PARALLEL COORDINATES : *VISUAL* Multidimensional Geometry and its Applications

Alfred Inselberg(©1992, 2004)

Senior Fellow San Diego SuperComputing Center, CA, USA

School of Mathematical Sciences
Tel Aviv University , Israel
aiisreal@post.tau.ac.il
&
Multidimensional Graphs Ltd
Raanana 43556, Israel

Chapter 1

The Plane with Parallel Coordinates

1.1 The Fundamental Duality

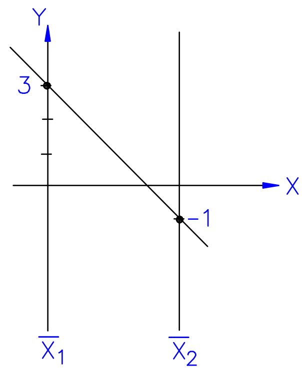

Figure 1.1: Points on the plane are represented by lines.

$\mathcal{P}$oints on the x_1x_2-plane are represented in parallel coordinates (abbreviated by $\|$-coords) by a line on the xy-plane, as shown in Fig. 1.1. To find out how a line might be represented a set of points on a line is selected, as shown in the top right of Fig. 1.2. The corresponding lines are plotted in $\|$-coords, shown in the top left of the figure, intersect at a point! Remarkably, this is true in general and for the line

$$\ell : x_2 = mx_1 + b, \tag{1.1}$$

the corresponding point is

$$\bar{\ell} : (\frac{d}{1-m}, \frac{b}{1-m}) \qquad m \neq 1\,, \tag{1.2}$$

where the distance between the parallel axes is taken as d. With reference to the later chapters we observe that the inter-axis distance is *directed* so that if the axes are interchanged the sign of d becomes negative. The point $\bar{\ell}$ is said to *represent* the line ℓ. It is evident that a point $\leftrightarrow$ line duality[1] is at work here and which must be properly considered in the *Projective*

[1] See portion on *Duality* in the "Geometry Background" chapter.

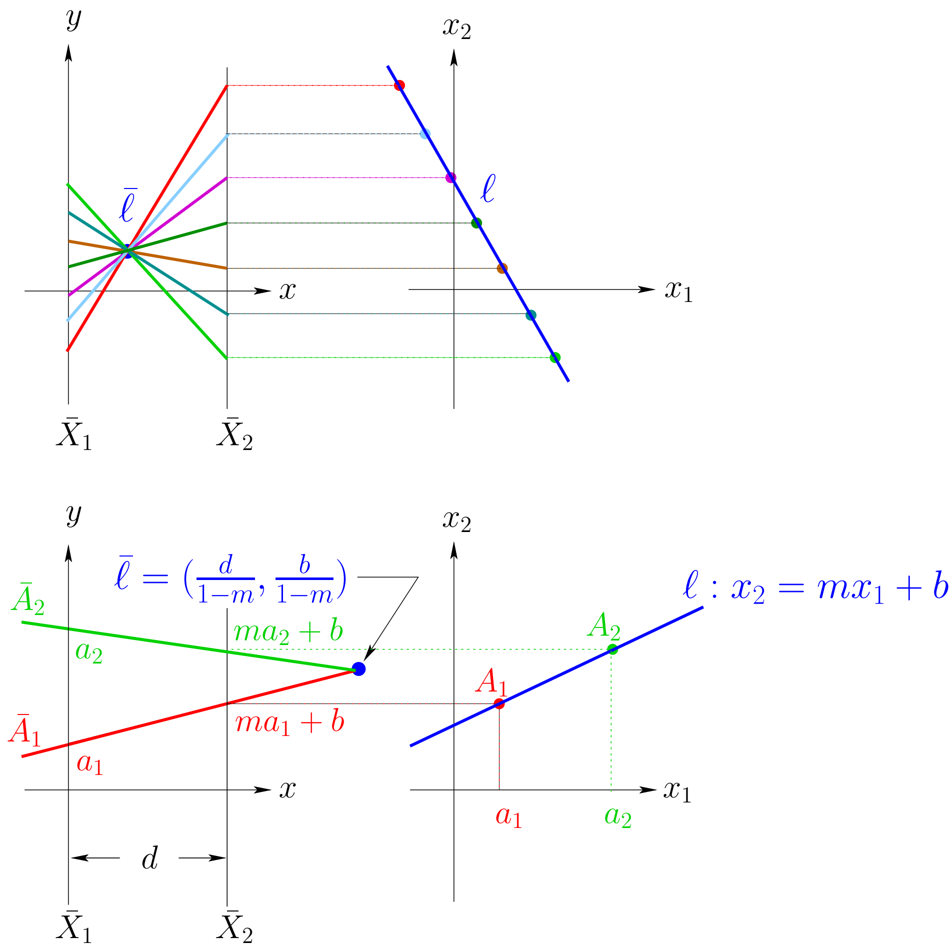

Figure 1.2: In the plane parallel coordinates induce a point $\longleftrightarrow$ line duality.

$\mathbb{P}^2$ rather than the Euclidean $\mathbb{R}^2$ plane. In eq. (1.2) as $m \to 1$,$\bar{\ell} \to \infty$ but in a constant direction. This is further clarified in Fig. 1.3 where the lines representing the points of a line with $m = 1$ are parallel with slope $\frac{b}{d}$. For this reason, a line with $m = 1$ is represented by the ideal point in the direction having slope $\frac{b}{d}$. The projective plane allows the inclusion of lines with $m = 1$ to complete the 1-1 correspondence between lines and points $\ell \leftrightarrow \bar{\ell}$ as given by eqs. 1.1 and 1.2.

From now on the x_1x_2 and xy planes are considered as two copies of the projective plane $\mathbb{P}^2$ and, in general, an object S in the x_1x_2-plane is represented by an object in the xy-plane denoted by $\bar{S}$. Expressing eq. 1.2, in homogeneous coordinates with triples within [...] and triples within (...) denoting line and point coordinates respectively, we get that

$$\ell : [m, -1, b] \longrightarrow \bar{\ell} : (d, b, 1 - m). \tag{1.3}$$

It is clear that lines with $m = 1$ are represented by the ideal points $(d, b, 0)$. The horizontal position of $\bar{\ell}$ depends only on the slope m as illustrated in Fig. 1.4. Specifically, for vertical lines ℓ (i.e. $m = \infty$) $\bar{\ell}$ is on the y-axis, while for horizontal lines ℓ (i.e. $m = 0$), $\bar{\ell}$ is on the x-axis. Lines with negative slope are represented by points within the parallel axes, lines

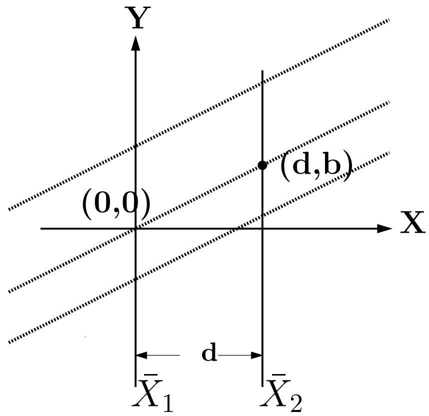

Figure 1.3: Lines representing points on the line $x_2 = x_1 + b$.

with $0 < m < 1$ with points to the right of the $\bar{X}_2$-axis and lines with $1 < m$ with points to the left of the $\bar{X}_1$-axis. Among other things, this points out the reason for representing a point P by the *whole* line $\bar{P}$, rather than just the segment between the parallel axes, for $\bar{\ell}$ may lie outside the strip between the axes.

The dependence of the horizontal position of $\bar{\ell}$ only on m shows that parallel lines are represented by points on the same *vertical* line. This property, as we see later, turns out to be very useful. Therefore, the ideal point P_m^∞ is represented by the vertical line $x = \frac{1}{1-m}$ shown in Fig. 1.5.

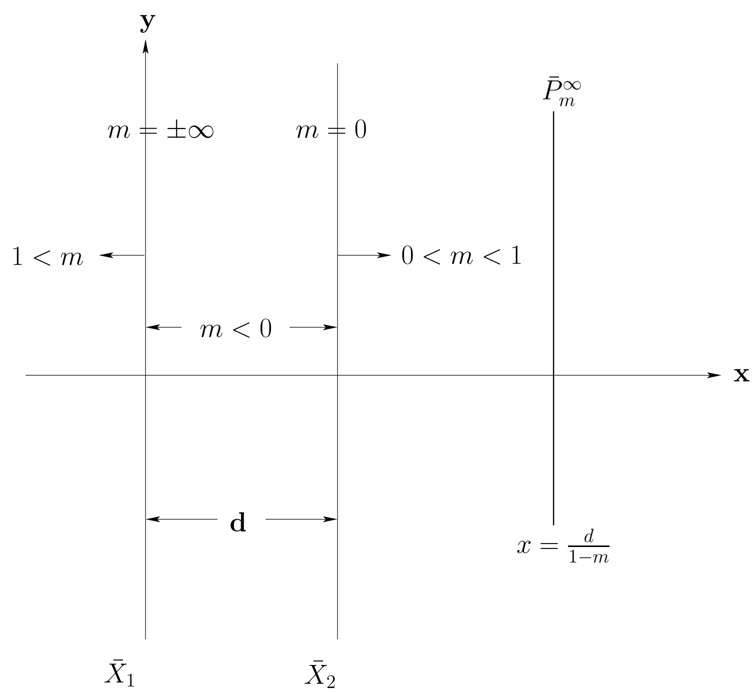

Figure 1.4: The horizontal position of $\bar{\ell}$ depends only on the slope m of ℓ. See eqs. (1.4) and (1.5) to understand the case $m = \pm\infty$.

For complete generality the line description below is used

$$\ell : a_1x_1 + a_2x_2 + a_3 = 0 \tag{1.4}$$

and for $a_2 \neq 0$, $m = -\frac{a_1}{a_2}$ and $b = -\frac{a_3}{a_2}$, providing the correspondence :

$$\ell : [a_1, a_2, a_3] \longrightarrow \bar{\ell} : (da_2, -a_3, a_1 + a_2). \tag{1.5}$$

In turn this specifies a linear transformation between the triples ℓ and $\bar{\ell}$:

$$\bar{\ell} = A\ell \quad , \quad \ell = A^{-1}\bar{\ell},$$

where ℓ and $\bar{\ell}$ are considered as column vectors. The 3×3 matrix is :

$$A \;=\; \begin{bmatrix} 0 & d & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \;,\; A^{-1} \;=\; \begin{bmatrix} -1/d & 0 & 1 \\ 1/d & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}. \tag{1.6}$$

which can be easily computed by taking 3 simple triples, like for example, [1,0,0], [0,1,0] and [0,0,1] for ℓ.

Since $\ell \rightarrow \bar{\ell}$ must be considered on the projective plane we can verify that the ideal line $l_\infty = [0,0,1]$ is mapped into $\bar{\ell}_\infty = (0,1,0)$, the ideal point in the vertical direction; this being the "intersection" of all the lines representing the ideal points which we have already seen are vertical. It is clear then that all lines of the projective plane P^2 are mapped into points of P^2 as given by eq. (1.6). Below we summarize the various cases :

$$\left\{ \begin{array}{ccc} \ell & \rightarrow & \bar{\ell} \\ [a_1, 0, a_2]\ , m = \infty & \rightarrow & (0, -a_3, a_1)\ , \mathit{on\ the}\ y - \mathit{axis} \\ [0, a_2, a_3]\ , m = 0 & \rightarrow & (d, -a_3, 1)\ , \mathit{on\ the}\ x - \mathit{axis} \\ [-a_2, a_2, a_3]\ , m = 1 & \rightarrow & (1, -\frac{a_3}{da_2}, 0)\ , \mathit{ideal\ point\ slope}\ -\frac{a_3}{da_2}\ (\frac{m}{d}) \\ [0,0,1]\ , \ell_\infty & \rightarrow & (0,1,0)\ , \bar{\ell}_\infty\ \mathit{vertical\ direction} \end{array} \right\}.$$

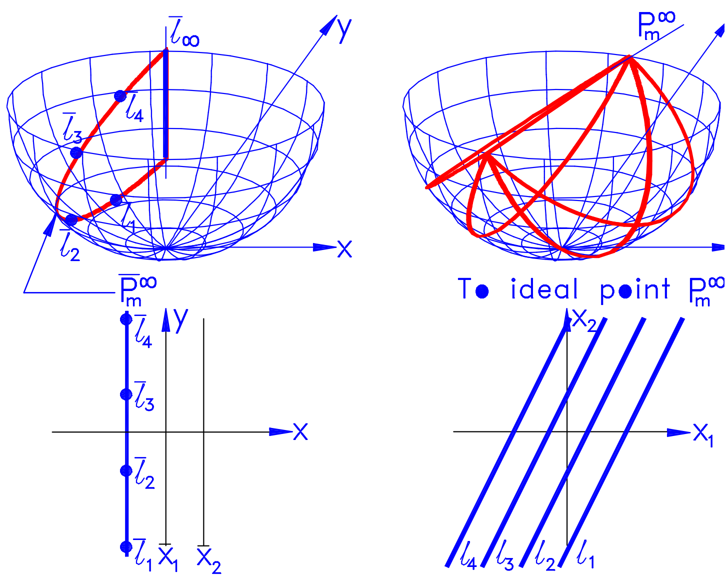

Figure 1.5: Under the duality parallel lines map into points on the same vertical line. On the projective plane model, the great semi-circles representing the lines share the same diameter since the lines have the same ideal point.

For the other half of the duality, using homogeneous coordinates we look carefully into the point $P \to \bar{P}$ line correspondence illustrated in Fig. 1.6. The point $P = (p_1, p_2, p_3) = (p_1/p_3, p_2/p_3, 1)$, $p_3 \neq 0$, is represented in the xy-plane by the line $\bar{P}$ with slope $(p_2 - p_1)/dp_3$. Hence,

$$\bar{P} \,:\, (dp_3)y = (p_2 - p_1)\,x + d\,p_1 \;,$$

so that

$$P = (p_1 \,,\, p_2 \,,\, p_3) \longrightarrow \bar{P} = [(p_1 - p_2)\,,\, dp_3 \,,\, -dp_1]. \tag{1.7}$$

The ideal point $P_m^\infty = (p_1 \,,\, p_2 \,,\, 0) = (1 \,,\, p_2/p_1 \,,\, 0)$ has direction with slope $m = p_2/p_1$, therefore, $\bar{P}_m^\infty = [(p_1 - p_2)\,,\, 0\,,\, -dp_1] = [1 - m\,,\, 0\,,\, -d]$ which is the vertical line at $x \;=\; d\,/\,(1 - m)$ as expected (see also Fig. 1.4). Considering P and $\bar{P}$ as column vectors we can write the above correspondence as a linear transformation:

$$\bar{P} = BP \;\;, P = B^{-1}\bar{P}$$

with

$$B \;=\; \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & -d \\ d & 0 & 0 \end{bmatrix} \;,\; B^{-1} \;=\; \begin{bmatrix} 0 & 0 & 1/d \\ 1 & 0 & 1/d \\ 0 & -1/d & 0 \end{bmatrix} \;. \tag{1.8}$$

So, in $\|$-coords there is a fundamental *Point* $\rightleftharpoons$ *Line* duality, formalized by Equations (1.5) and (1.7). Curiously for $d = 1$, $A^T = B^{-1}$.

Altogether then we have the transformations : *line* $\ell \xrightarrow{A} \bar{\ell}$ *point*, and *point* $P \xrightarrow{B} \bar{P}$ *line*. We can continue as $\ell \xrightarrow{A} \bar{\ell} \xrightarrow{B} \bar{\bar{\ell}}$ but in general the line ℓ is not the same as the line $\bar{\bar{\ell}}$. That is $B(A\ell) \neq \ell$ and likewise $A(BP) \neq P$ (See exercises 2 , 3). By the way, the matrices

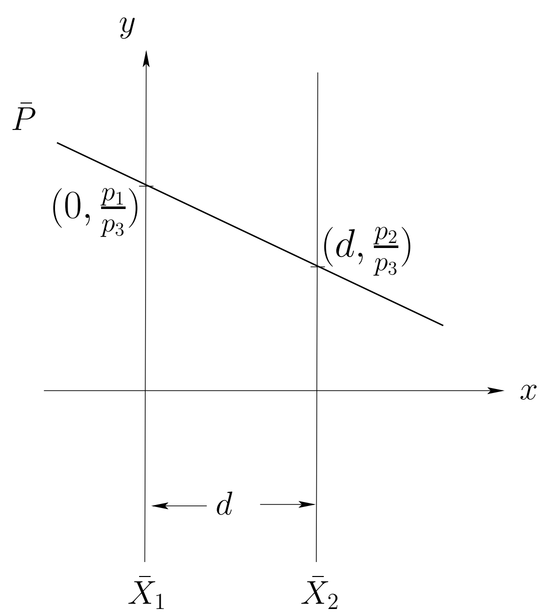

Figure 1.6: A point $P = (p_1, p_2, p_3) = (p_1/p_3 \,,\, p_2/p_3 \,,\, 1)\,,\, p_3 \neq 0\,,$ is represented by a line $\bar{P}$.

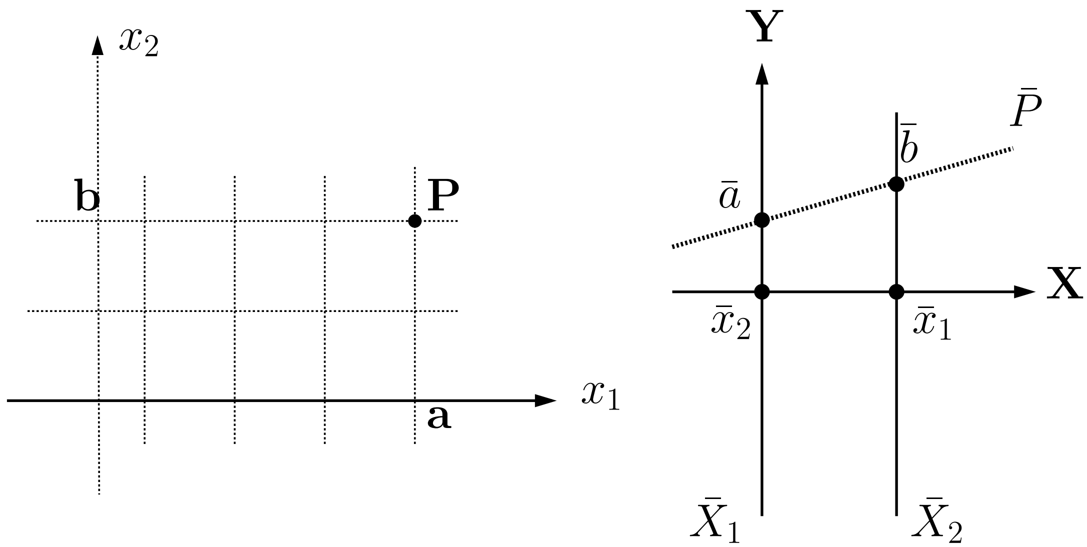

Figure 1.7: Motivation for the notation $\bar{X}_i$ of the axes, note the position of the $\bar{x}_i$. Two lines $x_1 = a, x_2 = b$ on the point P (left) whose images are two points $\bar{a}, \bar{b}$ on the line $\bar{P}$ (right). The axis $\bar{X}_1$ represents the ideal point in the vertical direction (hence $\bar{x}_2$ is on it) and $\bar{X}_2$ in the horizontal (with $\bar{x}_1$ on it).

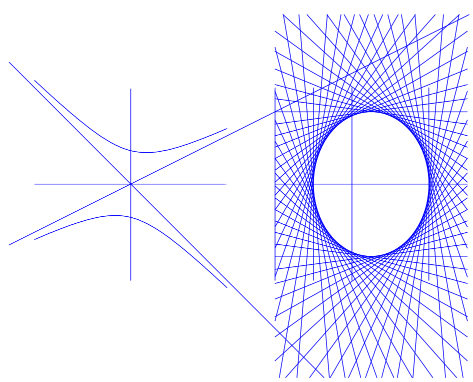

Figure 1.8: Hyperbola(point-curve) $\rightarrow$ Ellipse(line-curve).

A, B are quite sparse (i.e. have lots of zeros) and usually taking $d = 1$ so the computation to and from $\|$-coords is not too costly.

A few words about the notation shown in Fig. 1.7 are in order. Each of the axes $\bar{X}_i$, $i = 1, 2$ represents a *family* of parallel lines. Specifically, $\bar{X}_i$ represents the parallel lines $x_i =$ *constant* or the ideal point in the direction of x_i. Note the representation of x_1 and x_2 above.

The image(representation) $\bar{r}$ of a curve r can be obtained as the *envelope*[2] of the lines representing the points of r. This is distinguished by referring to the original curves as *point-curves* and their images as *line-curves* an example is shown in Fig. 1.8. Line-curves will be

[2]As explained in the Chapter on Envelopes

constructed from the *envelope* of the family of curves (for an old but very nice treatment of envelopes see [1]). In fact, the point $\bar{\ell}$ can also be found in this way to be the envelope of the infinite family of lines $\bar{A}$. In the chapter on Curves it is seen that

$$point - curve \leftrightarrow point - curve$$

directly by a transformation between the $x_1 x_2$ and xy projective planes [3]. Additional related topics can be found in [2].

Exercises

1. Verify eq. (1.2). Hint: The lines $\bar{A}_i,\ i = 1, 2$ are on the points $(0, a_i), (d, ma_i + b)$ of the xy-plane and $\bar{\ell} = A_1 \cap A_2$.

2. Apply $\ell \overset{A}{\rightarrow} \bar{\ell} \overset{B}{\rightarrow} \bar{\bar{\ell}} \ldots$. Does the pattern repeat? i.e. $\ell \rightarrow \ldots \rightarrow \ell$ after a finite number of transformations. What is meant here is the application of the line-to-point transformation expressed by the matrix A, then the point-to-line transformation expressed by the matrix B etc.

3. Repeat for $P \overset{B}{\rightarrow} \bar{P} \overset{A}{\rightarrow} \bar{\bar{P}} \ldots$. Does this pattern repeat?

1.2 Transformations under the duality

1.2.1 Rotations and Translations

With reference to Fig. 1.9, a line ℓ originaly horizontal, shown in position ℓ_o, is rotated counterclockwise about one of it's points which for convenience is taken as the origin O. Under the duality, the corresponding path of $\bar{\ell}$ is along the line $\bar{O}$ since in any of its rotated positions (say ℓ_1, ℓ_2, ℓ_3 etc.) ℓ still contains the point O. To better understand the corresponding motion of the point $\bar{\ell}$, recall that in the projective plane a line behaves like a "closed curve" (see Fig. 1.5) "joined" at its ideal point. Hence due to the counterclockwise rotation, the point $\bar{\ell}$ moves in the direction of increasing x (i.e. to the right) until the line's slope is 1 when $\bar{\ell}$ passes through the ideal point and "returns" to the Euclidean plane now on the left of the X_1-axis but still moving to the right. Dually, a translation of point A in the positive direction along a line ℓ (taken for convenience coincident with the x_1-axis) corresponds to the rotation of the line $\bar{A}$ in the clockwise direction. There are corresponding dualities and quite useful in higher dimensions which we will meet in due course.

1.2.2 Recognizing Orthogonality

There are two more transformations worth presenting at this stage.

1. $R_{\frac{1}{2}}$, *Reflection about the line* $x = \frac{1}{2}$. In the xy-plane the reflection of the vertical line $\bar{P}^{\infty}_m$ (the set of points representing the lines with slope m), about the line $x = \frac{1}{2}$ is the line $\bar{P}^{\infty}_{\frac{1}{m}}$(the set of points representing the lines with slope $\frac{1}{m}$), as shown in Fig. 1.10.

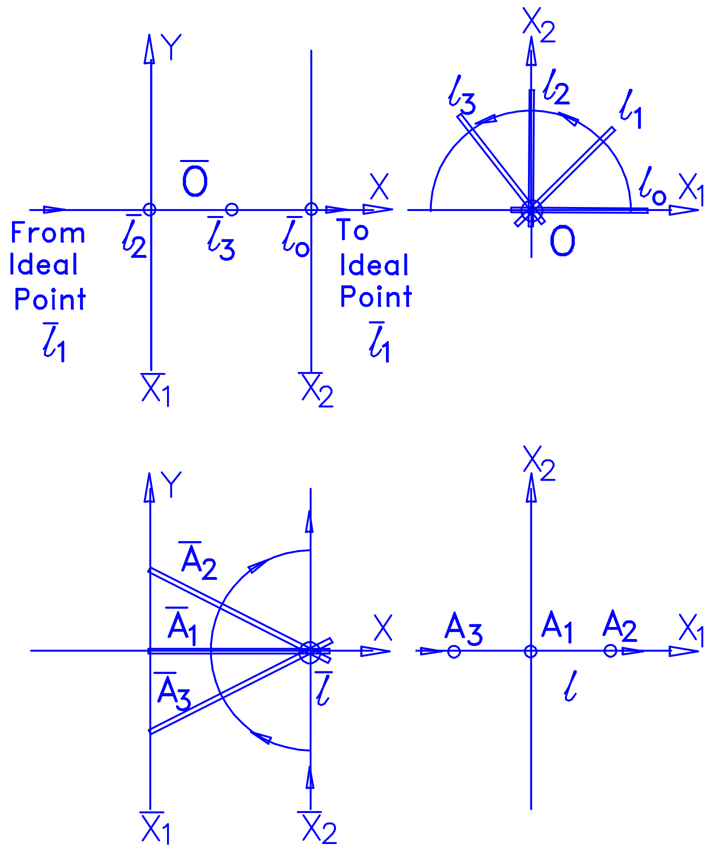

Figure 1.9: Duality between rotations and translations

That is, such a reflection finds the image of the lines with the reciprocal of the slope or

$$R_{\frac{1}{2}}(\bar{P}^{\infty}_{m}) = \bar{P}^{\infty}_{\frac{1}{m}}\,.$$

2. $C_{\frac{1}{2}}$, *Circle Inversion.* Consider the tangent from the point $\bar{\ell}_o = \bar{P}^{\infty}_{m} \cap x$-axis to the circle centered at $(\frac{1}{2}, 0)$ at the point $\bar{\ell}_1$ and radius $\frac{1}{2}$, see Fig. 1.11. Then $x(\bar{\ell}_1) = \frac{1}{(1+m)}\ C_{\frac{1}{2}} = \bar{P}^{\infty}_{-m}$.

With the composition

$$R_{\frac{1}{2}} C_{\frac{1}{2}}(\bar{P}^{\infty}_{m}) = \bar{P}^{\infty}_{-\frac{1}{m}}$$

the image of the lines *orthogonal* to the original ones is found. The x-distances a,b (shown in figures 1.10 and 1.11) of $\bar{P}^{\infty}_{m}$ from $(\frac{1}{2}, 0)$ are:

$$a = -\frac{(1-m)}{2(1+m)}\ , b = -\frac{1+m}{2(1-m)}$$

respectively. Therefore, $ab = 1/4$ is the invariance for the corresponding two lines (or families of lines) to be mutually orthogonal. All this shows that information on orthogonality is not "lost" when using ||-coords.

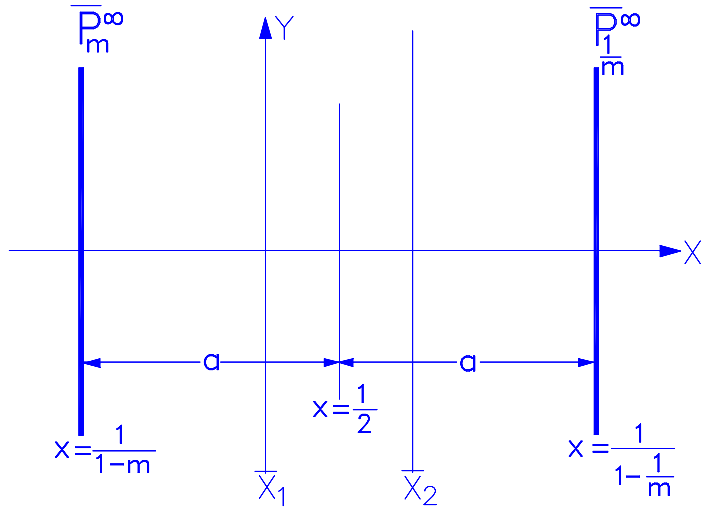

Figure 1.10: Reflection about the line $\frac{1}{2}$

1.2.3 A Preview

$\mathcal{A}$ glimpse of what lies ahead is seen in Fig. 1.12. In part (a) we see a square with unit side in Cartesian and dually in parallel coordinates. Note that the edge $\bar{AB}$ is the point represented by the intersection of the lines $\bar{A}$ with $\bar{B}$ representing the corresponding two vertices. The situation becomes more interesting in part (b) where a cube with unit side and its image are shown. The collection of all vertices has coordinates which are all possible ordered triples with 0's and 1's. The image of the cube consists of two adjacent copies of the square's image. Note the polygonal line indicated as the image of the vertex $E = (1, 0, 0)$ for example. It is clear that all of the cube's vertices are included in the image. Also the

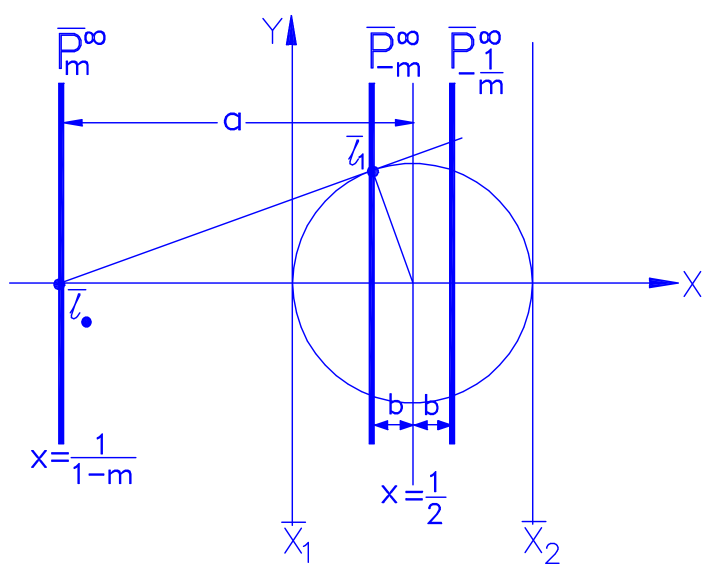

Figure 1.11: Circle Inversion

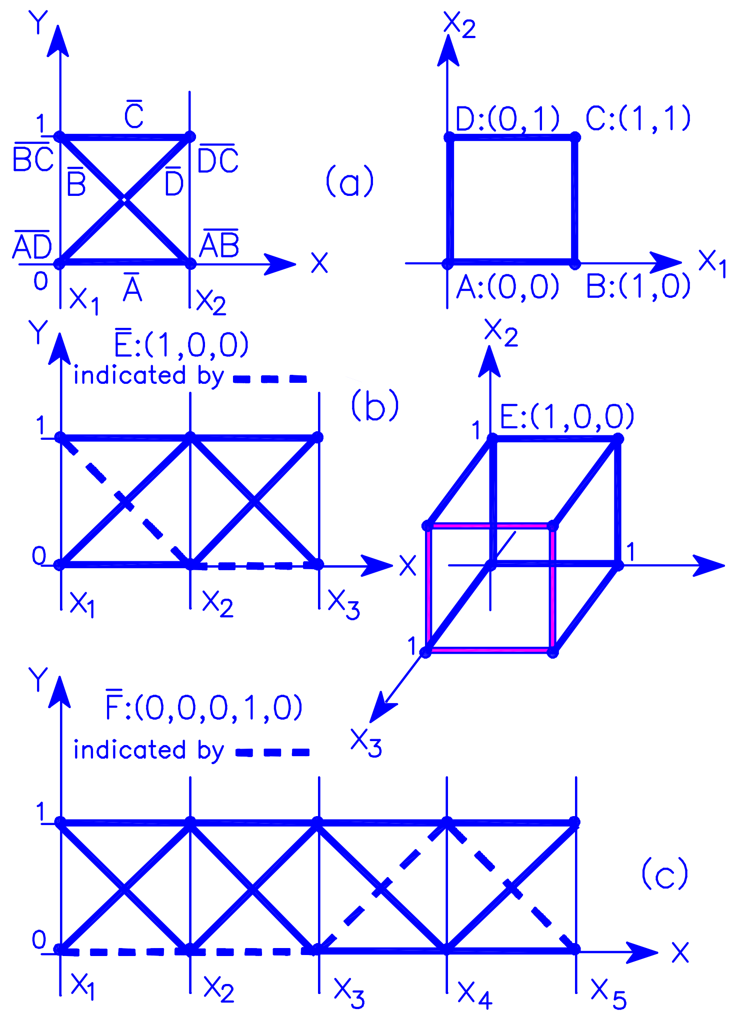

Figure 1.12: (a)Square,(b) 3-D cube (c) 5-D hypercube all with unit side

connection with the square as well as the symmetry of the cube are "suggested", but albeit not as of yet clearly understood, in the "double-square" image. In due course, we will see that this image contains **all** the information about the cube; namely, the edges, faces as well as inside and outside. Finally, whereas we do not know how to show a hypercube in 5-D or higher in Cartesian coordinates, there is no problem doing so in parallel coordinates as we can see in part (c) of the figure and still, as we shall eventually show, with all the information there. We will also see that showing objects with high-symmetry by repetitive patterns is more general than the hypercube. Now the reason this works is that our duality has nice generalizations to higher dimensions which is our next topic. Though the ensuing the development is carried out in the projective plane, it works out that we can still use all the Euclidean notions of angle, distance etc as long as ideal elements are not involved.

Exercises

Draw in Cartesian and Parallel coordinates a point $P = (p_1, p_2)$ and a line ℓ before and after :

1. the translation of the origin to the point (a_1, a_2),
2. the reflections about the x_1 and x_2 axes,
3. the rotation of the axes about the origin by an angle θ,
4. the composition of the translation followed by the rotation.

Index

References

[1] V. G. Boltyanskii. *Envelopes, R.B.Brown translator (original in Russian)*. Pergamon Press, New York, 1964.

[2] A. Inselberg. *N-Dimensional Graphics, Part I – Lines and Hyperplanes, IBM LASC Tech. Rep. G320-2711, 140 pages*. IBM LA Scientific Center, 1981.

[3] A. Inselberg. The plane with parallel coordinates. *Visual Computer*, 1:69–97, 1985.