TOPOLOGIA E GEOMETRIA DIFFERENZIALE

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① Sia $\Omega = \{ (x,y,z) \in \mathbb{R}^3 \,/\, y+z > 0 \}$ e

$$\omega(x,y,z) = \frac{1}{(y+z)^2}\left((y+z)\,dx + (z-x)\,dy - (x+y)\,dz\right)$$

1. Dimostrare che $\varphi : (u,v,w) \longmapsto \overset{x}{(w-v}, \overset{y}{u-w}, \overset{z}{w})$ è un diffeomorfismo da $\mathbb{R}^3$ in sé, e determinare $\varphi^*\omega$
2. Dire se $\varphi^*\omega$ è esatta e determinarne le primitive
3. Dimostrare che la distribuzione definita da ω è integrabile

② Calcolare la derivata di Lie di $X = \sin\theta \frac{\partial}{\partial\theta} + \cos\theta \frac{\partial}{\partial r}$ lungo il flusso del campo vettoriale $Y = e^{-r}\frac{\partial}{\partial\theta} + r^2\frac{\partial}{\partial r}$ e viceversa.

③ Sia data S^{2n}, $n > 1$. Dimostrare che essa non ammette nessuna struttura simplettica.

④ Si consideri $\mathbb{R}^3 \setminus \{0\}$, in coordinate sferiche (r, θ, φ)

1. Calcolare $d\omega$, con $\omega = r\,dr\wedge d\theta + r^2 \sin\theta\, dr\wedge d\varphi - \cos\varphi\, d\theta\wedge d\varphi$
2. Calcolare $\mathcal{L}_X\omega$, con $X = r\frac{\partial}{\partial\varphi} + \sin\theta\frac{\partial}{\partial r}$

⑤ 1. Determinare il gruppo fondamentale di $X =$ 

2. X è omeomorfo a T^2?

Tempo a disposizione 1h.15m - Le risposte vanno adeguatamente giustificate

(1)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \qquad \begin{cases} x = w - v \\ y = u - w \\ z = w . \end{cases}$$

$$x + y = u - v$$
$$z - x = v$$
$$y + z = u$$

trasf. lineare inv. $\Rightarrow$ diffeom.

$$dx = dw - dv$$
$$dy = du - dw$$
$$dz = dw$$

$\varphi^* \omega (u, v, w) =$

$$\frac{1}{u^2} \left(u (dw - dv) + v (du - dw) - (u - v) dw \right)$$

$$= \frac{1}{u^2} \left(u\,dw - u\,dv + v\,du - v\,dw - u\,dw + v\,dw \right)$$

$$= \frac{1}{u^2} \left(v\,du - u\,dv \right) = \overbrace{\frac{v}{u^2}}^{P} du \overbrace{- \frac{1}{u}}^{Q} dv$$

Per il lemma di Poincaré basta mostrare che $\varphi^* \omega$ è chiusa

$$\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} \stackrel{?}{=} 0 \qquad \frac{\partial Q}{\partial u} = + \frac{1}{u^2}$$

$$\omega = df \qquad \frac{\partial P}{\partial v} = \frac{1}{u^2} \quad ✓$$

$$P = \frac{\partial f}{\partial u} = \frac{v}{u^2} \qquad f = - \frac{v}{u} + \xi(v)$$

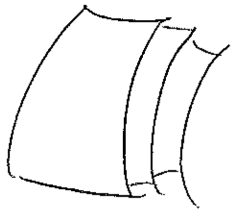
$$Q = \frac{\partial f}{\partial v} = - \frac{1}{u} + \xi' = - \frac{1}{u} \Rightarrow \xi' = 0 \Rightarrow \xi = \text{cost.}$$

ie $f = - \frac{v}{u}$ controllo: $\frac{\partial f}{\partial u} = \frac{v}{u^2}$ $\frac{\partial f}{\partial v} = - \frac{1}{u}$ ✓

Ovviamente è esatta anche ω

$$\omega = dF \qquad F(x,y,z) = -\frac{z-x}{y+z} \qquad \neq 0$$

La distr. è integrabile: sup. di livello $F = c$



(2) $X = \sin\vartheta \frac{\partial}{\partial\vartheta} + \cos\vartheta \frac{\partial}{\partial r}$

$Y = e^{-r} \frac{\partial}{\partial\vartheta} + r^2 \frac{\partial}{\partial r}$

$Y(f) = e^{-r} \frac{\partial f}{\partial\vartheta} + r^2 \frac{\partial f}{\partial r}$ $\qquad X(f) = \sin\vartheta \frac{\partial f}{\partial\vartheta} + \cos\vartheta \frac{\partial f}{\partial r}$

$X[Y(f)] = -\cos\vartheta\, e^{-r} \frac{\partial f}{\partial\vartheta} + 2\cos\vartheta\, r \frac{\partial f}{\partial r} + \underset{2^\circ \text{ ordine}}{\underline{\qquad}}$

si elidono

$Y[X(f)] = +e^{-r}\cos\vartheta \frac{\partial f}{\partial\vartheta} - e^{-r}\sin\vartheta \frac{\partial f}{\partial r} + \underset{2^\circ \text{ ordine}}{\underline{\qquad}}$

$[X,Y](f) = -2\cos\vartheta\, e^{-r} \frac{\partial f}{\partial\vartheta} + (2r\cos\vartheta + e^{-r}\sin\vartheta) \frac{\partial f}{\partial r}$

$[X,Y] = -2\cos\vartheta\, e^{-r} \frac{\partial}{\partial\vartheta} + \left(2r\cos\vartheta + e^{-r}\sin\vartheta\right) \frac{\partial}{\partial r}$

ora $[X,Y] = \mathcal{L}_X Y$

e $\mathcal{L}_Y X = [Y,X] = -[X,Y]$

③ Sia $\omega \in \Lambda^2(S^{2n})$ $n \geq 1$ una forma simplettica ($d\omega = 0$, ω non degenere).

$\underbrace{\omega^n := \omega \wedge \omega \wedge \dots \omega}_{n}$ è una $2n$-forma mai nulla (forma di volume).

$H^2(S^{2n}) = 0 \quad \Rightarrow \quad [\omega] = 0$, i.e. $\omega = d\alpha$

$\alpha \in \Lambda^1(S^{2n})$

$$\omega^n = \omega \wedge \omega^{n-1} = d\alpha \wedge \omega^{n-1}$$

ora, $d(\alpha \wedge \omega^{n-1}) = d\alpha \wedge \omega^{n-1} - \alpha \wedge d\omega^{n-1}$

è subito visto che $d\omega^{n-1} = 0$

(ex. per induzione. $d(\omega \wedge \omega) = \underset{\overset{\shortparallel}{0}}{d\omega} \wedge \omega + \omega \wedge \underset{\overset{\shortparallel}{0}}{d\omega} \dots$)

pertanto $\underset{\overset{\shortparallel}{\omega^n}}{d\alpha \wedge \omega^{n-1}} = d(\alpha \wedge \omega^{n-1})$

e quindi

$$0 < \int_{S^{2n}} \omega^n = \int_{S^{2n}} d(\alpha \wedge \omega^{n-1}) \underset{\text{Stokes}}{=} \int_{\underset{\overset{\shortparallel}{\emptyset}}{\partial S^{2n}}} \alpha \wedge \omega^{n-1} = 0$$

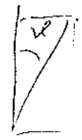
Assurdo.

(4) (r, ϑ, φ)

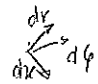
$$\omega = r\,dr\wedge d\vartheta + r^2\sin\vartheta\,dr\wedge d\varphi - \cos\varphi\,d\vartheta\wedge d\varphi$$

$$d\omega = \underbrace{dr\wedge dr\wedge d\vartheta}_{=0} + d(r^2\sin\vartheta)\,dr\wedge d\varphi - d(\cos\varphi)\,d\vartheta\wedge d\varphi$$

$$= (dr^2\sin\vartheta + r^2\,d\sin\vartheta)\wedge dr\wedge d\varphi - \underbrace{\sin\varphi\,d\varphi\wedge d\vartheta\wedge d\varphi}_{=0}$$



$$= \dots = r^2\cos\vartheta\,d\vartheta\wedge dr\wedge d\varphi$$



$$= -r^2\cos\vartheta\,dr\wedge d\vartheta\wedge d\varphi$$ (− f. di volume standard)

$$\mathcal{L}_X\omega = d\,i_X\omega + i_X\,d\omega$$

$X = r\frac{\partial}{\partial\varphi} + \sin\vartheta\frac{\partial}{\partial r}$

$$\boxed{i_X\omega} = \left(r\,dr\wedge d\vartheta + r^2\sin\vartheta\,dr\wedge d\varphi - \cos\varphi\,d\vartheta\wedge d\varphi,\ r\frac{\partial}{\partial\varphi} + \sin\vartheta\frac{\partial}{\partial r},\ \cdot\right)$$

$$= -r^3\sin\vartheta\,dr + r\cos\varphi\,d\vartheta + r\sin\vartheta\,d\vartheta + r^2\sin^2\vartheta\,d\varphi$$

$$= \boxed{-r^3\sin\vartheta\,dr + r(\cos\varphi + \sin\vartheta)\,d\vartheta + r^2\sin^2\vartheta\,d\varphi}$$

$$\boxed{d(i_X\omega)} = d(-r^3\sin\vartheta)\wedge dr + d\left(r(\cos\varphi+\sin\vartheta)\right)d\vartheta + d(r^2\sin^2\vartheta)\wedge d\varphi = -r^3\cos\vartheta\,d\vartheta\wedge dr$$
$$+ d(r\cos\varphi)\wedge d\vartheta + d(r^2\sin^2\vartheta)\wedge d\varphi + \sin\vartheta\,dr\wedge d\vartheta$$

$$+ \sin\vartheta \, dr \wedge d\vartheta$$

$$= -r^3 \cos\vartheta \, d\vartheta \wedge dr + \cos\varphi \, dr \wedge d\vartheta + r \sin\varphi \, d\varphi \wedge d\vartheta$$

$$+ 2r \sin^2\vartheta \, dr \wedge d\varphi + r^2 \cdot 2 \sin\vartheta \cos\vartheta \, d\vartheta \wedge d\varphi$$

$$= (r^3 \cos\vartheta + \cos\varphi + \sin\vartheta) \, dr \wedge d\vartheta + [2r^2 \sin\vartheta \cos\vartheta + r \sin\varphi] \, d\vartheta \wedge d\varphi$$

$$+ 2r \sin^2\vartheta \, dr \wedge d\varphi$$

$$i_X \, d\omega = \left(-r^2 \cos\vartheta \, dr \wedge d\vartheta \wedge d\varphi, \; r \frac{\partial}{\partial\varphi} + \sin\vartheta \frac{\partial}{\partial r}, \cdot, \cdot\right)$$

$$= -r^3 \cos\vartheta \, dr \wedge d\vartheta \quad \boxed{- r^2 \sin\vartheta \cos\vartheta \, d\vartheta \wedge d\varphi}$$

$$\mathcal{L}_X \omega = i_X d\omega + d \, i_X \omega =$$

$$\left[\cancel{-r^3 \cos\vartheta} + \cancel{r^3 \cos\vartheta} + \cos\varphi + \sin\vartheta\right] dr \wedge d\vartheta +$$

$$\left[2 r \sin^2\vartheta\right] dr \wedge d\varphi +$$

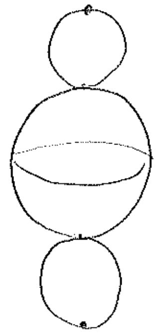
$$\left[2r^2 \sin\vartheta \cos\vartheta + r \sin\varphi - r^2 \sin\vartheta \cos\vartheta\right] d\vartheta \wedge d\varphi$$

$$= \left[\cos\varphi + \sin\vartheta\right] dr \wedge d\vartheta + \left[2 r \sin^2\vartheta\right] dr \wedge d\varphi +$$

$$+ r\left[r \sin\vartheta \cos\vartheta + \sin\varphi\right] d\vartheta \wedge d\varphi$$

(5)

N

$X =$  (top. relativa euclidea da $\mathbb{R}^3$)

S

$U = X \setminus \{N\}$

$V = X \setminus \{S\}$

connessi per archi, aperti

$U \cup V = X$

$U \cap V =$ 

è semplicemente connesso

$(U \cap V \underset{om}{\sim} S^2 \quad e \quad \pi_1(S^2) = \{e\})$

van Kampen:

$$\pi_1(X) = \pi_1(U) * \pi_1(V)$$

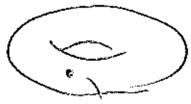
$$\pi_1(U) \cong \pi_1(S^2 \# S^2) \cong \mathbb{Z} \cong \pi_1(V)$$

$$\pi_1(X) \cong \mathbb{Z} * \mathbb{Z} \quad \text{(prodotto libero (non è abeliano..)}$$

$$\#$$

$$\pi_1(\mathbb{T}^2) \cong \mathbb{Z} \oplus \mathbb{Z} \quad \text{(abeliano)}$$

$$\Rightarrow \quad X \not\approx \mathbb{T}^2$$

[Altro modo, elementare: se esistesse f (omeo), $f: X \to \mathbb{T}^2$, allora $X \setminus \{e\} \approx$ 

$f(e)$

 e

Ma $X \setminus \{e\}$ è sconnessa, $\mathbb{T}^2 \setminus f(e)$ no.]