SUPERFICIE PARAMETRICHE

GEOMETRIA
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Addendum IV
Richiami di
Analisi

$\mathcal{U} \subset \mathbb{R}^2$ $\qquad \mathcal{U}$ aperto

$$\underline{r} : \begin{array}{ccc} \mathcal{U} & \to & \mathbb{R}^3 \\ \cup & & \\ (u,v) & \to & \underline{r} = \underline{r}(u,v) \end{array}$$

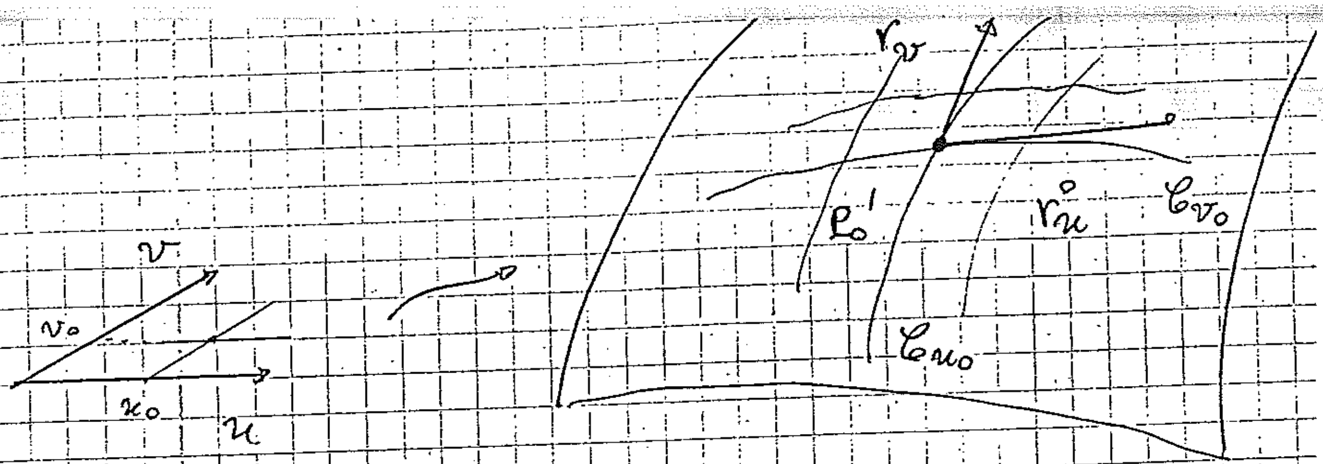
$$\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases} \qquad (u,v) \in \mathcal{U} \qquad \underline{r}(\mathcal{U}) =: \Sigma$$

$\underline{r}$ (o, con abuso di linguaggio Σ) è detta (porzione di) superficie parametrica (o parametrizzata ..)

Chiediamo poi

1. $\underline{r} \in \mathcal{C}^1(\mathcal{U})$ $\quad (x, y, z \in \mathcal{C}^1(\mathcal{U}))$
2. $\underline{r}$ iniettiva
3. $r_u \times r_v \neq 0$ in $\mathcal{U}$

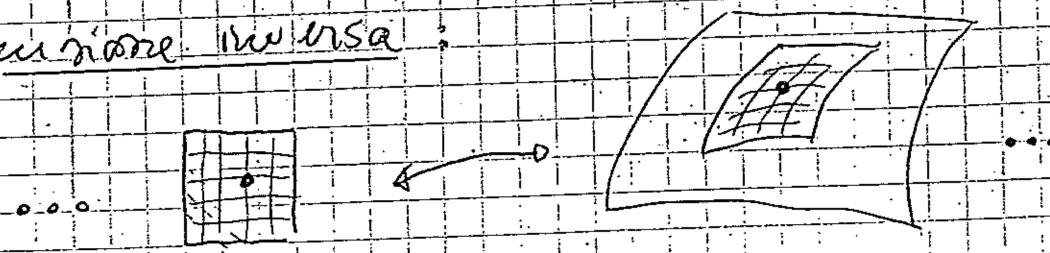
Le 1, 2, 3 definiscono una superficie (parametrica) regolare



$P_0 = (u_0, v_0)$

La 3. implica che C_{u_0}, C_{v_0} (curve regolari su Σ definite fissando u_0 e v_0 resp.) si intersecano trasversalmente in P_0 $\forall P_0 \in U$. C_u C_v : coordinate curvilinee su Σ

Vale anche un analogo del teorema della funzione inversa:



[ci stiamo liberando dell'"egemonia" cartesiana

"Il concetto di regolarità è invariante per trasformazioni piane regolari dei parametri"

Sia $\begin{cases} u' = u'(u,v) \\ v' = v'(u,v) \end{cases}$

$$\underline{R} = \underline{R}(u', v') \equiv \underline{r}(u(u',v'), v(u',v'))$$

è C^1 e iniettiva ...

$$\underline{R}_{u'} = \underline{r}_u u_{u'} + \underline{r}_v v_{u'}$$

$$\underline{R}_{v'} = \underline{r}_u u_{v'} + \underline{r}_v v_{v'}$$

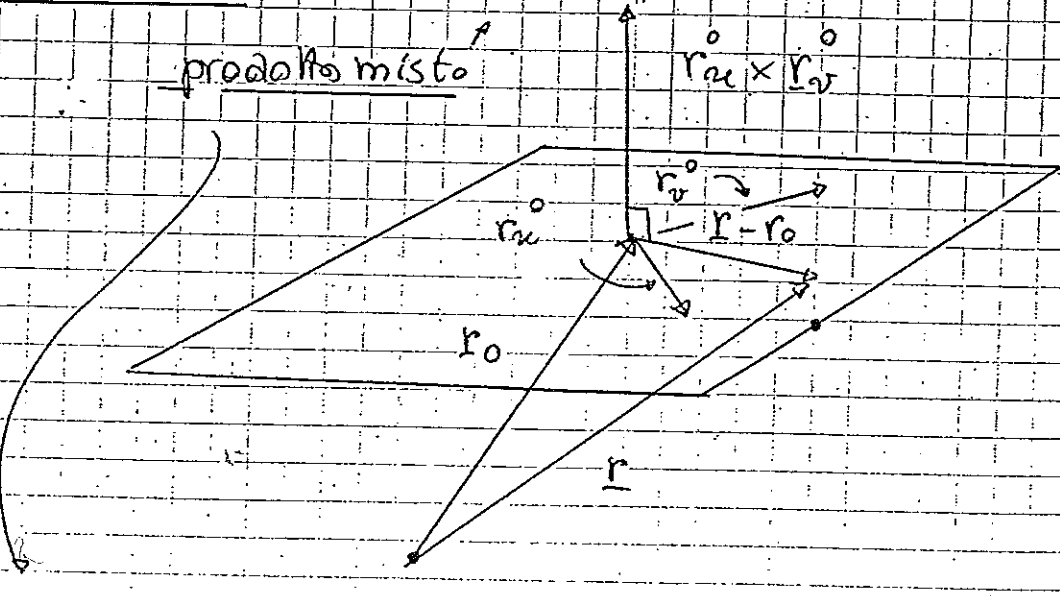
$$\Rightarrow \quad \underline{R}_{u'} \times \underline{R}_{v'} = \dots \underbrace{\frac{\partial(u,v)}{\partial(u',v')}}_{\neq 0} \underbrace{\underline{r}_u \times \underline{r}_v}_{\neq 0} \quad i$$

e sono paralleli...

piano tangente a Σ in P_0' : (è indipendente dalla parametrizzazione

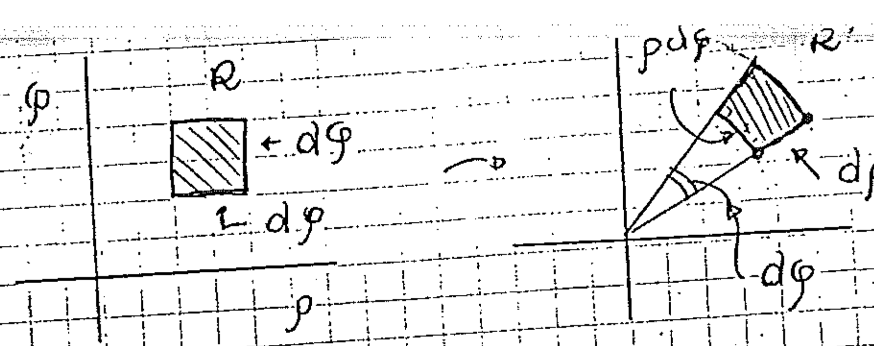
$$\boxed{\langle \underline{r}_u^0 \times \underline{r}_v^0 \mid \underline{r} - \underline{r}_0 \rangle = 0}$$

prodotto misto



$$\det(\underline{r} - \underline{r}_0, r_u^0, r_v^0) = 0$$

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ x_u^0 & y_u^0 & z_u^0 \\ x_v^0 & y_v^0 & z_v^0 \end{vmatrix} = 0$$



" Se $d\varphi$, $d\rho$ sono "infinitesimi", un rettangolo infinitesimo va in un rettangolo infinitesimo "
(ortogonalità)

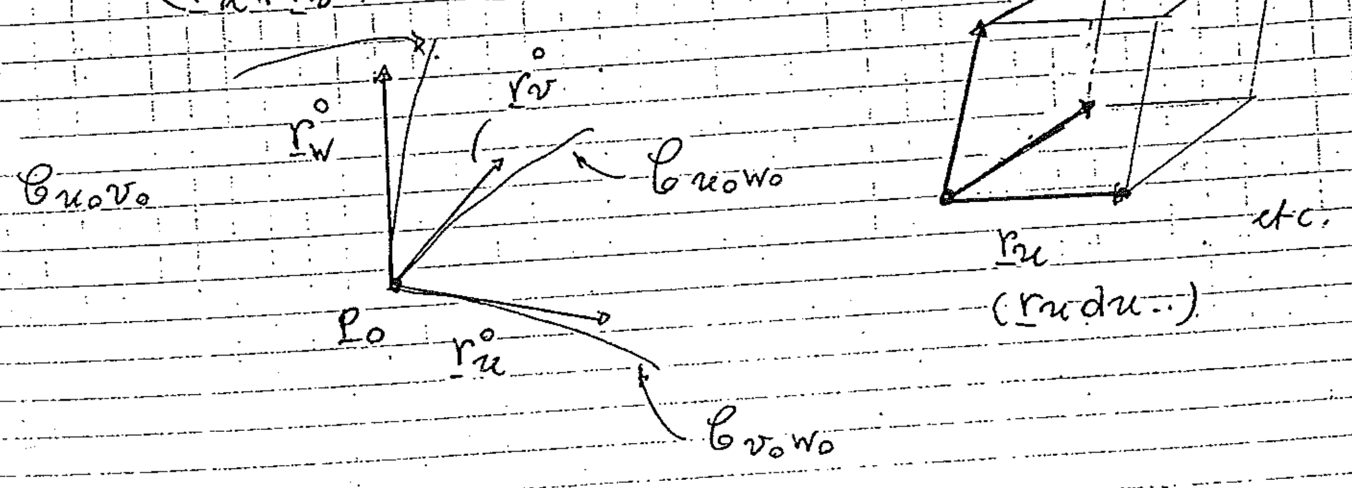
area $R' = \rho\, d\rho\, d\varphi$

area $R = d\rho\, d\varphi$

Coordinate curvilinee ortogonali in $\mathbb{R}^3$

$$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases} \qquad \underline{r} = \underline{r}(u,v,w)$$

$$\langle \underline{r}_u | \underline{r}_v \rangle = \langle \underline{r}_u | \underline{r}_w \rangle = \langle \underline{r}_v | \underline{r}_w \rangle = 0$$



" $|\det J|\, du\, dv\, dw$ = volume di un parallelepipedo "infinitesimo" di lati $r_u\, du$, $r_v\, dv$ $r_w\, dw$

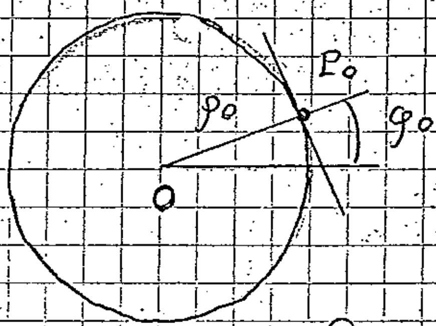
$|\det J|$ = fattore di distorsione locale dei volumi

Trasformazioni piane & superficie parametriche: esempi rilevanti

coordinate curvilinee ortogonali

$$\langle \mathring{r}_u \mid \mathring{r}_v \rangle = 0$$

(es. coordinate polari



$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases}$$

$\varphi \in [0, 2\pi)$
$\rho > 0$

C_{ρ_0} : $\rho = \rho_0$ circonfere centrate in

C_{φ_0} $\varphi = \varphi_0$ rette per

$$\begin{array}{cc} & \begin{array}{cc} \underline{r}_\rho & \underline{r}_\varphi \end{array} \\ J = & \begin{pmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{pmatrix} \end{array}$$

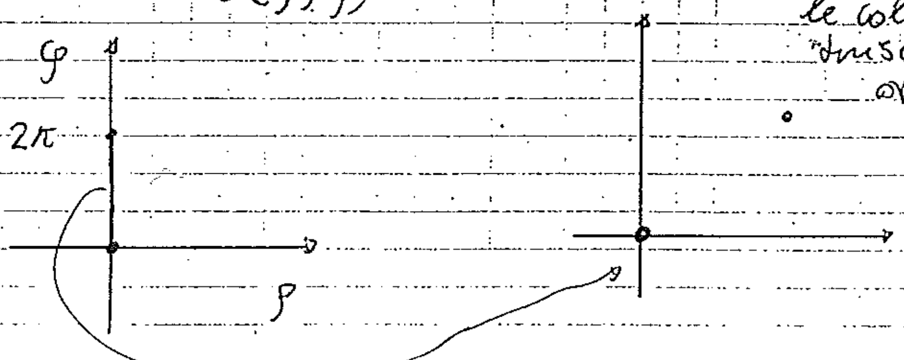
$\langle r_\rho | r_\varphi \rangle = 0$

$$\det J = \frac{\partial(x,y)}{\partial(\rho,\varphi)} = \rho$$

(! J non è orto- gonale le colonne non forniscono una ortonormale

$\| r_\varphi \| = \rho$

$\| \underline{r}_\rho \| = 1$

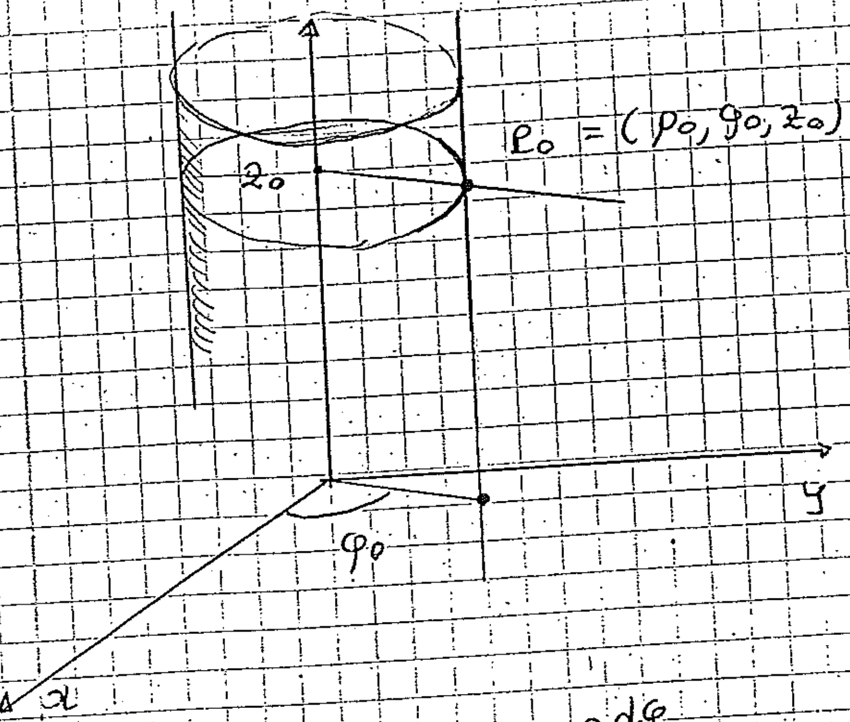


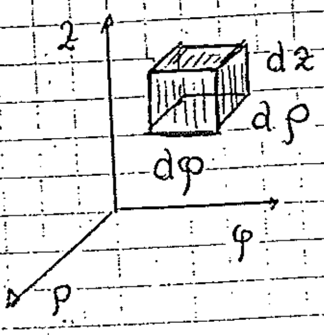
$$\mathbb{R}^+ \times [0, 2\pi) \leftrightarrow \mathbb{R}^2 - \{0,0\}$$

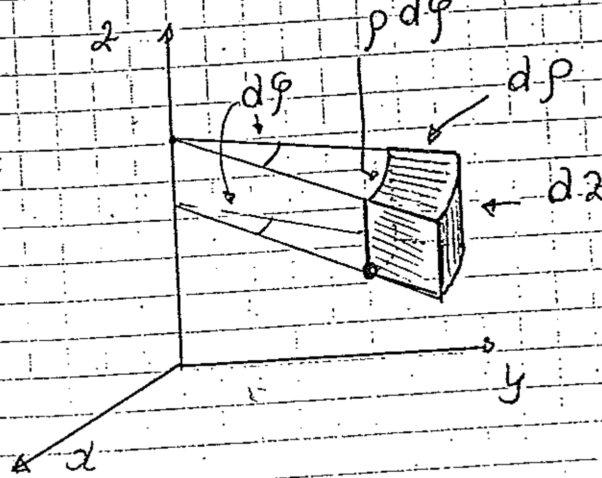
$\rho > 0$

☆ Coordinate cilindriche

$$\begin{cases} x = \rho \cos\varphi \\ y = \rho \sin\varphi \\ z = z \end{cases} \qquad \frac{\partial(x,y,z)}{\partial(\rho,\varphi,z)} = \rho$$





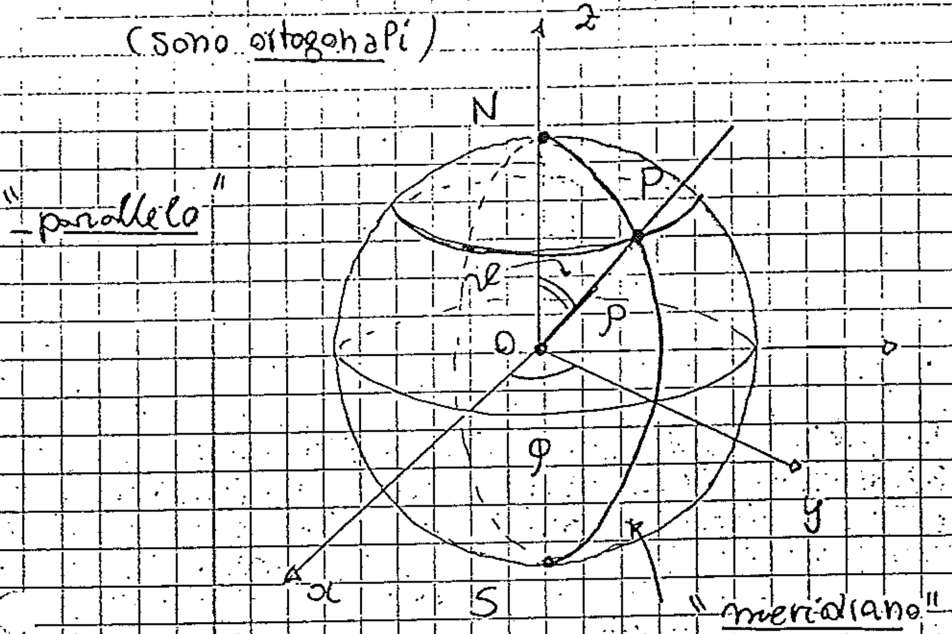


coordinate sferiche

$\rho \geq 0$, φ ...

$\vartheta \in [0, \pi]$

(sono ortogonali)



φ : longitudine

ϑ : colatitudine

$$\begin{cases} x = \rho \sin\vartheta \cos\varphi \\ y = \rho \sin\vartheta \sin\varphi \\ z = \rho \cos\vartheta \end{cases}$$

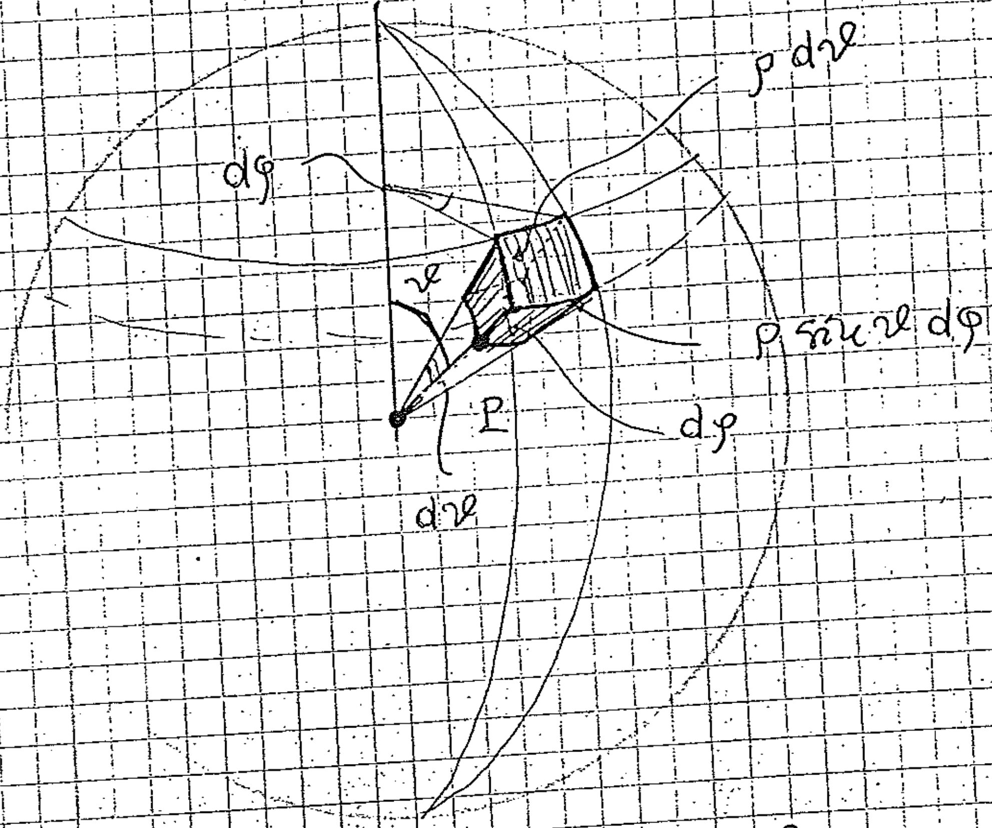
$$J = \begin{pmatrix} \sin\vartheta\cos\varphi & -\rho\sin\vartheta\sin\varphi & \rho\cos\varphi\cos\vartheta \\ \sin\vartheta\sin\varphi & \rho\sin\vartheta\cos\varphi & \rho\sin\varphi\cos\vartheta \\ \cos\vartheta & 0 & -\rho\sin\vartheta \end{pmatrix}$$

$$\det J = \frac{\partial(x,y,z)}{\partial(\rho,\varphi,\vartheta)} = \ldots -\rho^2 \sin\vartheta$$

$= 0$ se $\rho = 0$ o $\sin\vartheta = 0 \Rightarrow \vartheta = 0$, $\vartheta = \pi$

sull'asse z non c'è biunivocità

(.. φ è indeterminato



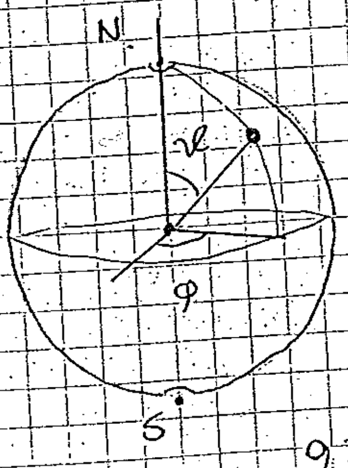
$$|\det J|\, d\rho\, d\varphi\, d\vartheta = \rho^2 \sin\vartheta\, d\rho\, d\varphi\, d\vartheta$$

Se $\rho = R > 0$ fissato, ho coordinate curvilinee su $x^2+y^2+z^2-R^2=0 \equiv S^2_R$

... non c'è biunivocità per N e S...

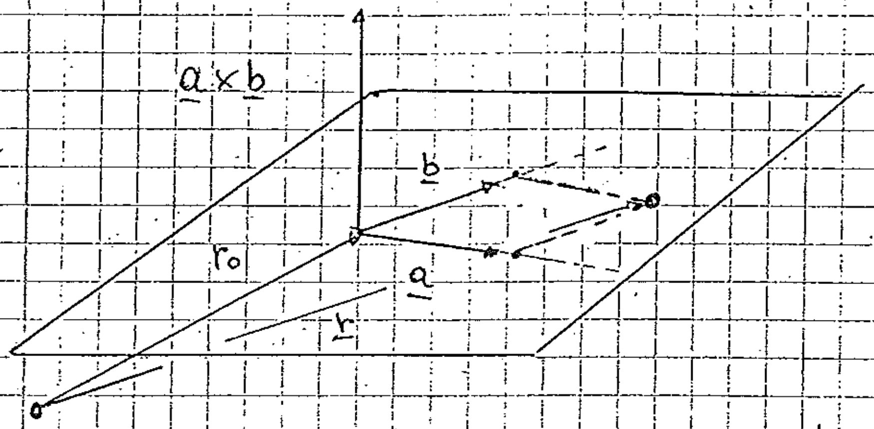
(φ è indeterminata)

$$S^2_R - \{N, S\} \longleftrightarrow [0, 2\pi) \times (0, \pi)$$



$\neq 0$

piano $\langle \underline{a} \times \underline{b} \mid \underline{r} - \underline{r}_0 \rangle = 0$



$$\underline{r} = \underline{r}_0 + u\,\underline{a} + v\,\underline{b}$$

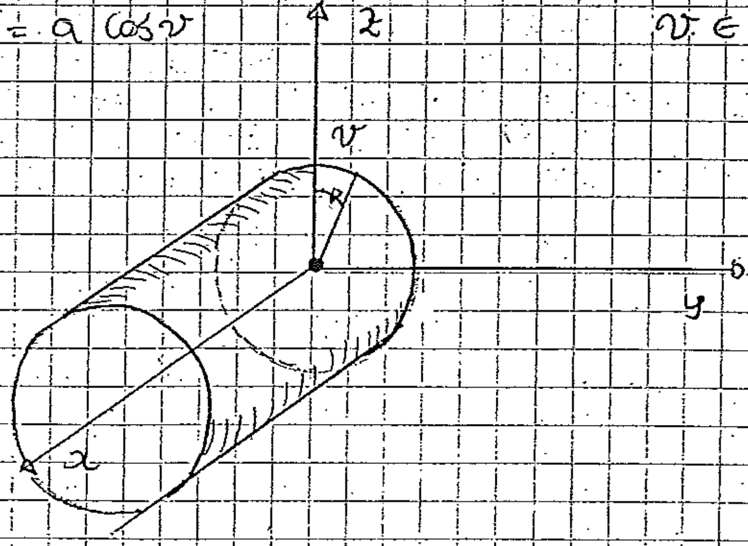
forma parametrica $\quad \underline{r}_u \times \underline{r}_v \neq 0$

$\Leftrightarrow \underline{a} \times \underline{b} \neq 0 \Leftrightarrow \underline{a}, \underline{b}$ linearmente indipendenti (i.e. non paralleli)

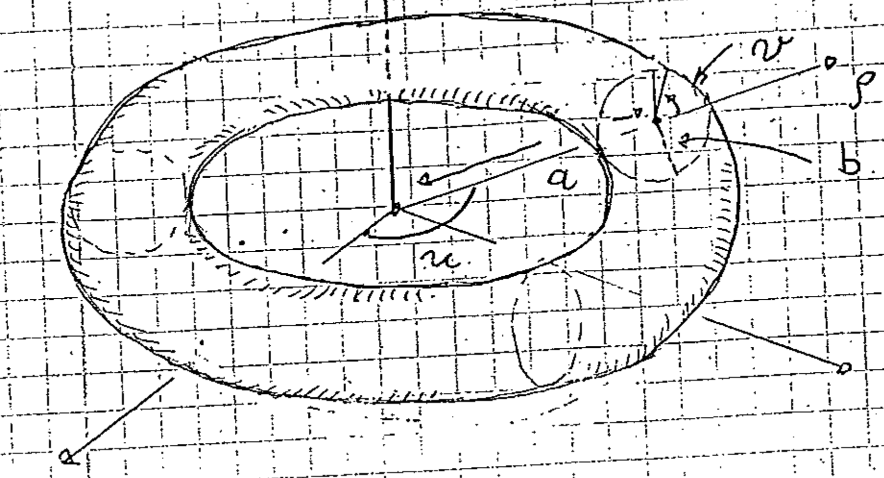
cilindro (asse di simmetria x)

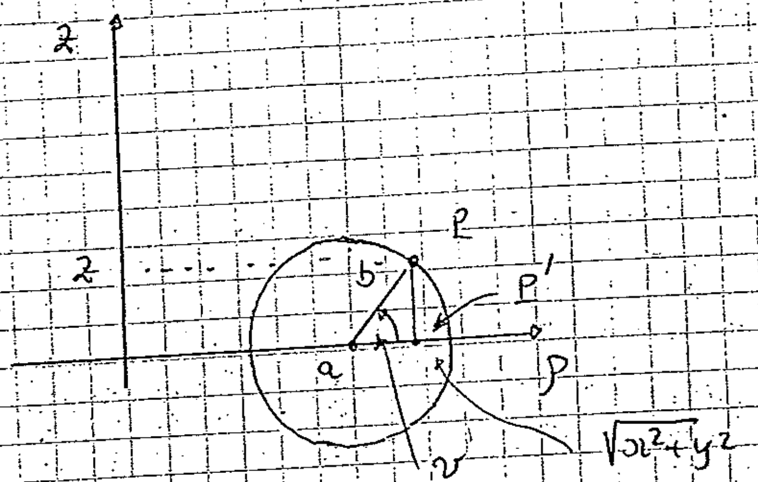
$$\underline{r}(u,v) = u\,\underline{i} + a \sin v\,\underline{j} + a \cos v\,\underline{k}$$

$$\begin{cases} x = u \\ y = a \sin v \\ z = a \cos v \end{cases} \qquad u \in \mathbb{R}, \quad v \in [0, 2\pi)$$

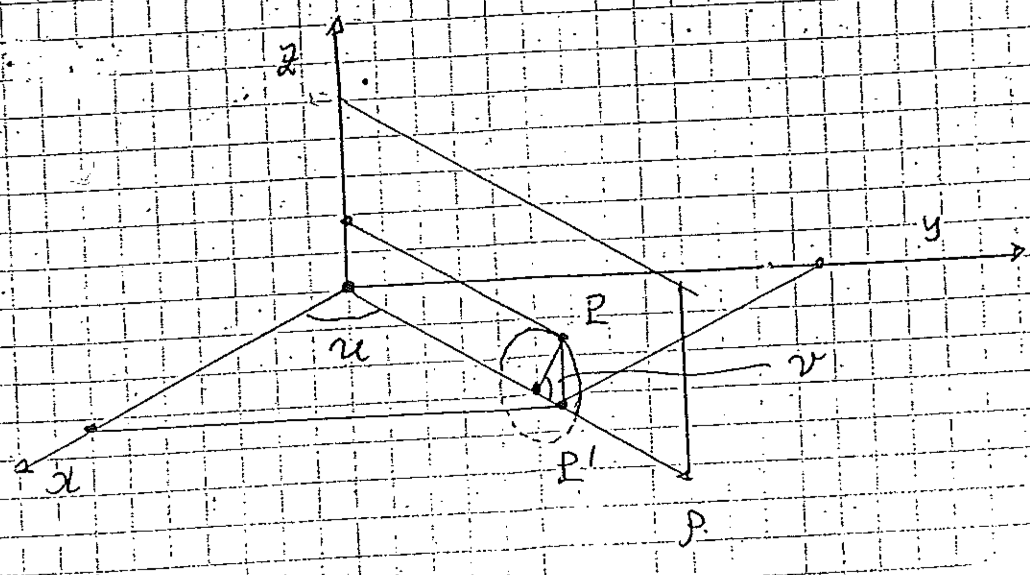


toro





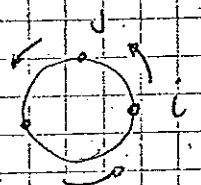
$$x = (a + b\cos v)\cos u \qquad u \in [0, 2\pi)$$
$$y = (a + b\cos v)\sin u \qquad v \in [0, 2\pi)$$
$$z = b\sin v$$

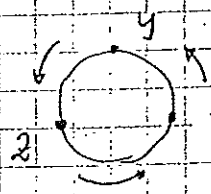


Controllare la regolarità ...

usare

$$\underline{r}_u \times \underline{r}_v = \underline{i}\,\frac{\partial(y,z)}{\partial(u,v)} + \underline{j}\,\frac{\partial(z,x)}{\partial(u,v)} + \underline{k}\,\frac{\partial(x,y)}{\partial(u,v)}$$





$$\|r_u \times r_v\| = \sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2}$$

Scriviamo l'eq. del toro in forma cartesiana implicita: $F(x,y,z)=0$

Si ha:

$$\boxed{\left(\sqrt{x^2+y^2}-a\right)^2 + z^2 = b^2}$$

$$F = \left(\sqrt{x^2+y^2}-a\right)^2 + z^2 - b^2 = 0$$

Studiare $\begin{cases} F=0 \\ \nabla F = 0 \end{cases}$ e concludere che

non ha soluzioni...

<u>Superficie parametriche cartesiane</u>

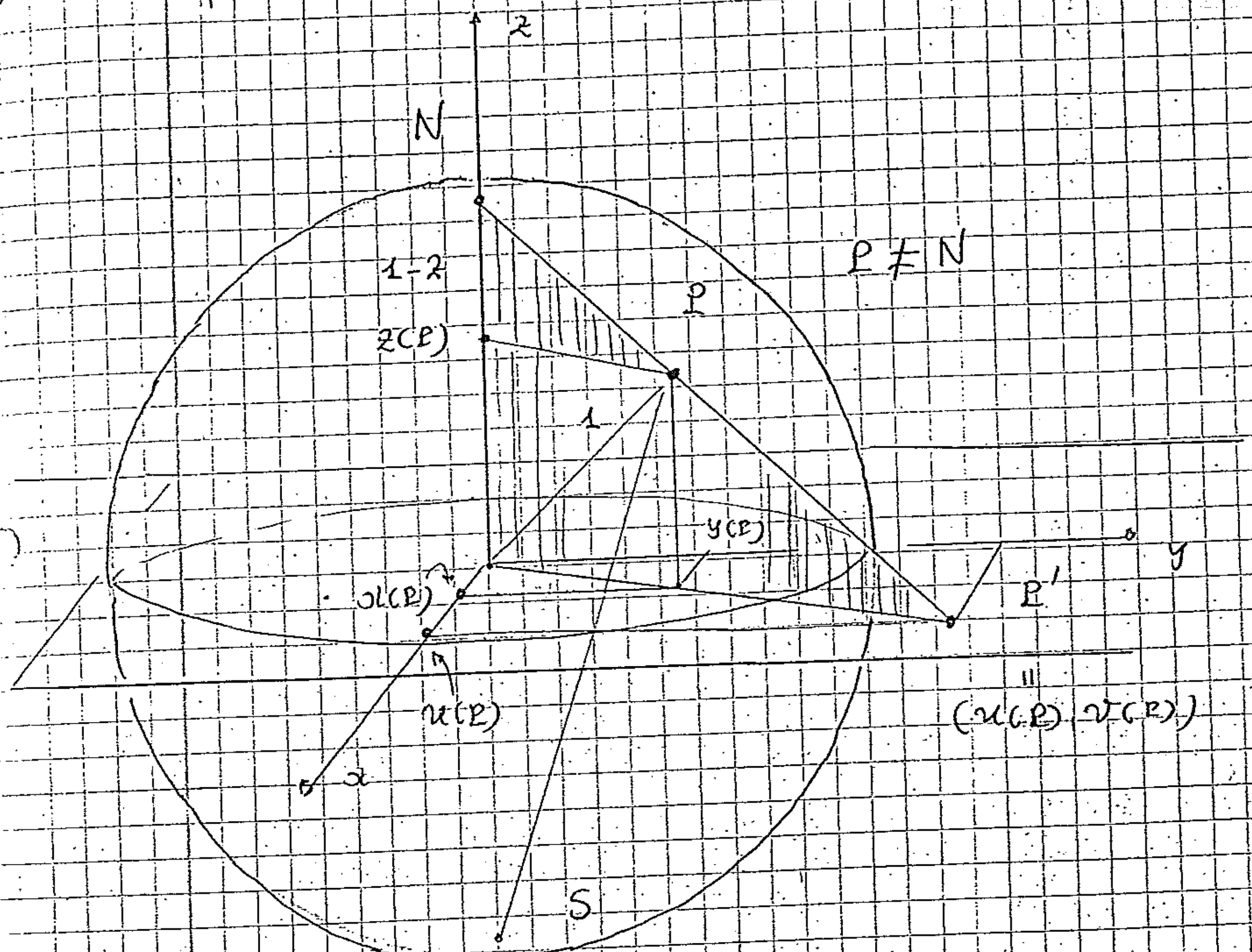
ovvero $z = f(x,y)$ ... $\begin{cases} x=u \\ y=v \\ z=f(u,v) \end{cases}$

$$\underline{r}_x \times \underline{r}_y = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x\,\underline{i} - f_y\,\underline{j} + 1\underline{k}$$

$$\|\underline{r}_x \times r_y\| = \underbrace{\sqrt{f_x^2 + f_y^2 + 1}}_{\neq 0}$$

sempre <u>regolare</u>

Sfera in proiezione stereografica



$$\begin{cases} u = \dfrac{x}{1-z} \\ v = \dfrac{y}{1-z} \end{cases}$$

$x = u(1-z)$

$y = v(1-z)$

$x^2 + y^2 + z^2 = 1 \quad \Rightarrow$

$$z = \frac{u^2+v^2-1}{1+u^2+v^2} \quad \Rightarrow$$

$$\begin{cases} x = \dfrac{2u}{1+u^2+v^2} \\ y = \dfrac{2v}{1+u^2+v^2} \\ z = \dfrac{u^2+v^2-1}{1+u^2+v^2} \end{cases}$$

In coordinate complesse

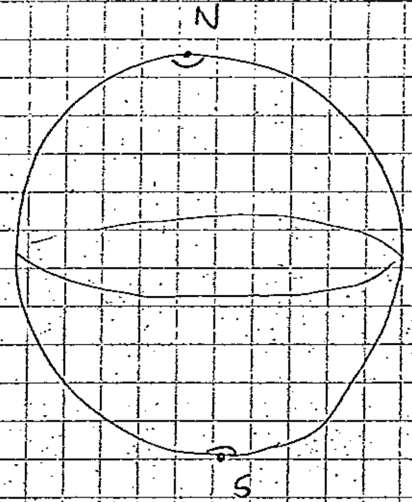
$$\zeta = u + iv$$

N : non è rappresentabile

S = (0, 0)

Viceversa, proiettando da S, P ≠ S

⇒ è $\zeta' = u' + iv' = \frac{1}{\zeta}$ per $P \in S^2$



⇒ ☆ $S^2 \simeq \mathbb{CP}^1$: retta proiettiva complessa

"Sfera di Riemann"

coordinate omogenee

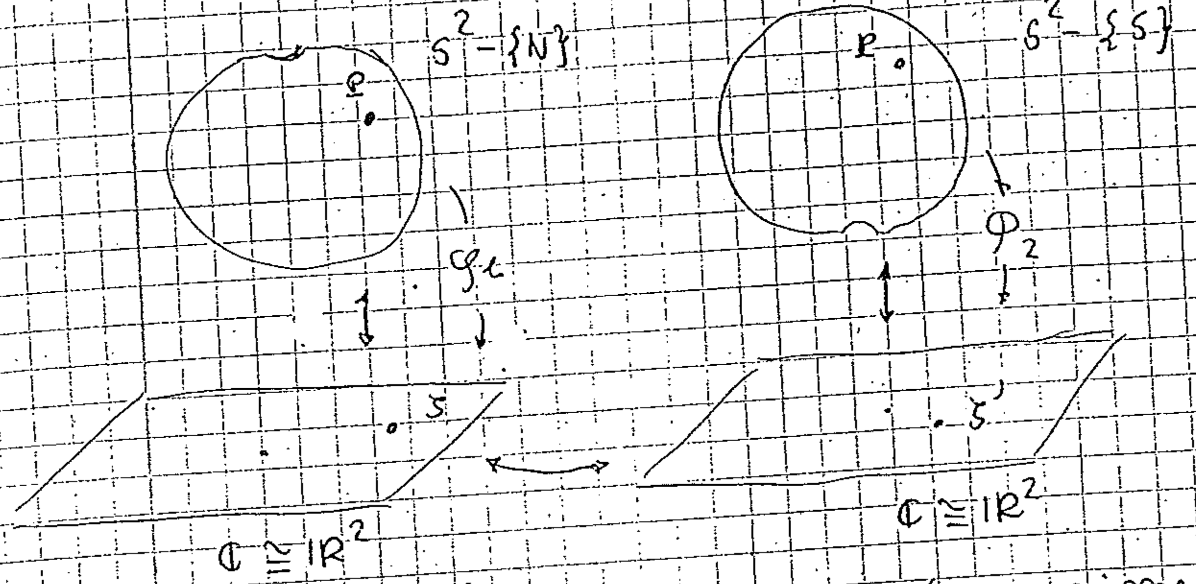
$$\mathbb{CP}^1 = \left\{ (z_0, z_1)/_\sim \;\middle|\; (z_0, z_1) \in \mathbb{C}^2 - \{(0,0)\} \right\}$$

$(z_0, z_1) \sim (z_0', z_1') \iff z_0' = \lambda z_0$, $z_1' = \lambda z_1$

$\mathbb{C} \ni \lambda \neq 0$ $\quad \zeta, \zeta'$ coordinate non omogenee

$$\zeta = \frac{z_1}{z_0}$$

S^2 è descritta da due carte

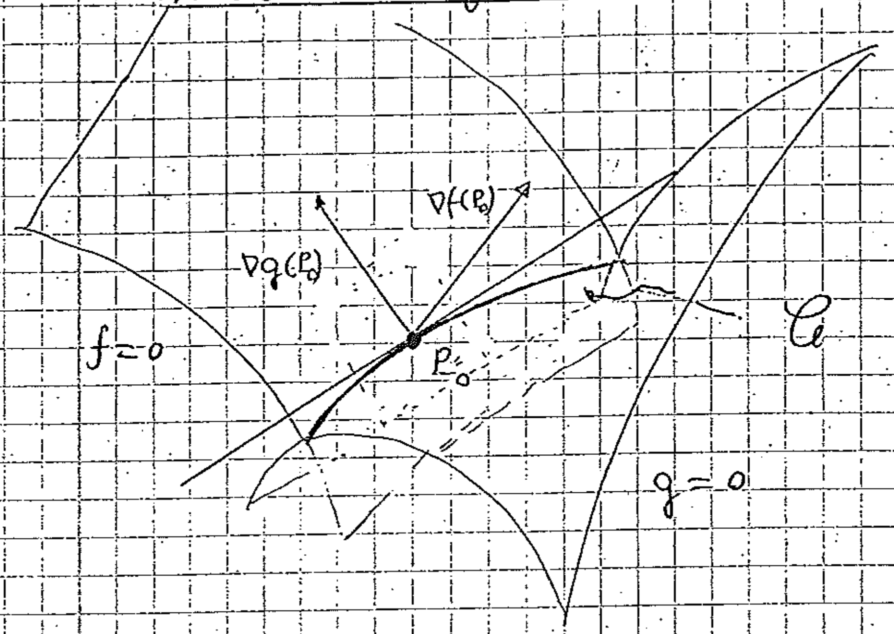


legate da una legge di trasformazione

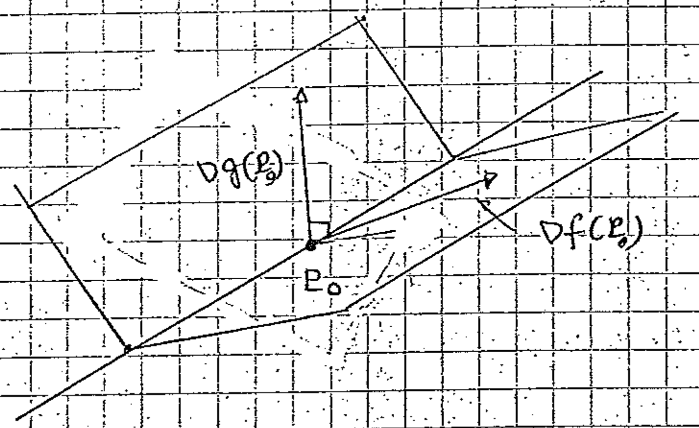
$$\ldots \quad \xi' = \frac{1}{\xi}$$

questo conduce al concetto di
<u>varietà differenziabile</u>

Descrizione implicita delle curve nello spazio



$P_0 = (x_0, y_0, z$



Discutiamo la seguente estensione del teorema del Dini (intuitivamente chiara)

Teorema Siano $f, g \in \mathcal{C}^1(\mathcal{U})$ $\mathcal{U}$ aperto

sia $P_0 \in \mathcal{C}$: $\begin{cases} f=0 \\ g=0 \end{cases}$, e sia

$$\overline{\nabla f(P_0) \times \nabla g(P_0) \neq 0}$$ (⇒ tale condizione sussiste localmente...)

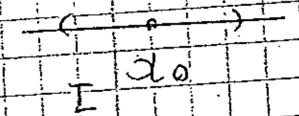
allora localmente $\mathcal{C}$ è una curva regolare precisamente, sia p. es. $\dfrac{\partial(f,g)}{\partial(y,z)} \neq 0$

$$\left(\nabla f \times \nabla g = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{vmatrix} = \right.$$

$$\left. = \underline{i} \, \frac{\partial(f,g)}{\partial(y,z)} + \underline{j} \, \frac{\partial(f,g)}{\partial(z,x)} + \underline{k} \, \frac{\partial(f,g)}{\partial(x,y)} \right)$$

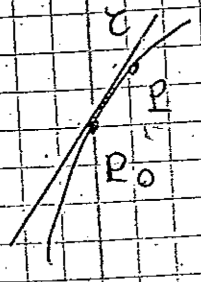
Allora, per $x \in I \ni x_0$ (I opportuno)

$$\mathcal{C}: \begin{cases} x = x \\ y = y(x) \\ z = z(x) \end{cases} \qquad \begin{matrix} y_0 = y(x_0) \\ z_0 = z(x_0) \end{matrix}$$



ovvero
$$f(x, y(x), z(x)) \equiv 0 \qquad \text{in } I$$
$$g(x, y(x), z(x)) \equiv 0$$
$$f(P_0) = g(P_0) = 0$$

(descrizione locale di $\mathcal{C}$)



L'equazione della retta tangente a $\mathcal{C}$ in P_0 (e per $P \in \mathcal{C}$ suff. vicini) è data da

$$\tau: \begin{cases} \langle \nabla f(P_0) \mid \underline{r} - \underline{r}_0 \rangle = 0 \\ \langle \nabla g(P_0) \mid \underline{r} - \underline{r}_0 \rangle = 0 \end{cases}$$

(intersezione dei piani tangenti alle superfici (loc. regolari) $f = 0$ e $g = 0$)

In forma parametrica:

$$\tau : \begin{cases} x = x_0 + (x - x_0) = x \\ y = y_0 + y'(x_0)(x - x_0) \\ z = z_0 + z'(x_0)(x - x_0) \end{cases}$$

☆ Dove y', z' si ottengono derivando rispetto a x le

$$f(x, y(x), z(x)) \equiv 0$$
$$g(x, y(x), z(x)) \equiv 0$$

$$\left(\Rightarrow \quad f_x + f_y y' + f_z z' = 0 \right.$$
$$g_x + g_y y' + g_z z' = 0$$

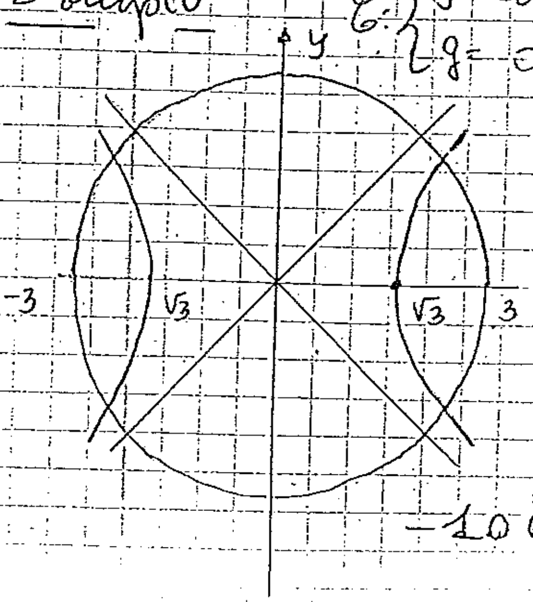
$\Rightarrow \dfrac{\partial(f,g)}{\partial(y,z)} \neq 0$ assicura la risolubilità del sistema ...)

□

☆ L'equazione del piano normale a $\mathcal{C}$ in P_0 è $\langle \nabla f(P_0) \times \nabla g(P_0) \mid r - r_0 \rangle = 0$

dà la direzione di r..

Esempio $\mathcal{C}$: $\begin{cases} f = x^2 + y^2 + z^2 - 9 = 0 & \text{sfera...} \\ g = x^2 - y^2 - 3 = 0 & \text{cilindro iperbolico} \end{cases}$



Sezione trasversale $z = 0$..

$$\nabla f = (2x, 2y, 2z)$$

$$\nabla g = (2x, -2y, 0)$$

$\nabla f = 0 \Rightarrow x = y = z = 0 \notin$ sfera

($\Rightarrow f = 0$ sup. regolare)

$\nabla g = 0 \quad x = y = 0 \quad \ldots \Rightarrow$ pertanto il

cilindro dato è regolare.

Esaminiamo

$$\begin{cases} f = 0 \\ g = 0 \\ \nabla f \times \nabla g = 0 \end{cases}$$

$$\nabla f \times \nabla g = (4yz, 4xz, -8xy) = 0$$

$$\Rightarrow \begin{cases} xy = 0 \\ xz = 0 \\ yz = 0 \\ x^2 + y^2 + z^2 = 9 \\ x^2 - y^2 = 3 \end{cases} \Rightarrow xy = 0$$

$x = 0 \Rightarrow -y^2 = 3$: NO $\Rightarrow x \neq 0$

$\Rightarrow y = 0$ e $z = 0$

$\Rightarrow x^2 = 9 = 3$ assurdo.

Poniamoci in $P = (2, 1, 2) \in \mathcal{C}$.

$$\frac{\partial(f,g)}{\partial(y,z)} = \begin{vmatrix} 2y & 2z \\ -2y & 0 \end{vmatrix} = 4yz = 4\cdot 2\cdot 1 = 8 \neq$$

$\Rightarrow$ posso esplicitare rispetto a x ...

Calcoliamo y' e z' $\quad (' = \frac{d}{dx})$.

$$\begin{cases} x^2 + y^2(x) + z^2(x) = 9 \\ x^2 - y^2(x) - 3 = 0 \end{cases}$$

$$\cancel{2}x + \cancel{2}yy' + \cancel{2}zz' = 0$$
$$\cancel{2}x - \cancel{2}yy' = 0$$

$\Rightarrow \quad 2 - 1\cdot y' = \quad y'(2) = 2$

$\Rightarrow \quad 2 + 2 + 2\cdot z' = 0 \Rightarrow z'(2) = -2$

$\Rightarrow$ <u>retta tangente</u> $\quad x = 2 + 1(x-2) = x$
$y = 1 + 2(x-2) = 2x-3$
$z = 2 - 2(x-2) = -2x+6$

$$\Rightarrow \begin{cases} y = 2x-3 \\ z = -2x+6 \end{cases}$$

oppure
$$\begin{cases} 2(x-2) + 1\cdot(y-1) + 2(z-2) = 0 \\ 2(x-2) - 1(y-1) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 2x + y + 2z - 9 = 0 \\ 2x - y - 3 = 0 \end{cases} \rightarrow \begin{array}{l} 2x + 2x - 3 + 2z \\ 2z + 4x - 12 = 0 \\ z + 2x - 6 = 0 \end{array}$$

Calcolo implicito

Data $f = z^3 - xz - y = 0$, calcolare, ove possibile, $\dfrac{\partial^2 z}{\partial x \partial y}$ $\quad (z = z(x,y))$

Sol. Supponiamo $\dfrac{\partial f}{\partial z} = 3z^2 - x \neq 0$

posto $z = z(x,y)$, è

$$G(x,y) \equiv z(x,y)^3 - x\,z(x,y) - y \equiv 0$$

$$\Rightarrow G_x = 3z^2 z_x - z - x z_x = 0$$

$$\Rightarrow z_x = \frac{z}{3z^2 - x}$$

$$G_y = 3z^2 z_y - x z_y - 1 = 0$$

$$\Rightarrow z_y = \frac{1}{3z^2 - x}$$

$G_{xy} = G_{yx} = 0$ (T. Schwarz)

$$\Rightarrow 3\,2z\,z_x z_y + 3z^2 z_{yx} - z_y - x z_{yx} = 0$$

$$\Rightarrow z_{xy} = z_{yx} = \frac{-3\,2z\,z_x z_y + z_y}{3z^2 - x} =$$

$$= z_y \frac{-3\,2z_x z + 1}{3z^2 - x} = \ldots\ldots$$