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HYPERBOLIC BOUND FOR RM

The feasibility analysis of the RM algorithm can also be performed using a different approach, called the Hyperbolic Bound [BBB01, BBB03]. The test has the same complexity as the original Liu and Layland bound but it is less pessimistic, so allowing to accept task sets that would be rejected using the original approach. Instead of minimizing the processor utilization with respect to task periods, the feasibility condition can be manipulated in order to find a tighter sufficient schedulability test as a function of the individual task utilizations.

The following theorem provides a sufficient condition for testing the schedulability of a task set under the RM algorithm.

Theorem 1.1 *Let $\Gamma = \{\tau_1, \dots, \tau_n\}$ be a set of n periodic tasks, where each task τ_i is characterized by a processor utilization U_i. Then, Γ is schedulable with the RM algorithm if*

$$\prod_{i=1}^{n} (U_i + 1) \leq 2. \tag{1.1}$$

Proof. Without loss of generality, we may assume that tasks are ordered by increasing periods, so that τ_1 is the task with the highest priority and τ_n is the task with the lowest priority. In [LL73], as well as in [DG00], it has been shown that the worst-case scenario for a set on n periodic tasks occurs when all the tasks start simultaneously (e.g., at time $t = 0$) and periods are such that

$$\forall i = 2, \dots, n \quad T_1 < T_i < 2T_1.$$

Moreover, the total utilization factor is minimized when computation times have the following relations:

$$\begin{cases} C_1 \; = \; T_2 - T_1 \\ C_2 \; = \; T_3 - T_2 \\ \dots \\ C_{n-1} \; = \; T_n - T_{n-1} \end{cases} \tag{1.2}$$

and the schedulability condition is given by:

$$\sum_{i=1}^{n} C_i \;\leq\; T_1. \tag{1.3}$$

From equations (1.2), the schedulability condition can also be written as

$$C_n \;\leq\; 2T_1 - T_n \tag{1.4}$$

Now, defining

$$R_i = \frac{T_{i+1}}{T_i} \quad \text{and} \quad U_i = \frac{C_i}{T_i}.$$

equations (1.2) can be written as follows:

$$\begin{cases} U_1 = R_1 - 1 \\ U_2 = R_2 - 1 \\ \dots \\ U_{n-1} = R_{n-1} - 1. \end{cases} \tag{1.5}$$

Now we notice that:

$$\prod_{i=1}^{n-1} R_i = \frac{T_2}{T_1}\frac{T_3}{T_2}\cdots\frac{T_n}{T_{n-1}} = \frac{T_n}{T_1}.$$

If we divide both sides of the feasibility condition (1.4) by T_n, we get:

$$U_n \leq \frac{2T_1}{T_n} - 1.$$

Hence, the feasibility condition for a task set which fully utilizes the processor can be written as:

$$U_n + 1 \leq \frac{2}{\prod_{i=1}^{n-1} R_i}.$$

Since $R_i = U_i + 1$ for all $i = 1, \dots, n-1$, we have

$$(U_n + 1)\prod_{i=1}^{n-1}(U_i + 1) \leq 2$$

and finally

$$\prod_{i=1}^{n}(U_i + 1) \leq 2,$$

which proves the theorem. $\square$

The new test can be compared with the Liu and Layland one in the task utilization space, denoted as the U-space. Here, the Liu and Layland bound for RM is represented by a n-dimensional plane which intersects each axis in $U_{\mathrm{lub}}(n) = n(2^{1/n} - 1)$. All points below the RM surface represent periodic task sets that are feasible by RM. The new bound expressed by equation (1.1) is represented by a n-dimensional hyperbolic surface tangent to the RM plane and intersecting the axes for $U_i = 1$ (this is the reason why it is referred to as the hyperbolic bound). Figure 1.1 illustrates such bounds for $n = 2$. Notice that the asymptotes of the hyperbole are at $U_i = -1$. From the plots, we can clearly see that the feasibility region below the H-bound is larger than that below the LL-bound, and the gain is given by the dark gray area.

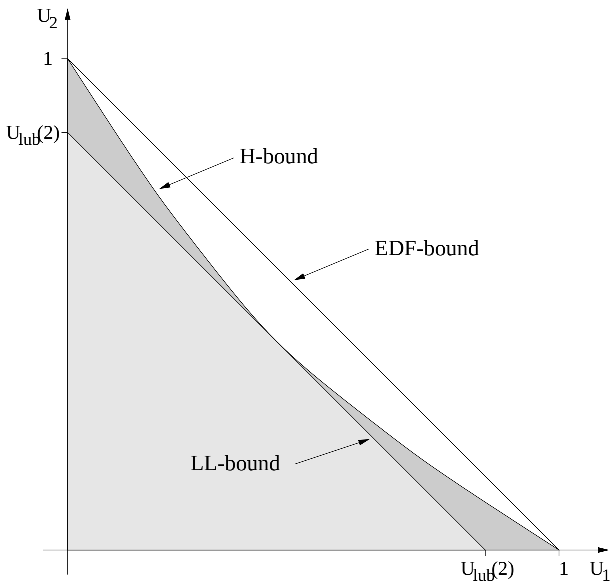

Figure 1.1 Schedulability bounds for RM and EDF in the utilization space.

It has been shown [BBB03] that the hyperbolic bound is tight, meaning that it is the best possible bound that can be found using the individual task utilization factors U_i as a task set knowledge.

Moreover, the gain (in terms of schedulability) achieved by the hyperbolic test over the classical Liu and Layland test increases as a function of the number of tasks, and tends to $\sqrt{2}$ for n tending to infinity.

REFERENCES

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