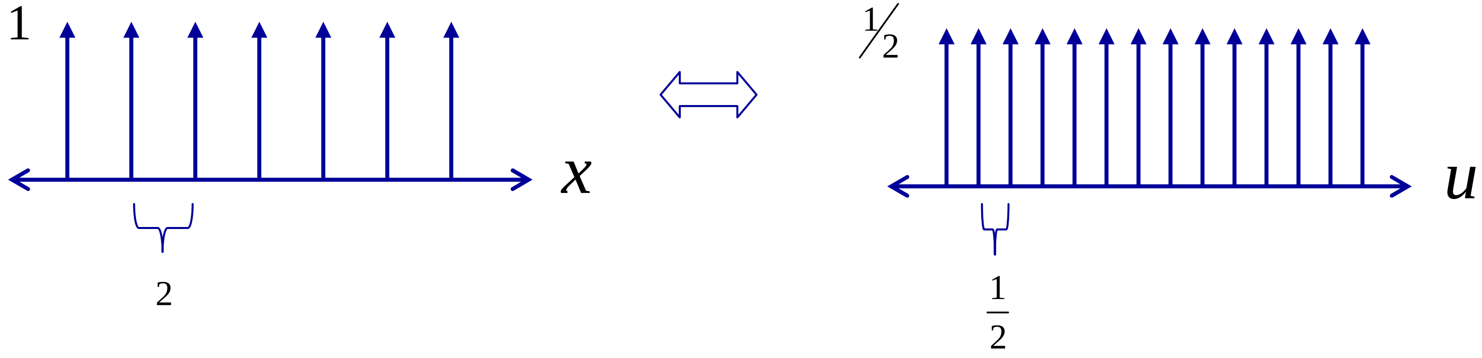
$$comb_{M,N}(x,y) \triangleq \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta\left(x-kM,y-lN\right)$$

- In the case of continuous signals:

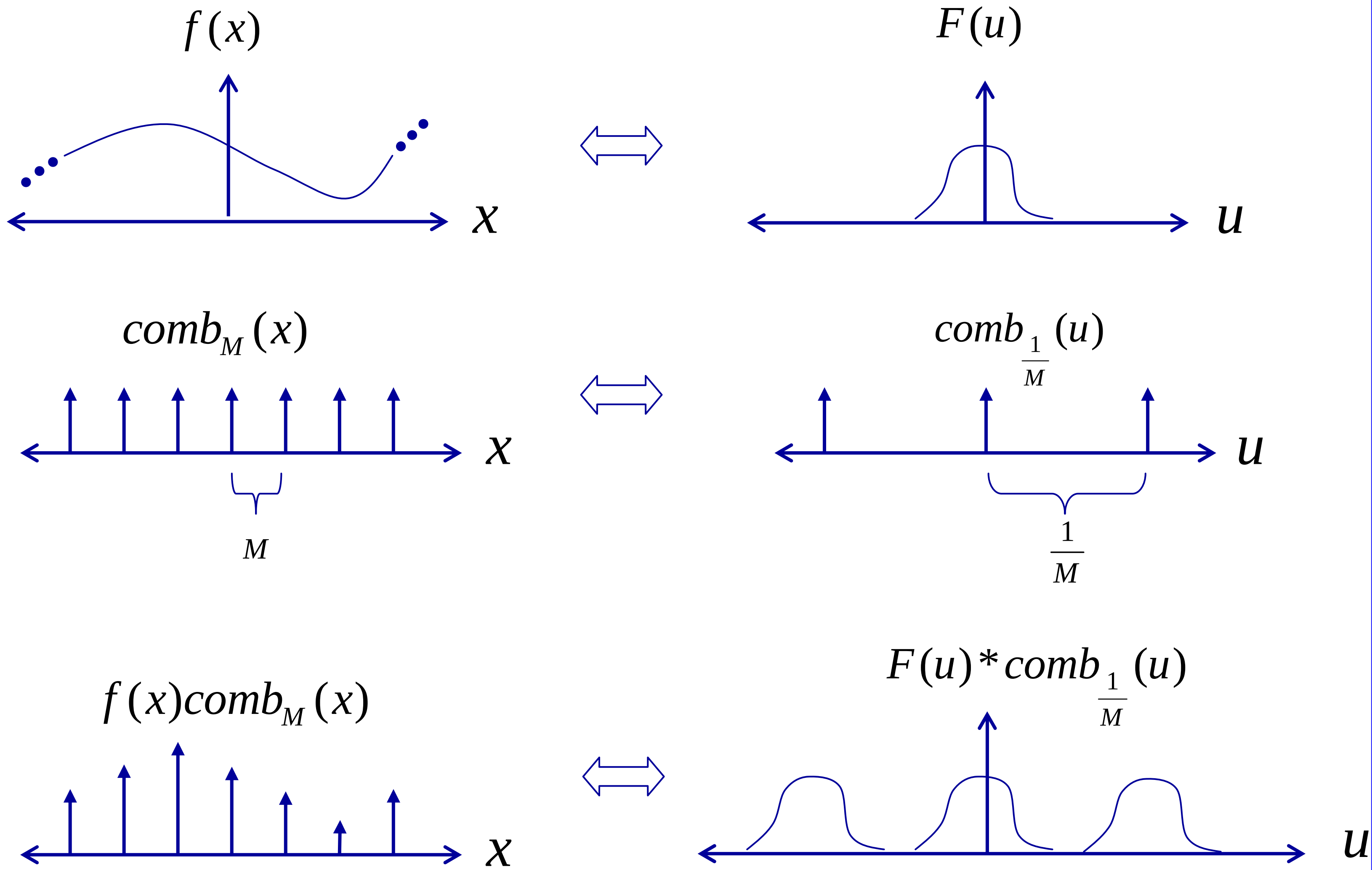
$$\underbrace{\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta\left(x-kM,y-lN\right)}_{comb_{M,N}(x,y)} \Leftrightarrow \underbrace{\frac{1}{MN}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta\left(u-\frac{k}{M},v-\frac{l}{N}\right)}_{comb_{\frac{1}{M},\frac{1}{N}}(u,v)}$$

Impulse Train

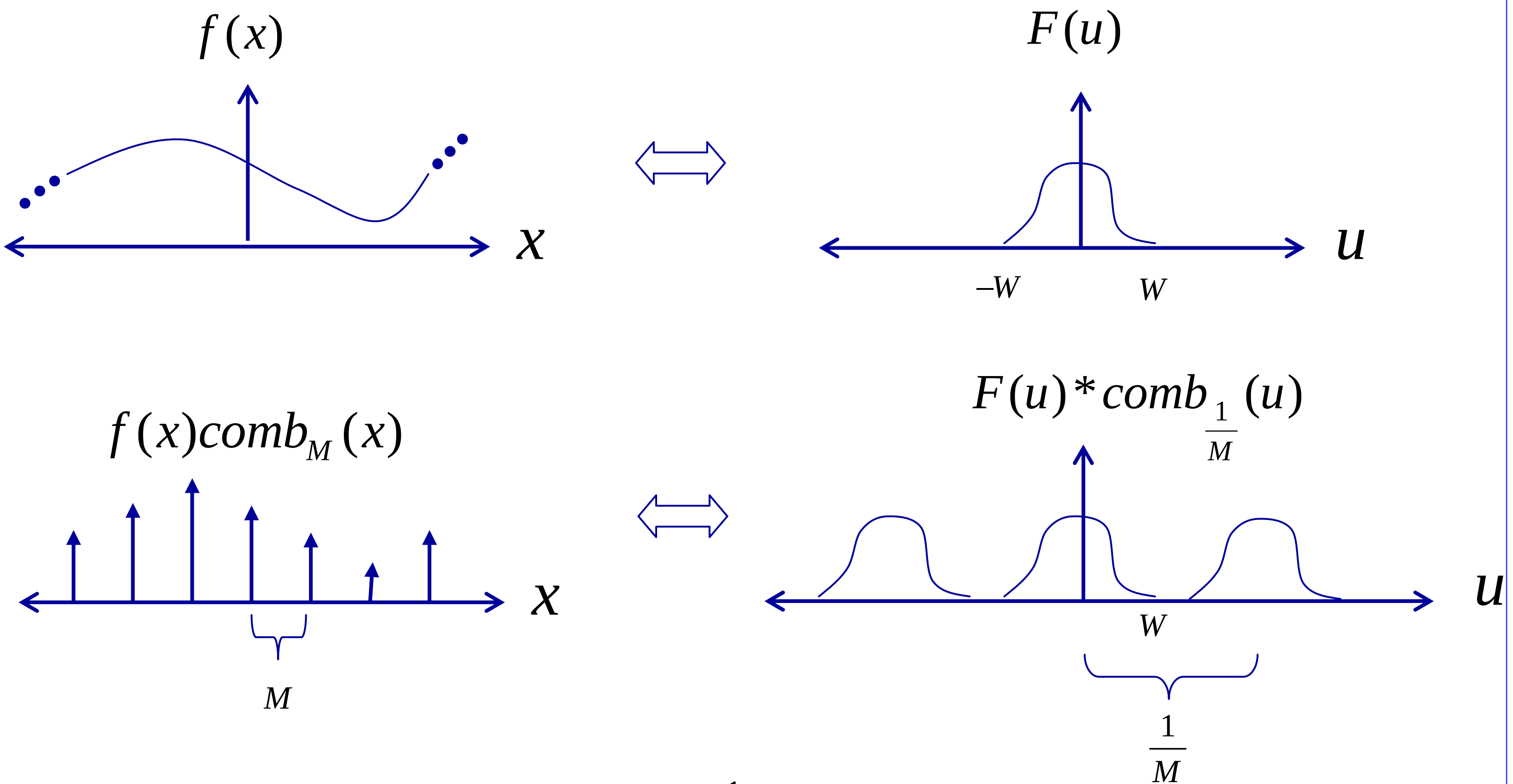
$$comb_2(x) \quad \Longleftrightarrow \quad \frac{1}{2} comb_{\frac{1}{2}}(u)$$



Sampling revisitation

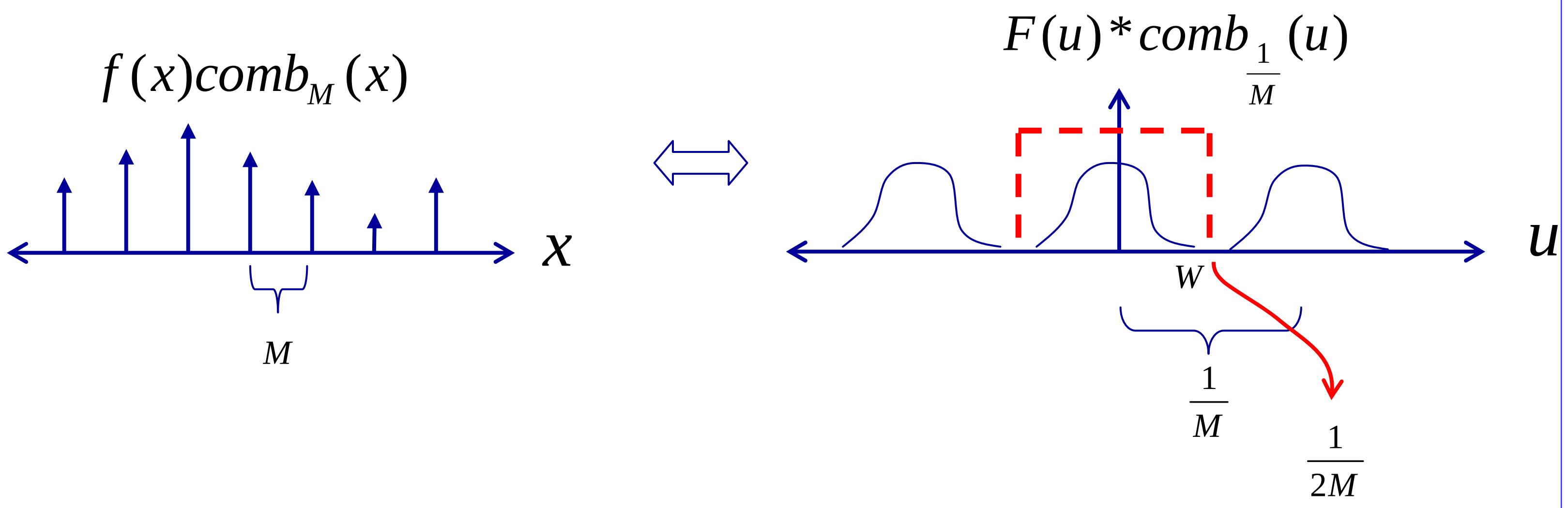


Sampling revisitation



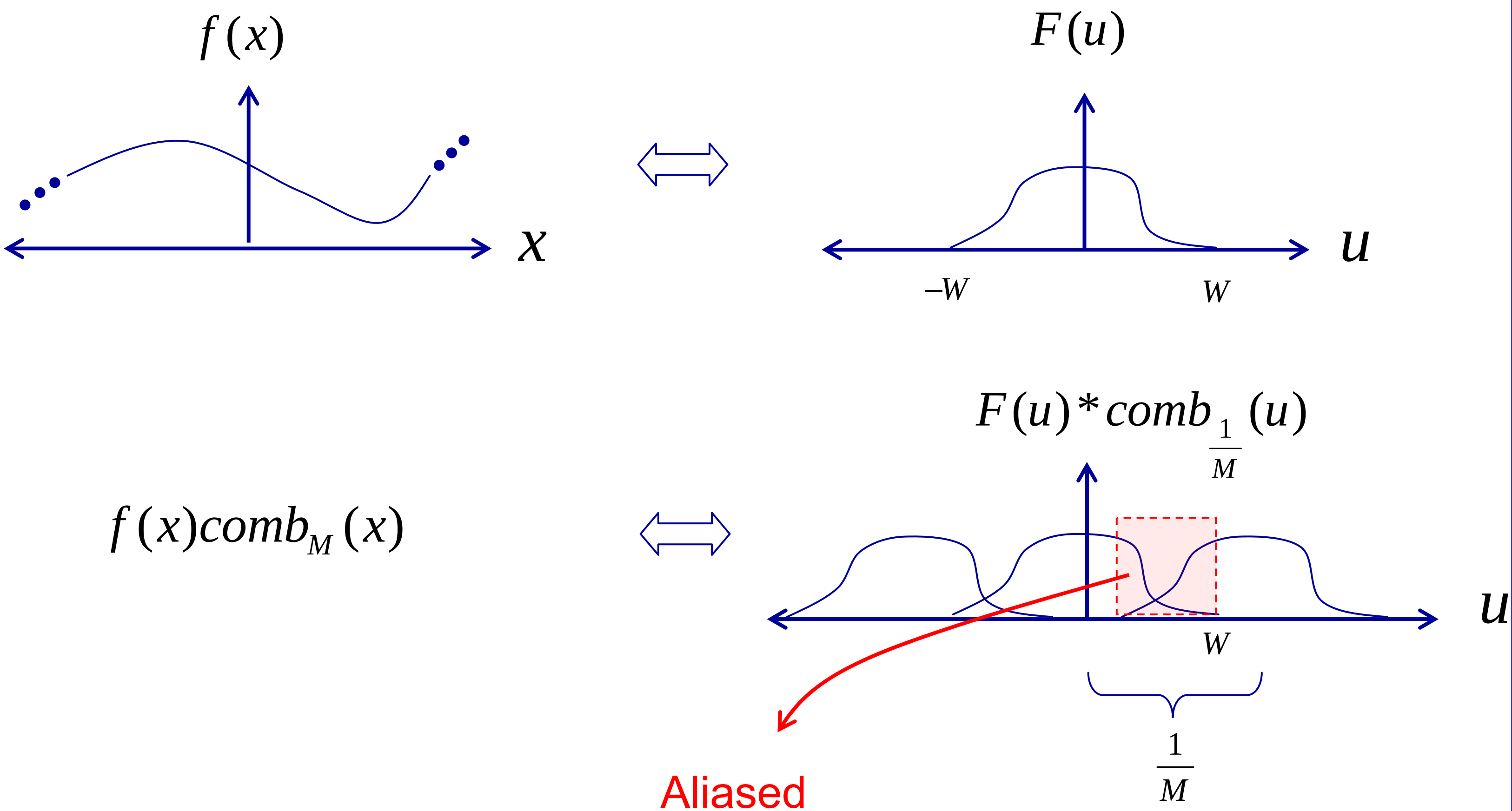
No aliasing if $\dfrac{1}{M} > 2W$

Sampling and aliasing

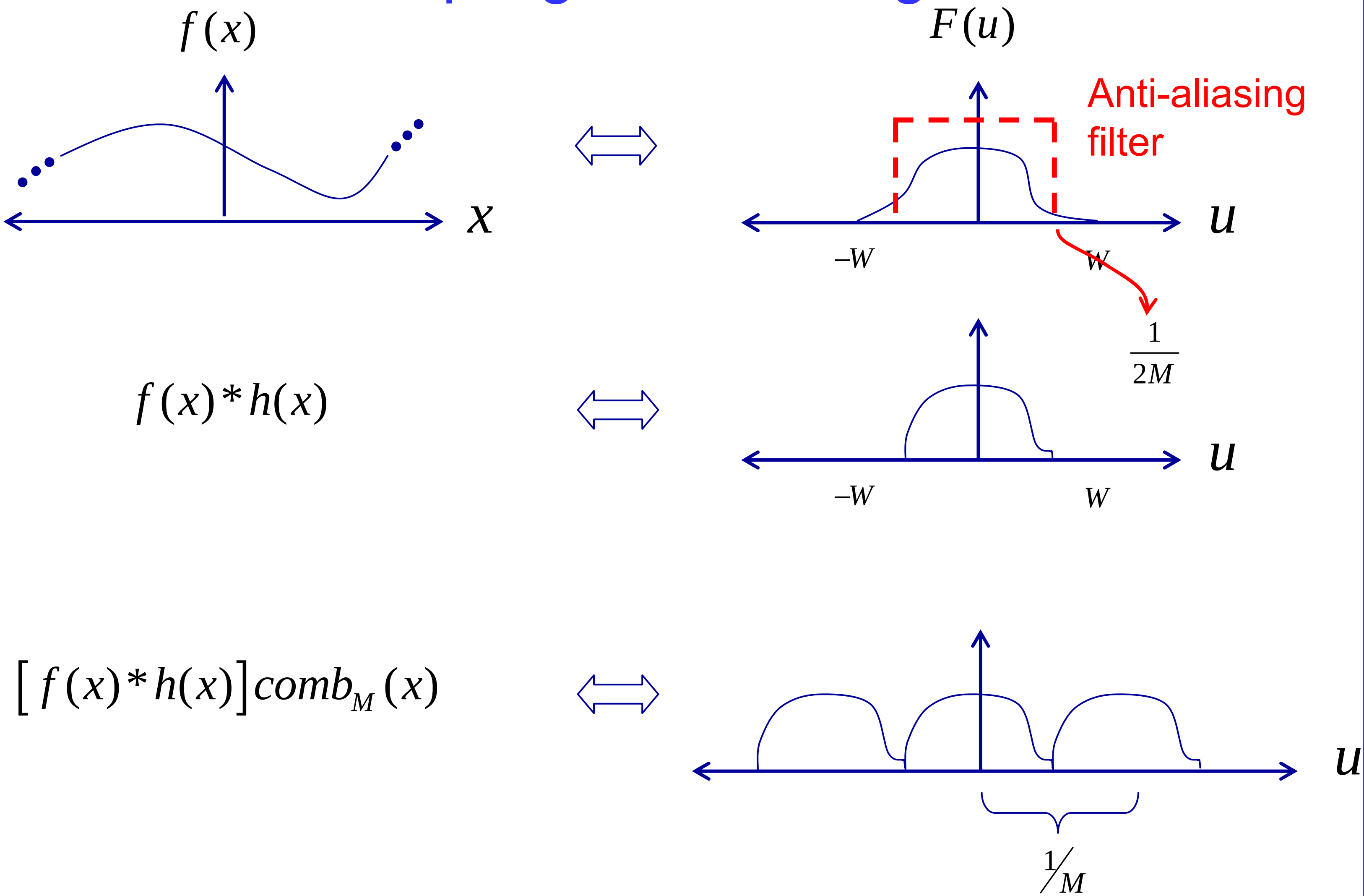


If there is no aliasing, the original signal can be recovered from its samples by low-pass filtering.

Sampling and aliasing



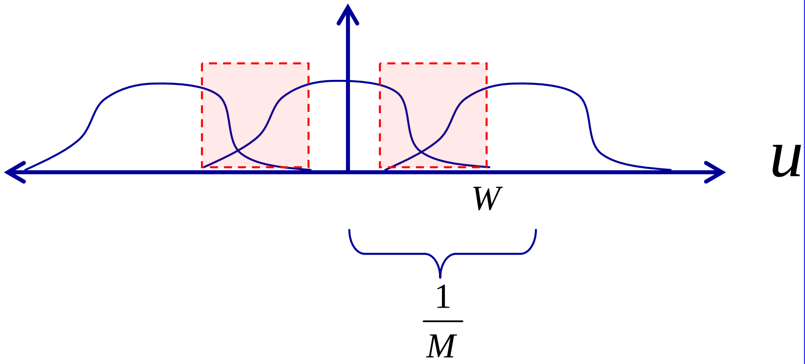
Sampling and aliasing



Sampling and aliasing

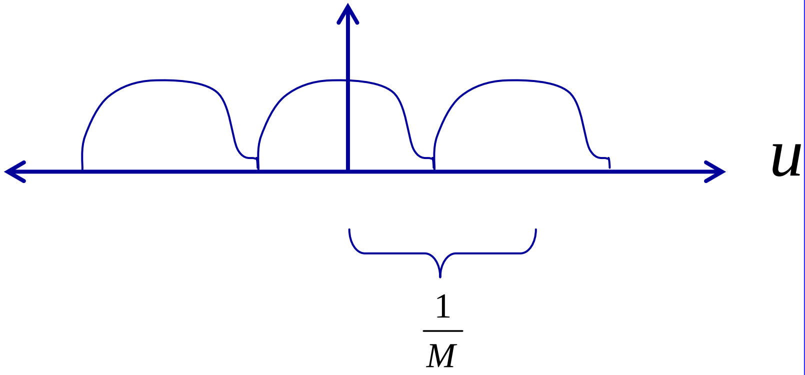
- Without anti-aliasing filter:

$$f(x)comb_M(x) \Leftrightarrow$$



- With anti-aliasing filter:

$$[f(x) * h(x)]comb_M(x) \Leftrightarrow$$



More formally qui

- Let $g[m,n] = f(x,y)\, comb_{M,N}(x,y)$

we start in the continuous
in signal domain
m,n: discrete indexes
x,y: continuous (real) coordinates

Using the multiplication property:

$$G(u,v) = F(u,v) * \frac{1}{MN} comb_{\frac{1}{M},\frac{1}{N}}(u,v)$$

in frequency domain

$$comb_{M,N}(u,v) \triangleq \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta(u - kM, v - lN)$$

$$comb_{\frac{1}{M},\frac{1}{N}}(u,v) \triangleq \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta\left(u - \frac{k}{M}, v - \frac{l}{N}\right)$$

$$comb_{\frac{1}{M},\frac{1}{N}}(\eta,\nu) \triangleq \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta\left(\eta - \frac{k}{M}, \nu - \frac{l}{N}\right)$$

eta, nu: generic variables, used to calculate the integral in the convolution

Sampling

Convolution integral:

$$F*H(u,v)=\int_{\frac{-1}{2}}^{\frac{1}{2}}\int_{\frac{-1}{2}}^{\frac{1}{2}}F(\eta,\nu)H\left(u-\eta,v-\nu\right)d\eta d\nu$$

$$H\left(u,v\right)=\frac{1}{MN}comb_{\frac{1}{M},\frac{1}{N}}(u,v)=\frac{1}{MN}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta\left(u-\frac{k}{M},v-\frac{l}{N}\right)$$

$$H\left(\eta,\nu\right)=\frac{1}{MN}comb_{\frac{1}{M},\frac{1}{N}}(\eta,\nu)=\frac{1}{MN}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta\left(\eta-\frac{k}{M},\nu-\frac{l}{N}\right)$$

$$F*H\left(u,v\right)=\frac{1}{MN}\int_{\frac{-1}{2}}^{\frac{1}{2}}\int_{\frac{-1}{2}}^{\frac{1}{2}}F(\eta,\nu)\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta\left(u-(\eta-\frac{k}{M}),v-(\nu-\frac{l}{N})\right)d\eta d\nu=$$

$$=\frac{1}{MN}\int_{\frac{-1}{2}}^{\frac{1}{2}}\int_{\frac{-1}{2}}^{\frac{1}{2}}F(\eta,\nu)\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta\left(u-\eta+\frac{k}{M},v-\nu+\frac{l}{N}\right)d\eta d\nu$$

Sampling

- Since k,l goes from –infty to +infity, we can change their sign in the formula

$$G(u,v) = F * H(u,v) = \frac{1}{MN} \int_{\frac{-1}{2}}^{\frac{1}{2}} \int_{\frac{-1}{2}}^{\frac{1}{2}} F(\eta,\nu) \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta\left(u - \eta - \frac{k}{M}, v - \nu - \frac{l}{N}\right) d\eta d\nu$$

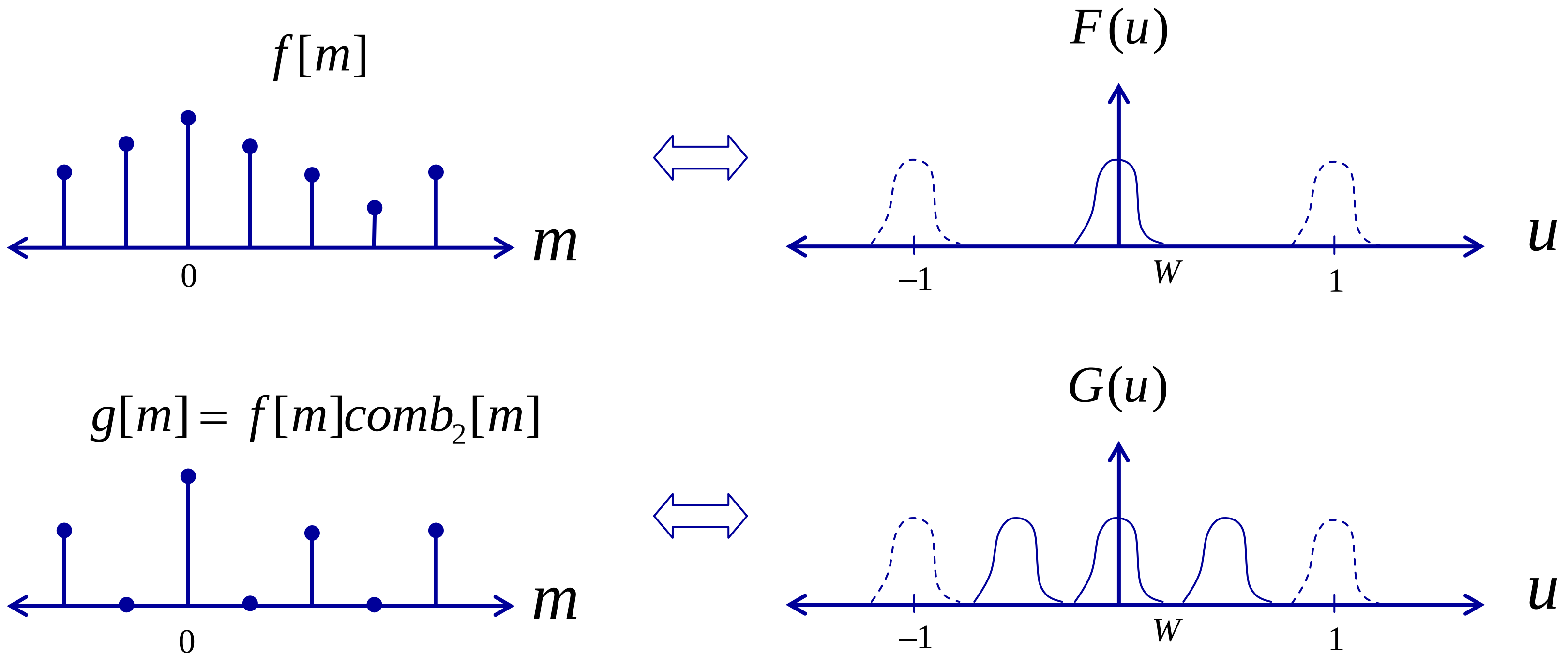
Sampling

$$G(u,v)=\frac{1}{MN}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\int_{\frac{-1}{2}}^{\frac{1}{2}}\int_{\frac{-1}{2}}^{\frac{1}{2}}F(\eta,\nu)\delta\left(u-\eta-\frac{k}{M},v-\nu-\frac{l}{N}\right)d\eta d\nu$$

sampling property of the delta function:

$$G(u,v)=\frac{1}{MN}\sum_{k=-\infty}^{+\infty}\sum_{l=-\infty}^{+\infty}F\left(u-\frac{k}{M},v-\frac{l}{N}\right)$$

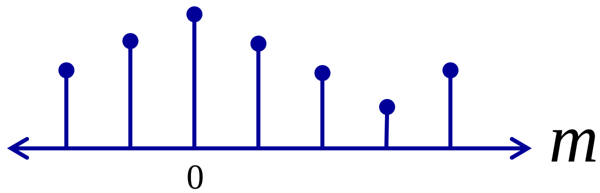
Downsampling



Downsampling makes the repetitions get closer to each other → risk of aliasing

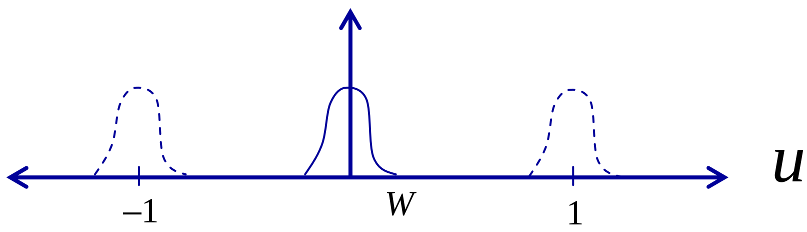
Downsampling

$f[m]$

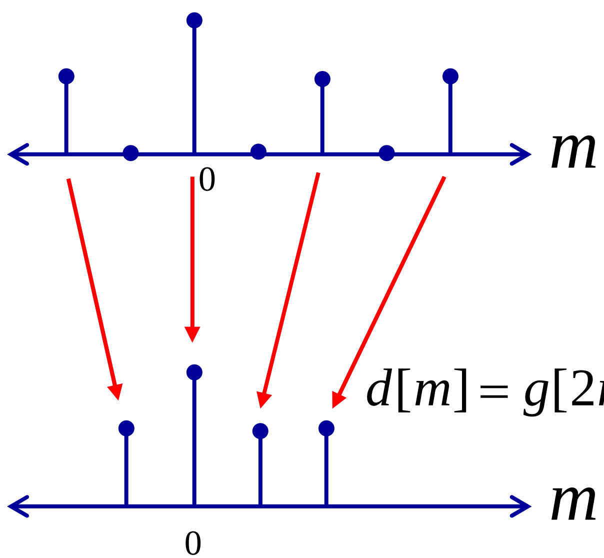


$\Longleftrightarrow$

$F(u)$

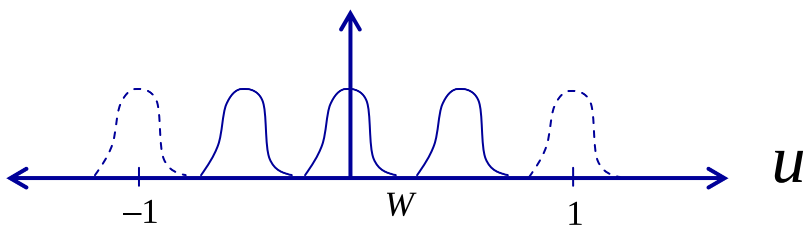


$g[m] = f[m]comb_2[m]$



$\Longleftrightarrow$

$G(u)$



$d[m] = g[2m] \Longleftrightarrow G_d(u) = \frac{1}{2}\left(G(u) + G\left(u + \frac{1}{2}\right)\right)$

Gloria Menegaz

Downsampling

$$g[m,n] = f[m,n]comb_{M,N}[m,n]$$
$$d[m,n] = g[Mm, Nn]$$

$$G(u,v) = \frac{1}{MN}\sum_{k\in \mathrm{N}_k}\sum_{l\in \mathrm{N}_l} F\left(u-\frac{k}{M}, v-\frac{l}{N}\right)$$

where

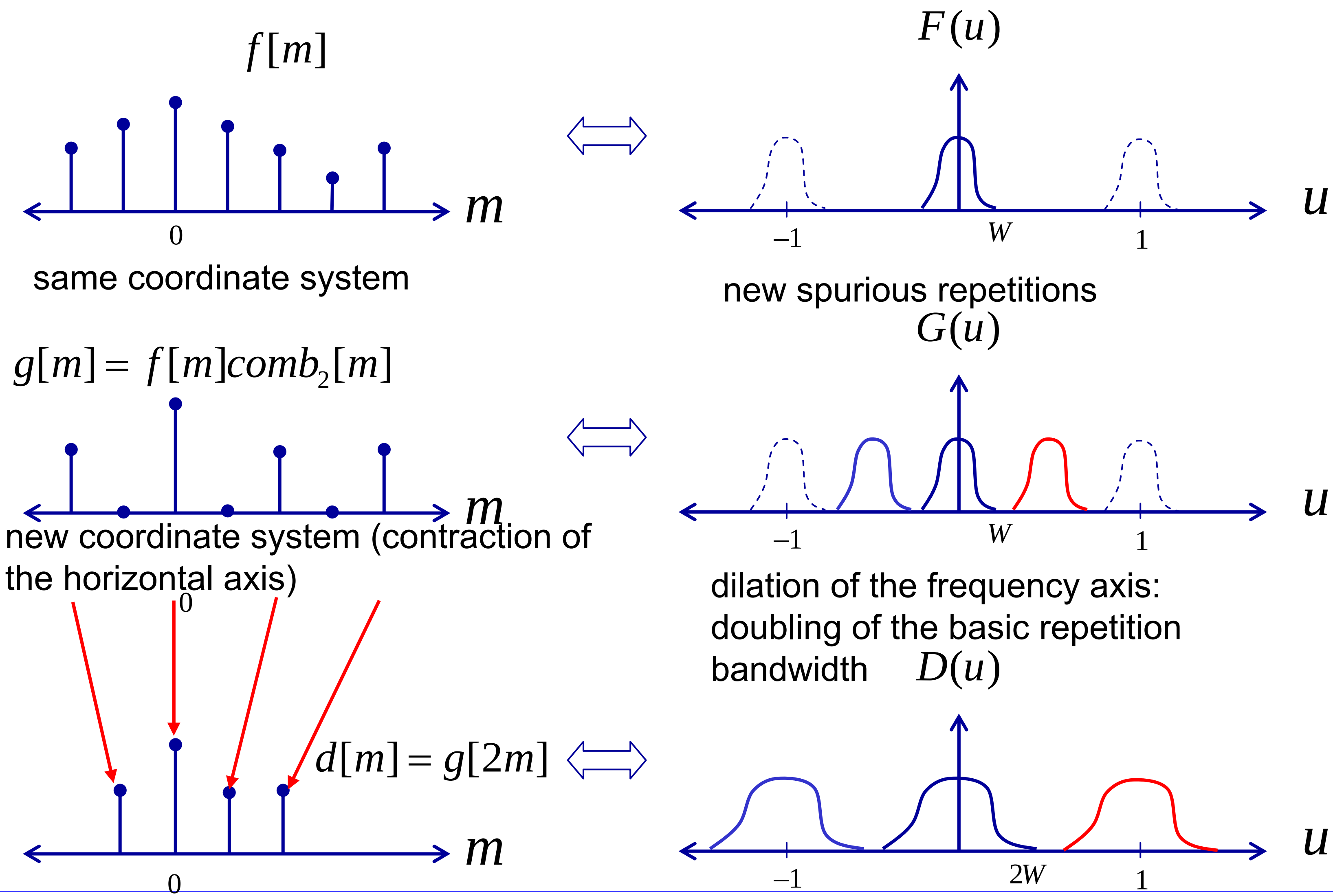
$$\mathrm{N}_k = \left\{k \text{ such that } \frac{-1}{2} \le u - \frac{k}{M} < \frac{1}{2}\right\}$$
$$\mathrm{N}_l = \left\{l \text{ such that } \frac{-1}{2} \le v - \frac{l}{N} < \frac{1}{2}\right\}$$

Additional repetitions in the basic period (-1/2,1/2)

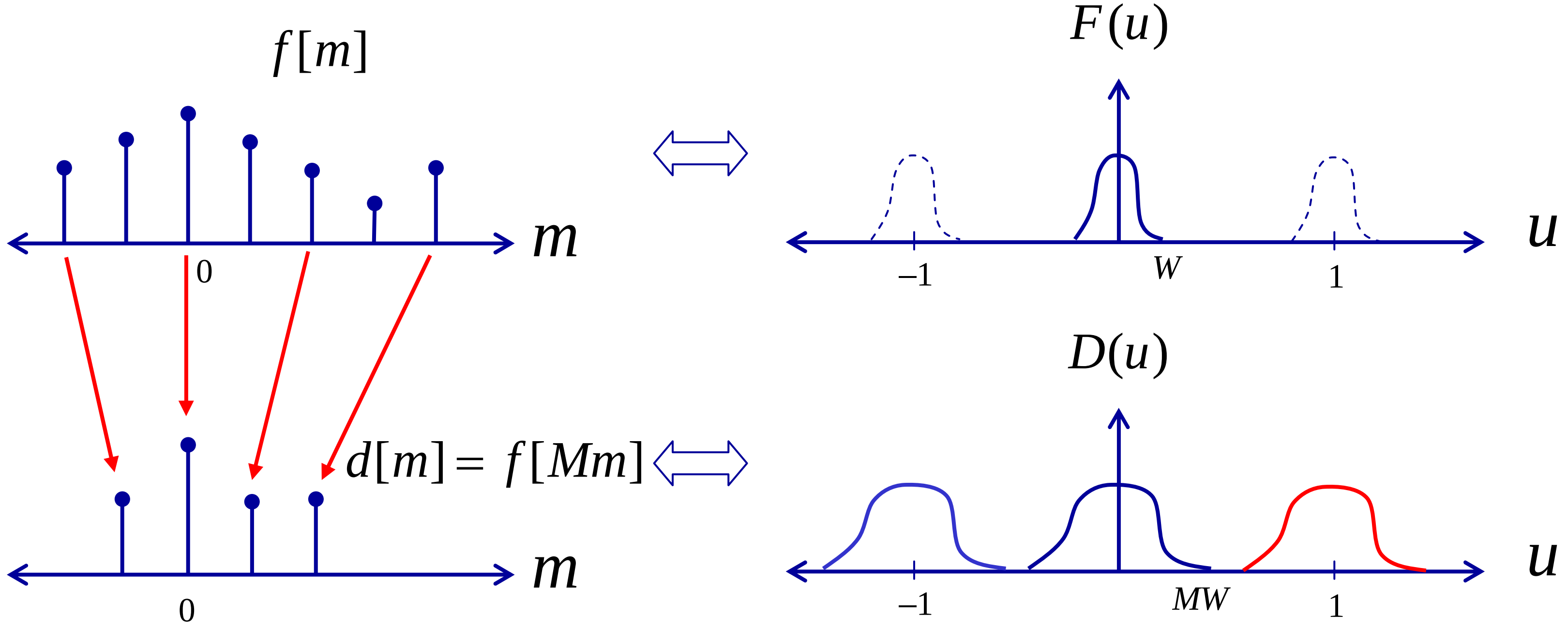
$$M = 2, N = 2$$

$$G(u,v) = \frac{1}{4}\left\{F(u,v) + F\left(u, v-\frac{1}{2}\right) + F\left(u-\frac{1}{2}, v\right) + F\left(u-\frac{1}{2}, v-\frac{1}{2}\right)\right\}$$

Example


$f[m]$
$F(u)$
m
u
0
−1
W
1
same coordinate system
new spurious repetitions
$g[m] = f[m]comb_2[m]$
$G(u)$
new coordinate system (contraction of the horizontal axis)
dilation of the frequency axis: doubling of the basic repetition bandwidth
$D(u)$
$d[m] = g[2m]$
2W

Example

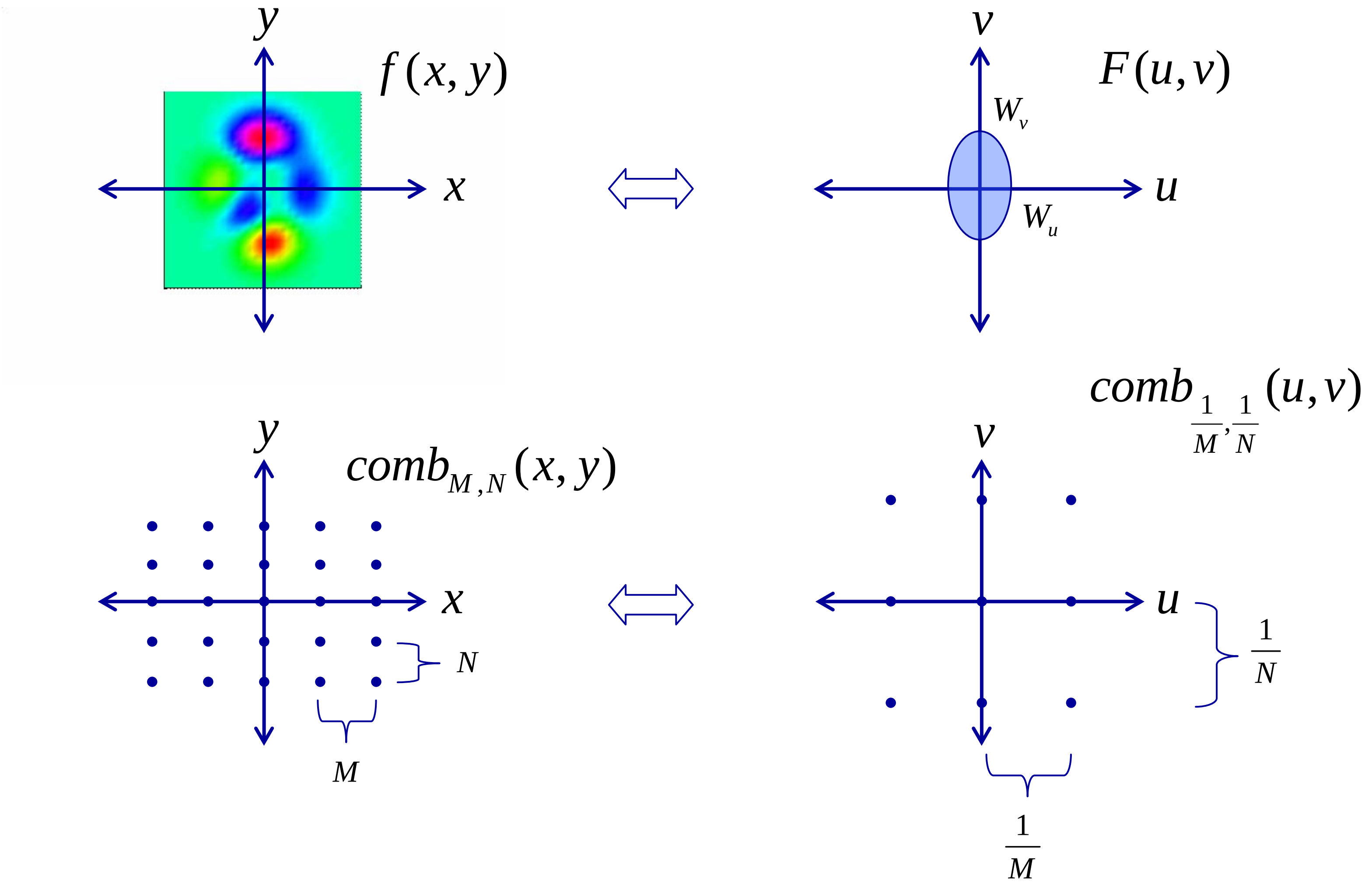


No aliasing if $MW < \frac{1}{2}$

Aliasing in images

- Without the anti-aliasing filter the recovered image (subsampling+upsampling) is different from the original.
- With anti-aliasing filter (low-pass), the *smoothed* version of the original image can be recovered by interpolation

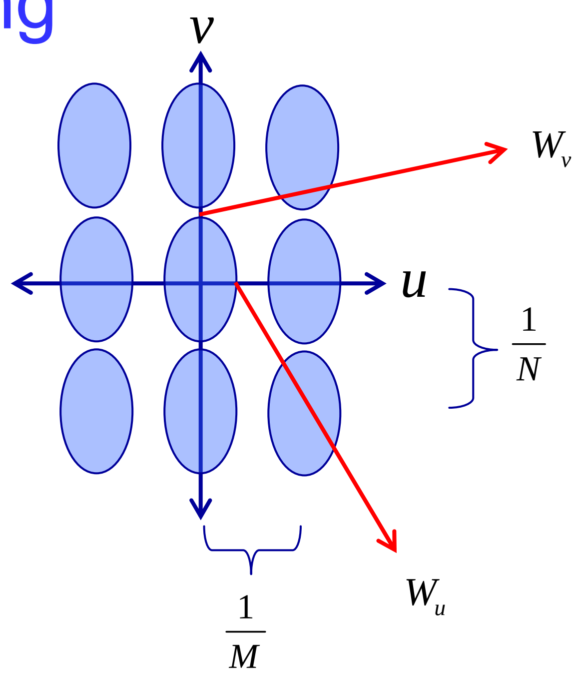
Sampling



y
$f(x,y)$
x
v
$F(u,v)$
W_v
u
W_u
y
$comb_{M,N}(x,y)$
x
N
M
$comb_{\frac{1}{M},\frac{1}{N}}(u,v)$
v
u
$\frac{1}{N}$
$\frac{1}{M}$

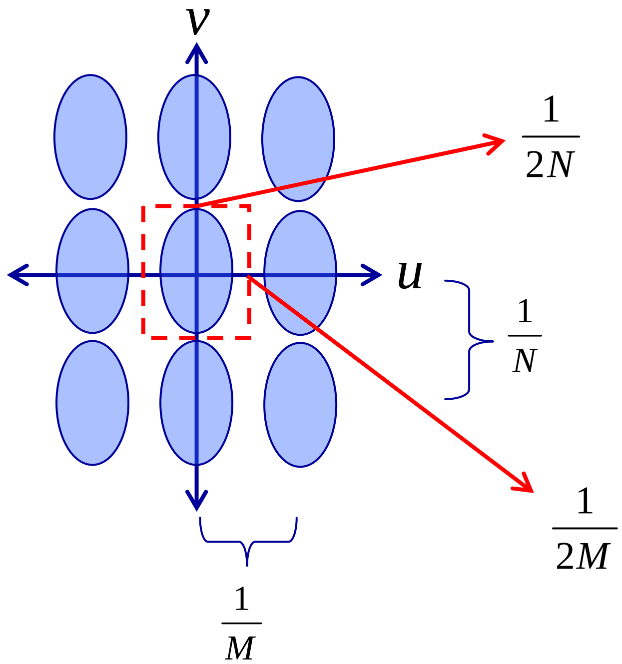
Sampling

$$f(x,y)\,comb_{M,N}(x,y) \Leftrightarrow$$



No aliasing if $\frac{1}{M} > 2W_u$ *and* $\frac{1}{N} > 2W_v$

Interpolation



Ideal reconstruction filter:

$$H(u,v)=\begin{cases} MN, & \text{for } u \le \dfrac{1}{2M} \text{ and v} \le \dfrac{1}{2N} \\ 0, & \text{otherwise} \end{cases}$$

Anti-Aliasing

```
a=imread('barbara.tif');
```

Anti-Aliasing

```
a=imread('barbara.tif');
b=imresize(a,0.25);
c=imresize(b,4);
```

Anti-Aliasing

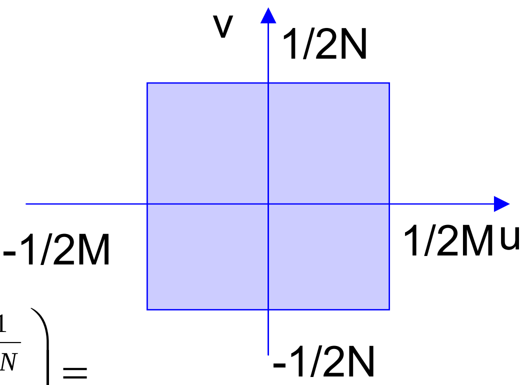
```
a=imread('barbara.tif');
b=imresize(a,0.25);
c=imresize(b,4);

H=zeros(512,512);
H(256-64:256+64, 256-64:256+64)=1;

Da=fft2(a);
Da=fftshift(Da);
Dd=Da.*H;
Dd=fftshift(Dd);
d=real(ifft2(Dd));
```

Ideal Reconstruction Filter: 2D box

$$h(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(u,v)e^{j2\pi(ux+vy)}dudv=\int_{\frac{1}{2N}}^{\frac{1}{2N}}\int_{\frac{-1}{2M}}^{\frac{1}{2M}}MNe^{j2\pi(ux+vy)}dudv$$



$$=\int_{\frac{-1}{2M}}^{\frac{1}{2M}}Me^{j2\pi ux}du\int_{\frac{-1}{2N}}^{\frac{1}{2N}}Ne^{j2\pi vy}dv$$

$$=MN\frac{1}{j2\pi x}\left(e^{j2\pi x\frac{1}{2M}}-e^{-j2\pi y\frac{1}{2M}}\right)\times\frac{1}{j2\pi y}\left(e^{j2\pi y\frac{1}{2N}}-e^{-j2\pi y\frac{1}{2N}}\right)=$$

$$=\frac{1}{\frac{\pi}{M}x}\frac{1}{2j}\left(e^{j\pi x\frac{1}{M}}-e^{-j\pi x\frac{1}{M}}\right)\times\frac{1}{\frac{\pi}{N}y}\frac{1}{2j}\left(e^{j\pi y\frac{1}{N}}-e^{-j\pi y\frac{1}{N}}\right)=$$

$$=\frac{\sin\left(\frac{\pi}{M}x\right)}{\frac{\pi}{M}x}\times\frac{\sin\left(\frac{\pi}{N}y\right)}{\frac{\pi}{N}y}$$

$$\sin(x)=\frac{1}{2j}\left(e^{jx}-e^{-jx}\right)$$

2D sinc