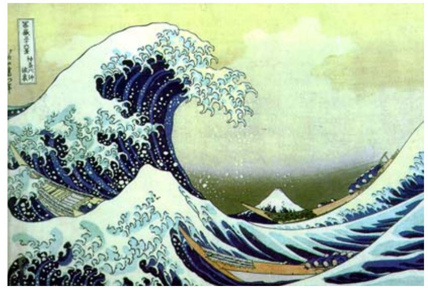


Multimedia communications

Comunicazioni multimediali

G. Menegaz

gloria.menegaz@univr.it

The Fourier kingdom

- CTFT
  - Continuous time signals

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

$$f(t) = \int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} dt$$

  - The amplitude F*(ω)*, also called Fourier transform, of each sinusoidal wave $e^{\omega j t}$ is equal to its correlation with *f*
  - If *f (t)* is uniformly regular, then its Fourier transform coefficients also have a fast decay when the frequency increases, so it can be easily approximated with few low-frequency Fourier coefficients.

The Fourier kingdom

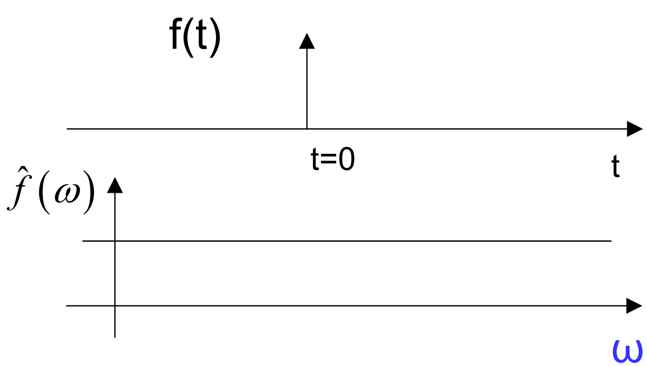
- DTFT
  - Over discrete signals, the Fourier transform is a decomposition in a discrete orthogonal Fourier basis $\{e^{i2kn/N}\}_{0 \leq k<N}$ of C^N , which has properties similar to a Fourier transform on functions.
  - Its embedded structure leads to fast Fourier transform(FFT) algorithms, which compute discrete Fourier coefficients with $O(N \log N)$ instead of N^2. This FFT algorithm is a cornerstone of discrete signal processing.
- The Fourier transform is unsuitable for representing transient phenomena
  - the support of $e^{\omega it}$ covers the whole real line, so $\hat{f}(\omega)$ depends on the values $f(t)$ for all times $t \in R$. This global "mix" of information makes it difficult to analyze or represent any local property of $f(t)$ from $\hat{f}(\omega)$.
    - As long as we are satisfied with *linear time-invariant* operators or *uniformly regular signals*, the Fourier transform provides simple answers to most questions. Its richness makes it suitable for a wide range of applications such as signal transmissions or stationary signal processing. However, to represent a *transient* phenomenon—a word pronounced at a particular time, an apple located in the left corner of an image—the Fourier transform becomes a cumbersome tool that requires *many coefficients* to represent a *localized* event.

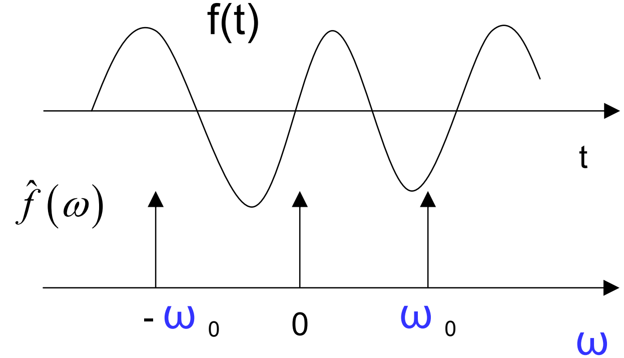
The Fourier kingdom

- The F-transform is not suitable for representing transient phenomena
  - Intuition

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

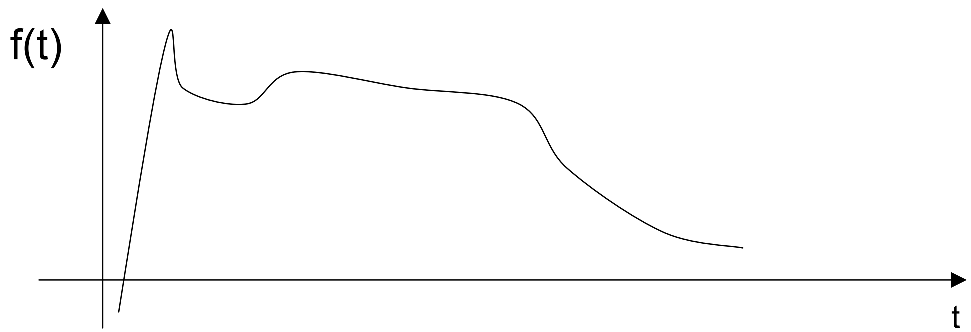
  - F(ω) depends on the values taken by f(t) on the entire temporal axis, which is not suitable for analyzing local properties
  - Need of a transformation which is well localized in *time and frequency*





The Fourier kingdom

- Transient phenomena

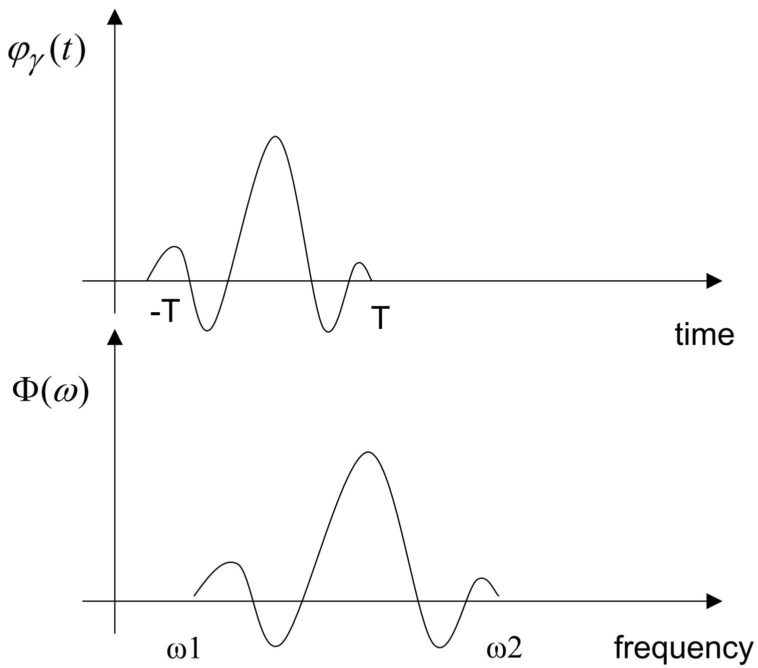


The two transients present in the signal contribute **differently** to the spectrum. The F-transform does not allow to characterize them **separately** to get a local description of the frequency content of the signal.

The basis functions of the FT are complex sinusoids, thus **F(ω) is a measure of the correlation of the signal f(t) with the complex exponential at frequency ω**, which spreads over the whole frequency axis.

Time-frequency localization

- Time-frequency atoms: basis functions that are well localized in *both* time and frequency



$\langle f(t), \varphi_\gamma(t) \rangle$ only depends on the values of f in the neighborhood of T

$F(\omega)\Phi(\omega)$ only depends on the values of F in the neighborhood of ω

Windowed Fourier Transform

Windowed Fourier atoms were introduced in 1946 by Gabor to measure localized frequency components of sounds
Also *short time Fourier transform*

$$g_{u,\xi}(t) = e^{j\xi t} g(t-u)$$

$$\|g\| = 1 \rightarrow \|g_{u,\xi}\| = 1 \qquad \forall (u,\xi)$$

$$Sf(u,\xi) = \langle f, g_{u,\xi} \rangle = \int_{-\infty}^{+\infty} f(t) g_{u,\xi}^{*}(t) dt = \int_{-\infty}^{+\infty} f(t) g(t-u) e^{-j\xi t} dt$$

The Fourier integral is *localized in the neighborhood of u by the window* $g(t-u)$.

The transform $Sf(u,\xi)$ depends only on the values of $f(t)$ and $f(\omega)$ in the time and frequency neighborhoods where the energy is concentrated, providing information on the behavior of the function within a bounded time-frequency interval.

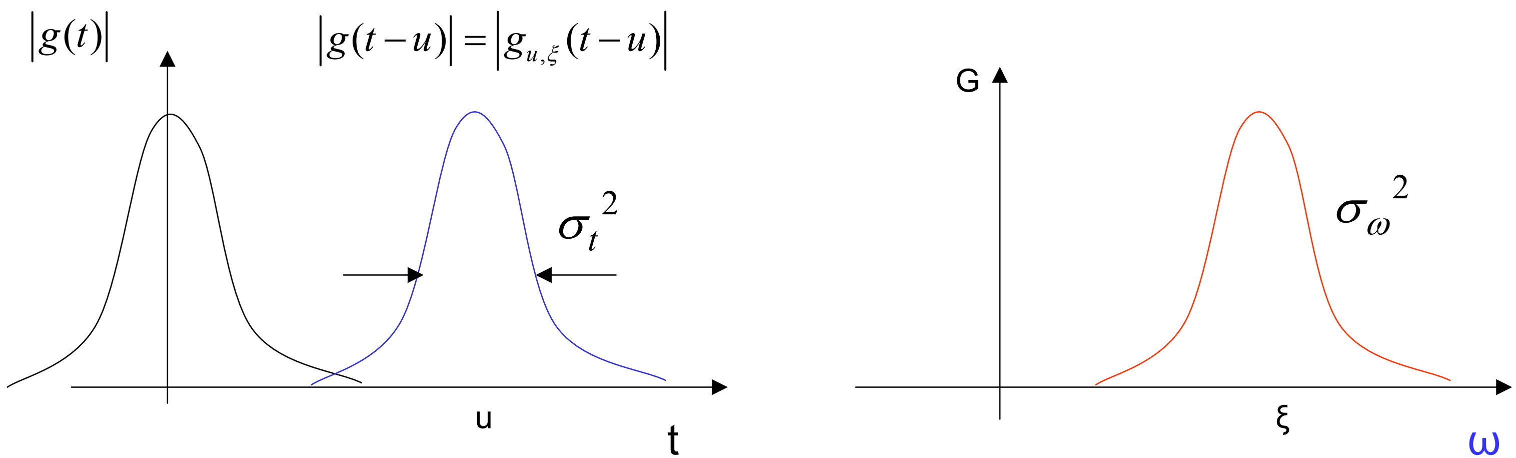
Wavelet genesis : the WT was designed following the same approach, with the goal of characterizing transient phenomena in signals by a mapping into the time/frequency domain.

Windowed Fourier Transform

$$g_{u,\xi(t)} = g(t-u)e^{j\xi t}$$

$$G_{u,\xi}(\omega) = G(\omega-\xi)e^{-ju(\omega-\xi)}$$

$$E = \int_{-\infty}^{+\infty} |\, g(t)\,|^2 \; dt = (Plancherel) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\, G(\omega)\,|^2 \; d\omega$$



Time-frequency atoms

$g_{u,\xi}(t)$ is centered in u

$$\left\|\sigma^2{}_t\right\| = \int_{-\infty}^{+\infty} (t-u)^2 \left|g_{u,\xi}(t)\right|^2 dt = \int_{-\infty}^{+\infty} t^2 \left|g(t)\right|^2 dt$$

time spread around u

interpreted as a probability distribution

independent of u and ξ

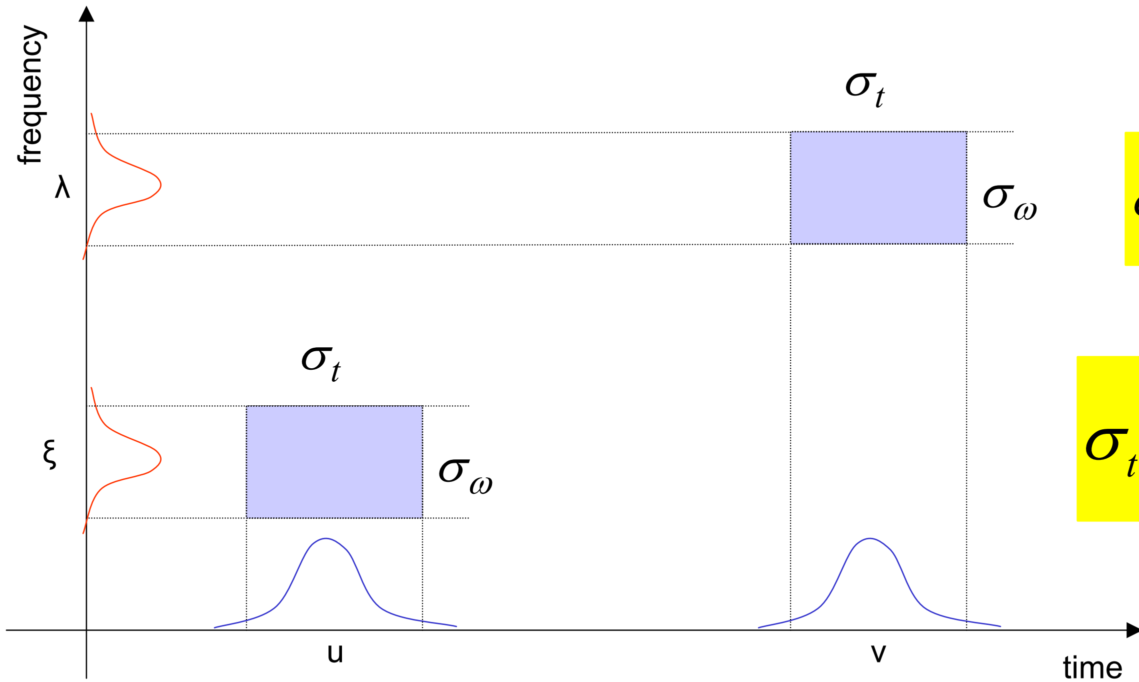
$$\left\|\sigma^2{}_\omega\right\| = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (\omega-\xi)^2 \left|G_{u,\xi}(\omega)\right|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \omega^2 \left|G(\omega)\right|^2 d\omega$$

frequency spread around ξ

$g_{u,\xi}(t)$ corresponds to a Heisemberg box of surface $\sigma_t \sigma_\omega$

that is independent of u and ξ, hence the windowed Fourier transform has the same resolution across the time-frequency plan

Heisemberg boxes



$$\sigma_t \sigma_\omega \geq \frac{1}{2}$$

$$\sigma_t^{\,2} \sigma_\omega^{\,2} \geq \frac{1}{4}$$

When g is a Gaussian the atoms are called Gabor functions.
Since in this case the equality holds, these minimize the area of the Heisemberg box, Gabor atoms are considered as optimal for the time-frequency characterization of signals.

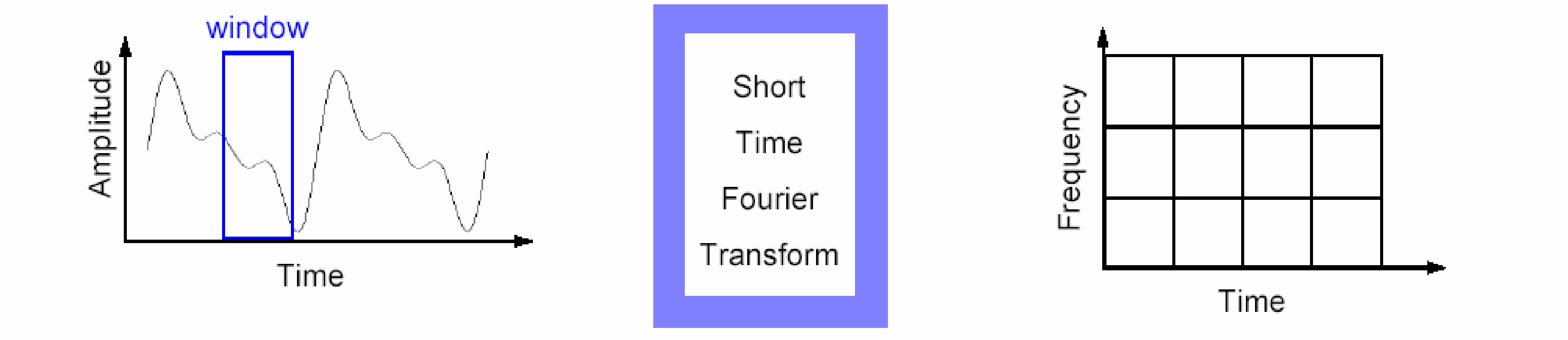
Windowed Fourier Transform

- It can be interpreted as a Fourier transform of f at the frequency ξ, localized by the window $g(t-u)$ in the neighborhood of u. This windowed Fourier transform is highly redundant and represents one-dimensional signals by a time-frequency image in (u, ξ). It is thus necessary to understand how to select many fewer time frequency coefficients that represent the signal efficiently.
- A windowed Fourier transform decomposes signals over waveforms that have the same time and frequency resolution. It is thus effective as long as the signal does not include structures having different time-frequency resolutions, some being very localized in time and others very localized in frequency.
- Wavelets address this issue by changing the time and frequency resolution.

STFT

- The STFT (windowed FT) represents a sort of compromise between the time- and frequency-based views of a signal. It provides some information about both *when* and at *what* frequencies a signal event occurs.
- However, you can only obtain this information with limited *precision*, and that *precision is determined by the size of the window*
- While the STFT compromise between time and frequency information can be useful, the drawback is that once you choose a particular size for the time window, that window is the same for all frequencies
- Many signals require a more flexible approach, one where we can vary the window size to determine more accurately either time or frequency.

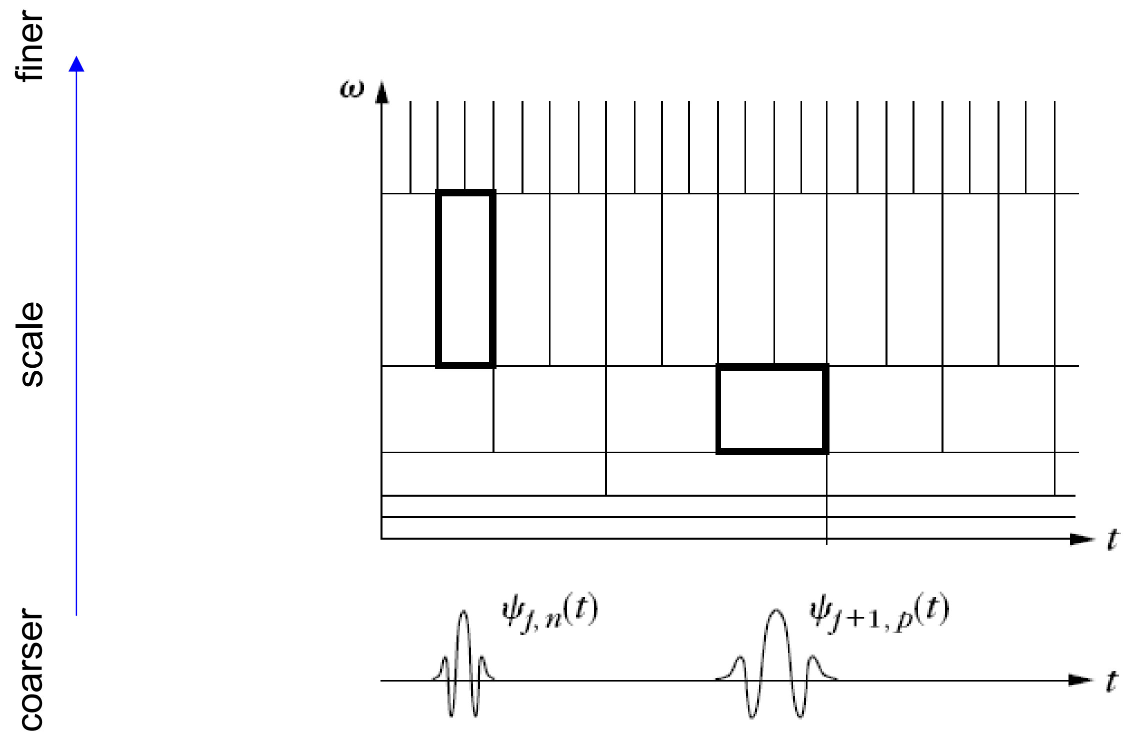
Windowed Fourier transform



Uniform tiling of the time-frequency plan

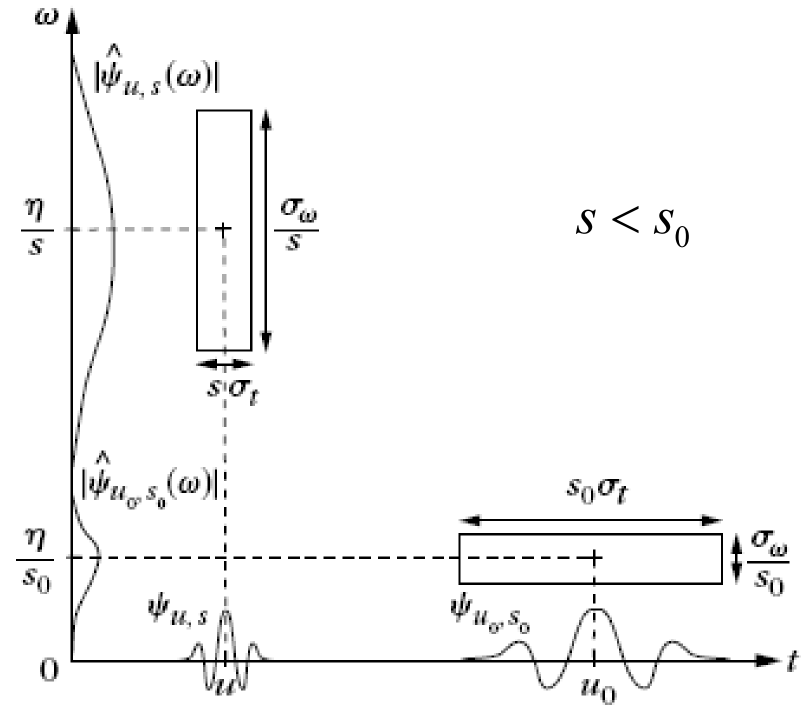
Target

- Non uniform tiling of the time-frequency space
  - This kind of tiling is adapted to analyze the scaling evolution of transients with zooming procedures across scales.



Wavelet basis

- As opposed to windowed Fourier atoms, wavelets have a *time-frequency resolution that changes*.
- The wavelet $\Psi_{u,s}$ has a time support centered at *u* and proportional to *s*. Let us choose a wavelet whose Fourier transform $\hat{\Psi}_{u,s}(\omega)$ is nonzero in a positive frequency interval centered at η. The Fourier transform $\hat{\Psi}_{u,s}(\omega)$ is dilated by 1/*s* and thus is localized in a positive frequency interval centered at η/*s*; its size is scaled by 1/*s*.



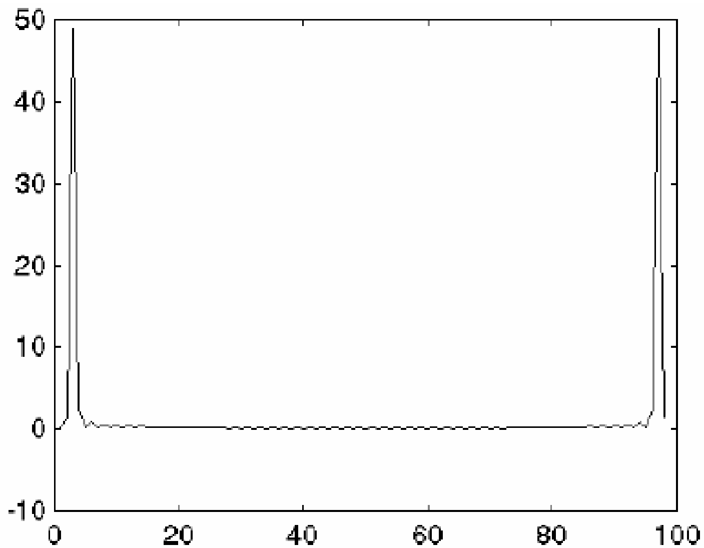
$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right) \Leftrightarrow \Psi_{u,s}(\omega) = e^{-j\omega u}\sqrt{s}\Psi(s\omega)$$

Multiscale zooming

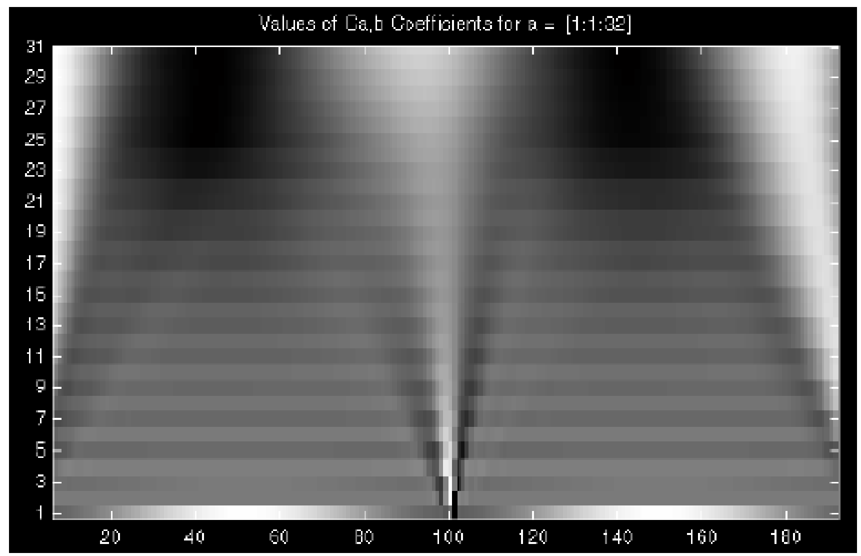
- In the time-frequency plane, the Heisenberg box of a wavelet atom *u,s* is therefore a rectangle centered at $(u, \eta/s)$, with time and frequency widths, respectively, proportional to *s* and 1/*s*.
- When *s* varies, the time and frequency width of this time-frequency resolution cell changes, but its area remains constant
  - Large-amplitude wavelet coefficients can detect and measure short high frequency variations because they have a narrow time localization at high frequencies.
- At low frequencies their time resolution is lower, but they have a better frequency resolution.
  - This modification of time and frequency resolution is adapted to represent sounds with sharp attacks, or radar signals having a frequency that may vary quickly at high frequencies.

Multiscale zooming

- Signal singularities have specific scaling invariance characterized by Lipschitz exponents.
  - Pointwise regularity of *f* can be characterized by the asymptotic decay of the wavelet transform amplitude $|Wf(u, s)|$ when s goes to zero.
  - Singularities are detected by following the local maxima of the wavelet transform across scales.



Fourier Coefficients

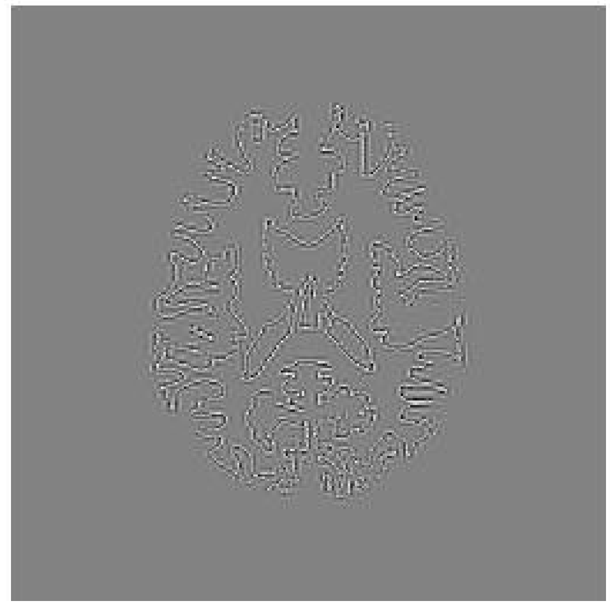


Wavelet Coefficients

Multiscale zooming

- In images, wavelet local maxima indicate the *position of edges*, which are sharp variations of image intensity.
  - At different scales, the geometry of this local maxima support provides contours of image structures of varying sizes.
  - This multiscale edge detection is particularly effective for pattern recognition in computer vision.

Multiscale edge detection



Wavelet transforms

Wavelet transforms

- A wavelet is a function of zero average centered in the neighborhood of t=0 and is normalized

$$\int_{-\infty}^{+\infty} \psi(t)\,dt = 0$$

$$\|\psi\| = 1$$

- The translations and dilations of the wavelet generate a family of time-frequency atoms

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right)$$

- Wavelet transform of f in L2(R) at position u and scale s is

$$Wf(u,s) = \langle f, \psi_{u,s} \rangle = \int_{-\infty}^{+\infty} f(t)\frac{1}{\sqrt{s}}\psi^*\left(\frac{t-u}{s}\right)dt$$

Wavelet transforms

- Real wavelets: suitable for detecting sharp signal transitions

$$Wf(u,s) = \left\langle f, \psi_{u,s} \right\rangle = \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) dt$$

  - When s goes to zero the decay of the wavelet coefficients characterize the regularity of f in the neighborhood of u
  - Edges in images
  - Example: Mexican hat (second derivative of a Gaussian)

$$\psi(t) = \frac{2}{\pi^{1/4}\sqrt{3\sigma}} \left(\frac{t^2}{\sigma^2} - 1\right) \exp\left(-\frac{t^2}{2\sigma^2}\right)$$

$$\hat{\psi}(\omega) = \frac{-\sqrt{8}\sigma^{5/2}\pi^{1/4}}{\sqrt{3}} \omega^2 \exp\left(-\frac{\sigma^2\omega^2}{2}\right)$$

$-\psi(t)$

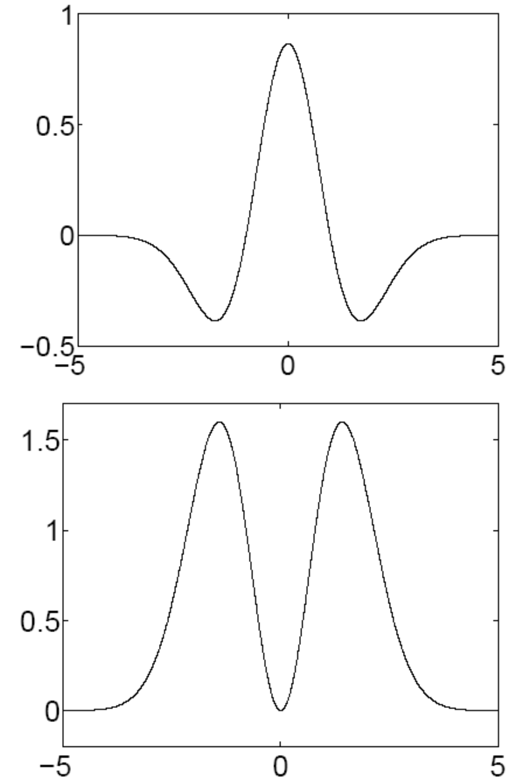


Fig. 4.6. A Wavelet Tour of Signal Processing, 3rd ed. Mexican hat wavelet for $\sigma = 1$ and its Fourier transform.

$-\hat{\psi}(\omega)$

Real wavelets: example

- The wavelet transform was calculated using a Mexican hat wavelet

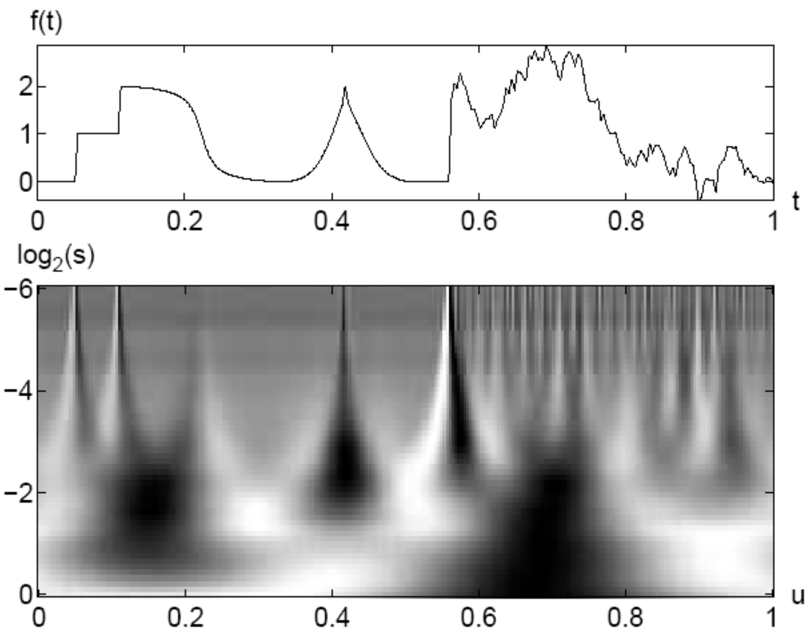


Fig. 4.7. A Wavelet Tour of Signal Processing, 3rd ed. Real wavelet transform $Wf(u,s)$ computed with a Mexican hat wavelet The vertical axis represents $\log_2 s$. Black, grey and white points correspond respectively to positive, zero and negative wavelet coefficients.

Real wavelets: Admissibility condition

- Theorem 4.3 (Calderon, Grossman, Morlet)

Let ψ in $L^2(R)$ be a real function such that

$$C_\psi = \int_0^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < +\infty$$

Admissibility condition

Any f in $L^2(R)$ satisfies

$$f(t) = \frac{1}{C_\psi} \int_0^{+\infty} \int_{-\infty}^{+\infty} Wf(u,s) \frac{1}{\sqrt{s}} \psi\left(\frac{t-u}{s}\right) du \frac{ds}{s^2}$$

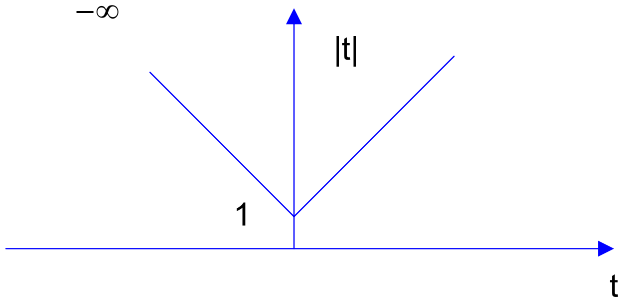
and

$$\int_{-\infty}^{+\infty} |f(t)|^2 dt = \frac{1}{C_\psi} \int_0^{+\infty} \int_{-\infty}^{+\infty} |Wf(u,s)|^2 du \frac{1}{s^2} ds$$

Admissibility condition

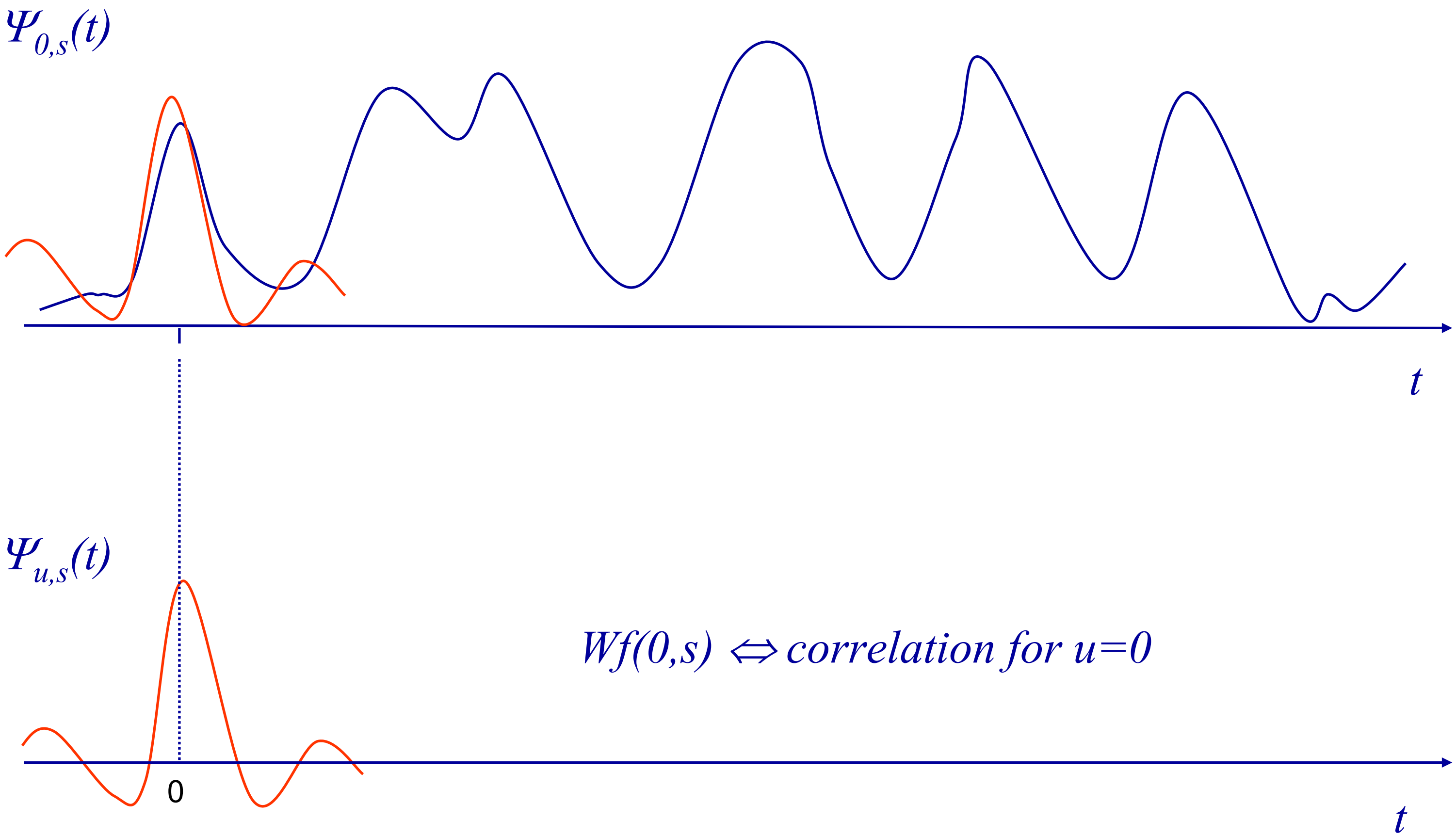
- Consequences
  - The integral is finite if the wavelet has zero average $\hat{\psi}(0)=0$
    - This condition is nearly sufficient →
  - If $\hat{\psi}(0)=0$ and $\hat{\psi}(\omega)$ is continuously differentiable, than the admissibility condition is satisfied
    - This happens if it has a sufficient time decay

$$\int_{-\infty}^{+\infty}(1+|t|)\,|\psi(t)|\,dt<+\infty$$

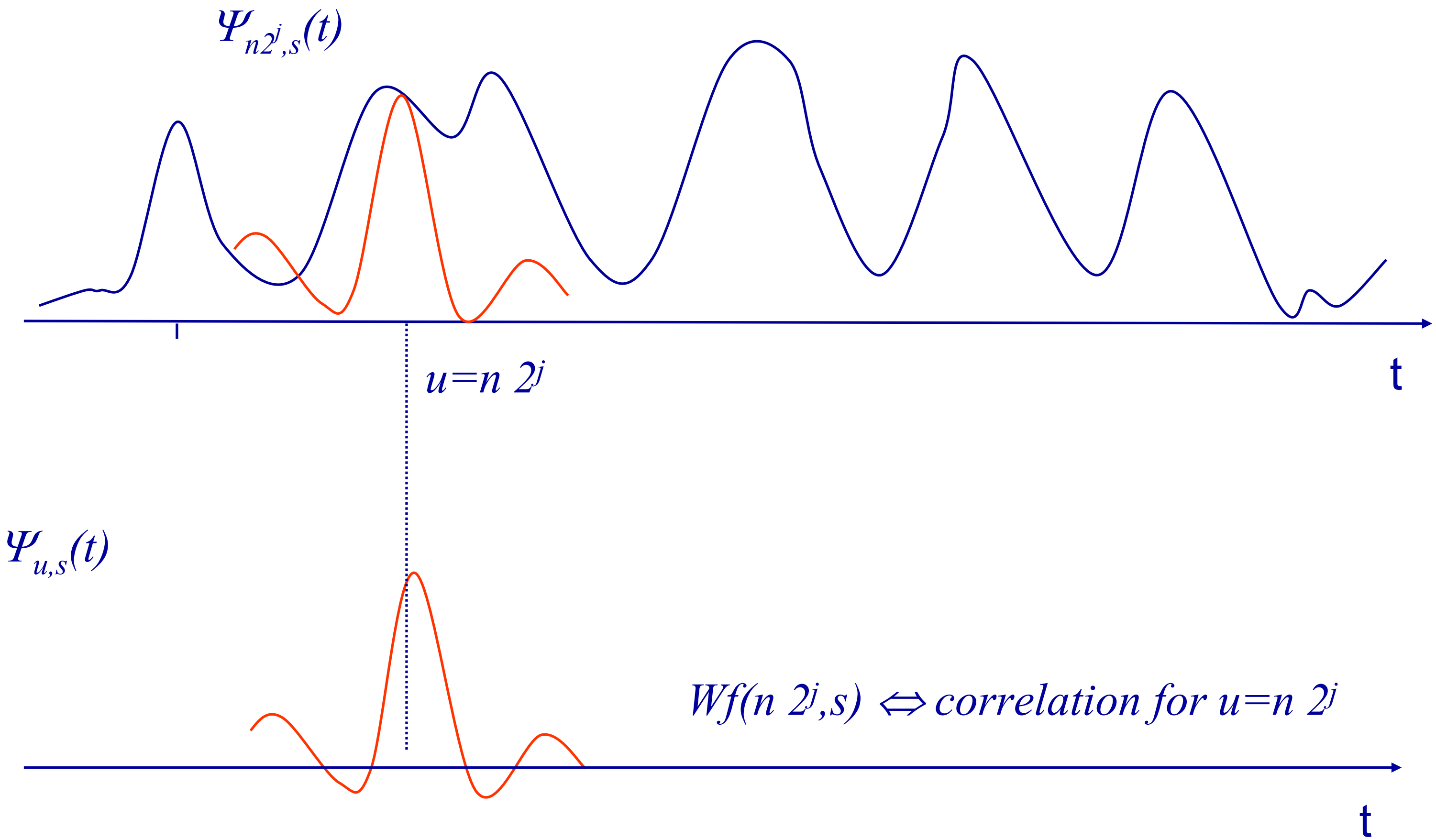


→ The wavelet function must decay **sufficiently fast** in both time and frequency

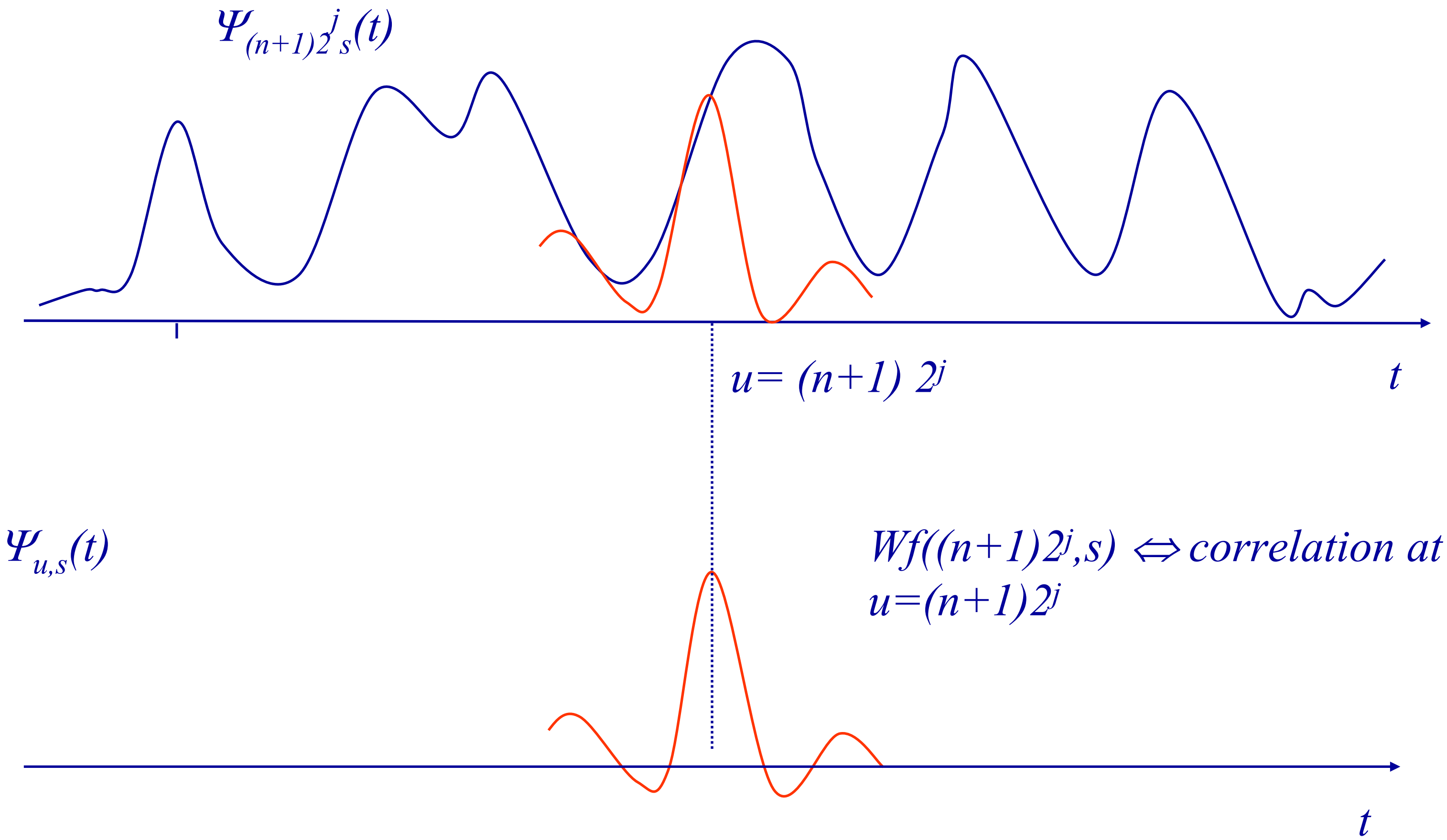
Wavelet transform



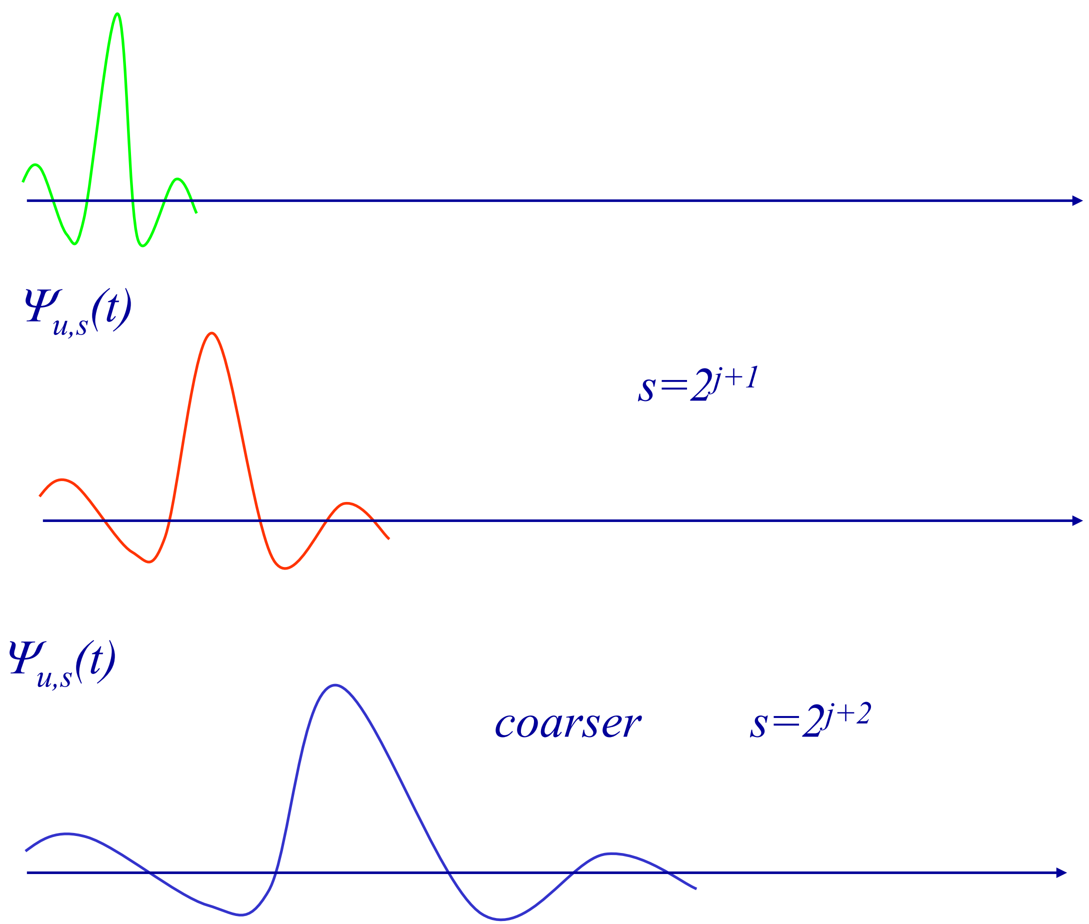
Wavelet transform



Wavelet transform



Changing the *scale*



multiresolution

Fourier versus Wavelets

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$C(scale, position) = \int_{-\infty}^{\infty} f(t) \psi(scale, position, t) dt$$

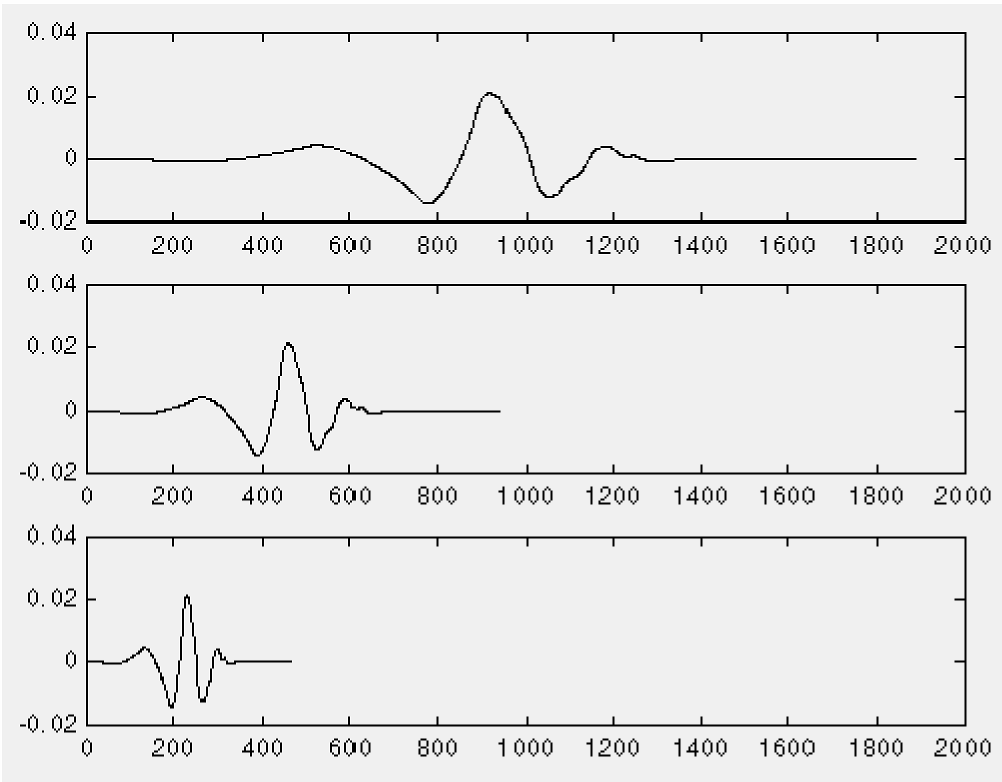
Sine Wave

Wavelet (db10)

Signal

Constituent wavelets of different scales and positions

Scaling

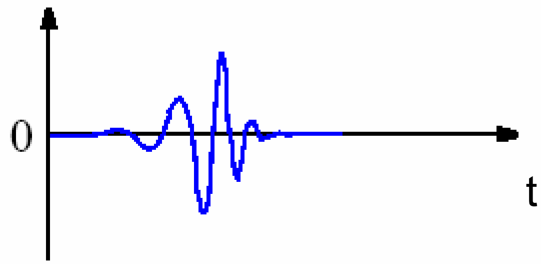


$$f(t) = \psi(t) \quad ; \quad a = 1$$

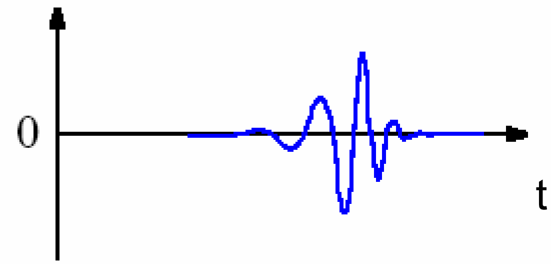
$$f(t) = \psi(2t) \quad ; \quad a = \frac{1}{2}$$

$$f(t) = \psi(4t) \quad ; \quad a = \frac{1}{4}$$

Shifting



Wavelet function
$\psi(t)$



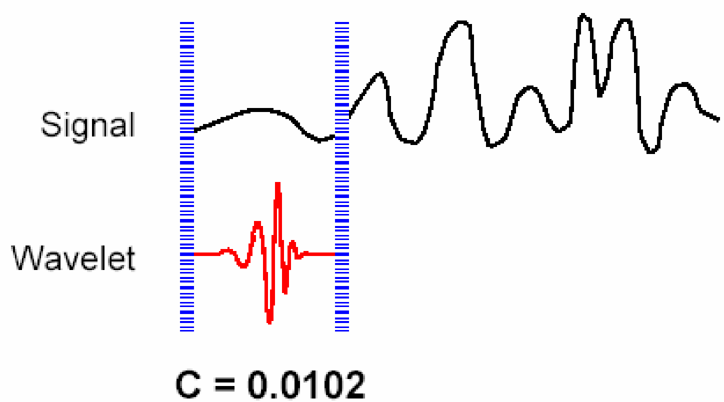
Shifted wavelet function
$\psi(t-k)$

Recipe

1 Take a wavelet and compare it to a section at the start of the original signal.

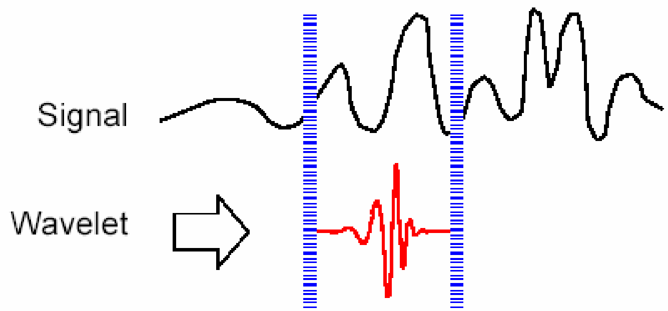
2 Calculate a number, C, that represents how closely correlated the wavelet is with this section of the signal. The higher C is, the more the similarity. More precisely, if the signal energy and the wavelet energy are equal to one, C may be interpreted as a correlation coefficient.

Note that the results will depend on the shape of the wavelet you choose.

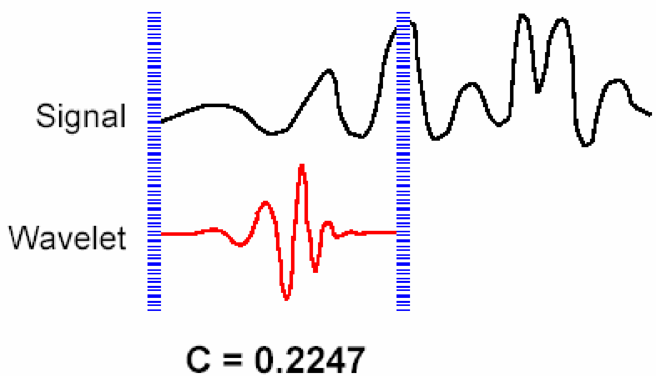


Recipe

3 Shift the wavelet to the right and repeat steps 1 and 2 until you've covered the whole signal.



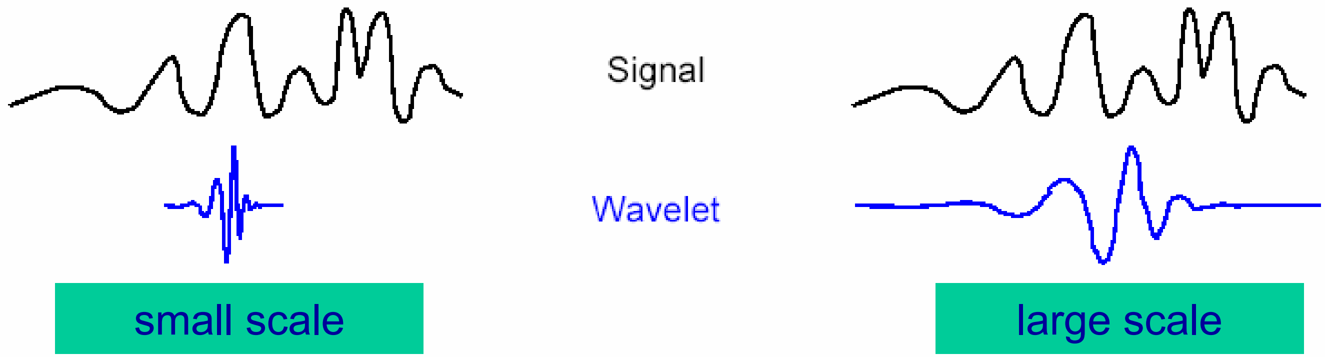
4 Scale (stretch) the wavelet and repeat steps 1 through 3.



5 Repeat steps 1 through 4 for all scales.

Wavelet Zoom

- WT at position u and scale s measures the local correlation between the signal and the wavelet



Thus, there is a correspondence between wavelet scales and frequency as revealed by wavelet analysis:

(small) • Low scale $a \Rightarrow$ Compressed wavelet $\Rightarrow$ Rapidly changing details $\Rightarrow$ High frequency ω.

(large) • High scale $a \Rightarrow$ Stretched wavelet $\Rightarrow$ Slowly changing, coarse features $\Rightarrow$ Low frequency ω.

Frequency domain

- Parseval $$Wf(u,s) = \int_{-\infty}^{+\infty} f(t)\psi^*_{u,s}(t)dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\Psi^*_{u,s}(\omega)d\omega$$

  The wavelet coefficients Wf(u,s) depend on the values of f(t) (and F(ω)) in the time-frequency region where the energy of the corresponding wavelet function (respectively, its transform) is concentrated

  - *time/frequency localization*
  - The *position and scale* of high amplitude coefficients allow to characterize the *temporal evolution* of the signal

  - Time domain signals (1D) : Temporal evolution
  - Spatial domain signals (2D) : Localize and characterize spatial singularities

$$\psi_{u,s}(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t-u}{s}\right) \Leftrightarrow \Psi_{u,s}(\omega) = e^{-j\omega u}\sqrt{s}\,\Psi(s\omega)$$

Stratching in time ↔ Shrinking in frequency (and viceversa)

Parseval & Plancherel

Theorem 2.3. If f and h are in $\mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$, then

$$\int_{-\infty}^{+\infty} f(t)\, h^*(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\omega)\, \hat{h}^*(\omega)\, d\omega. \tag{2.25}$$

For $h = f$ it follows that

$$\int_{-\infty}^{+\infty} |f(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{f}(\omega)|^2\, d\omega. \tag{2.26}$$

Proof. Let $g = f \star \bar{h}$ with $\bar{h}(t) = h^*(-t)$. The convolution, Theorem 2.2, and property (2.23) show that $\hat{g}(\omega) = \hat{f}(\omega)\, \hat{h}^*(\omega)$. The reconstruction formula (2.8) applied to $g(0)$ yields

$$\int_{-\infty}^{+\infty} f(t)\, h^*(t)\, dt = g(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{g}(\omega)\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\omega)\, \hat{h}^*(\omega)\, d\omega. \quad \blacksquare$$

by definition of convolution

inverse transform in t=0

Note to Plancherel's formula

Table 2.1 Fourier Transform Properties

Property	Function	Fourier Transform	
	$f(t)$	$\hat{f}(\omega)$	
Inverse	$\hat{f}(t)$	$2\pi f(-\omega)$	(2.15)
Convolution	$f_1 \star f_2(t)$	$\hat{f}_1(\omega)\hat{f}_2(\omega)$	(2.16)
Multiplication	$f_1(t)\, f_2(t)$	$\frac{1}{2\pi}\hat{f}_1 \star \hat{f}_2(\omega)$	(2.17)
Translation	$f(t-u)$	$e^{-iu\omega}\hat{f}(\omega)$	(2.18)
Modulation	$e^{i\xi t} f(t)$	$\hat{f}(\omega-\xi)$	(2.19)
Scaling	$f(t/s)$	$\lvert s\rvert \hat{f}(s\,\omega)$	(2.20)
Time derivatives	$f^{(p)}(t)$	$(i\omega)^p \hat{f}(\omega)$	(2.21)
Frequency derivatives	$(-it)^p f(t)$	$\hat{f}^{(p)}(\omega)$	(2.22)
Complex conjugate	$f^*(t)$	$\hat{f}^*(-\omega)$	(2.23)
Hermitian symmetry	$f(t) \in \mathbb{R}$	$\hat{f}(-\omega) = \hat{f}^*(\omega)$	(2.24)

For real signals f(t)

$$f(t) \to \hat{f}(\omega)$$

$$f(-t) \to \hat{f}(-\omega) = \hat{f}^*(\omega)$$

Proof

$$\Im\{f(-t)\} = \int_{-\infty}^{+\infty} f(-t)e^{-j\omega t}dt = \int_{-\infty}^{+\infty} f(t')e^{j\omega t'}dt' = \hat{f}(-\omega)$$

Wavelets and linear filtering

- *The WT can be rewritten as a convolution product and thus the transform can be interpreted as a linear filtering operation*

$$Wf(u,s) = \langle f, \psi_{u,s} \rangle = \int_{-\infty}^{+\infty} f(t) \frac{1}{\sqrt{s}} \psi^* \left(\frac{t-u}{s} \right) dt =$$

$$\bar{\psi}_s(t) = \frac{1}{\sqrt{s}} \psi^* \left(\frac{-t}{s} \right)$$

$$\hat{\bar{\psi}}_s(\omega) = \sqrt{s} \hat{\psi}^*(s\omega)$$

$$\hat{\psi}(0) = 0$$

$\rightarrow$ *band-pass filter*

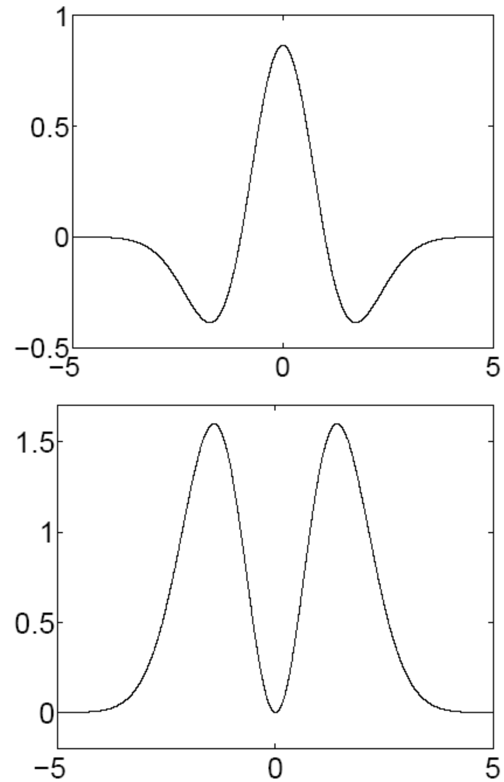
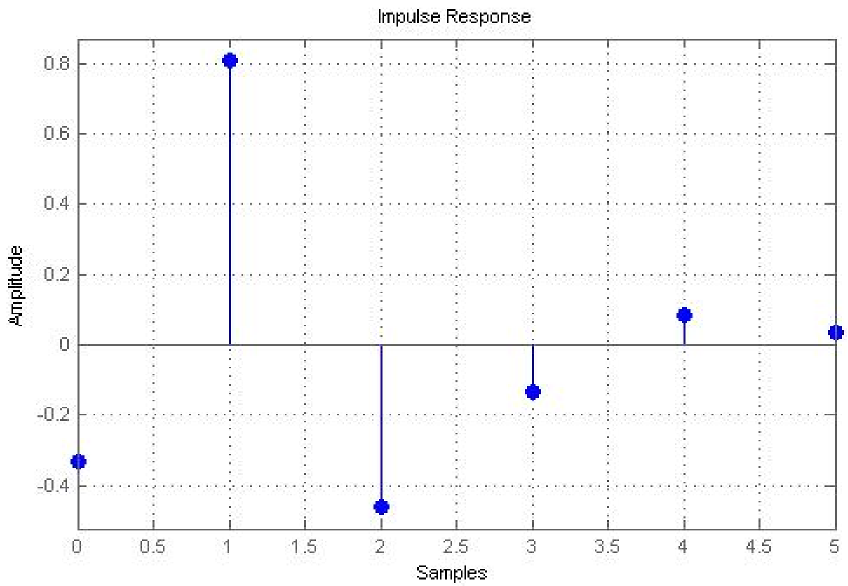
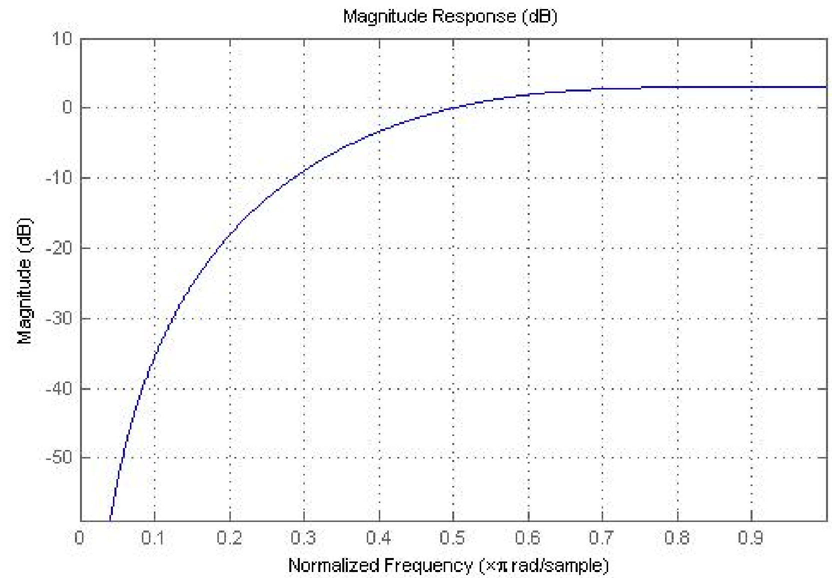


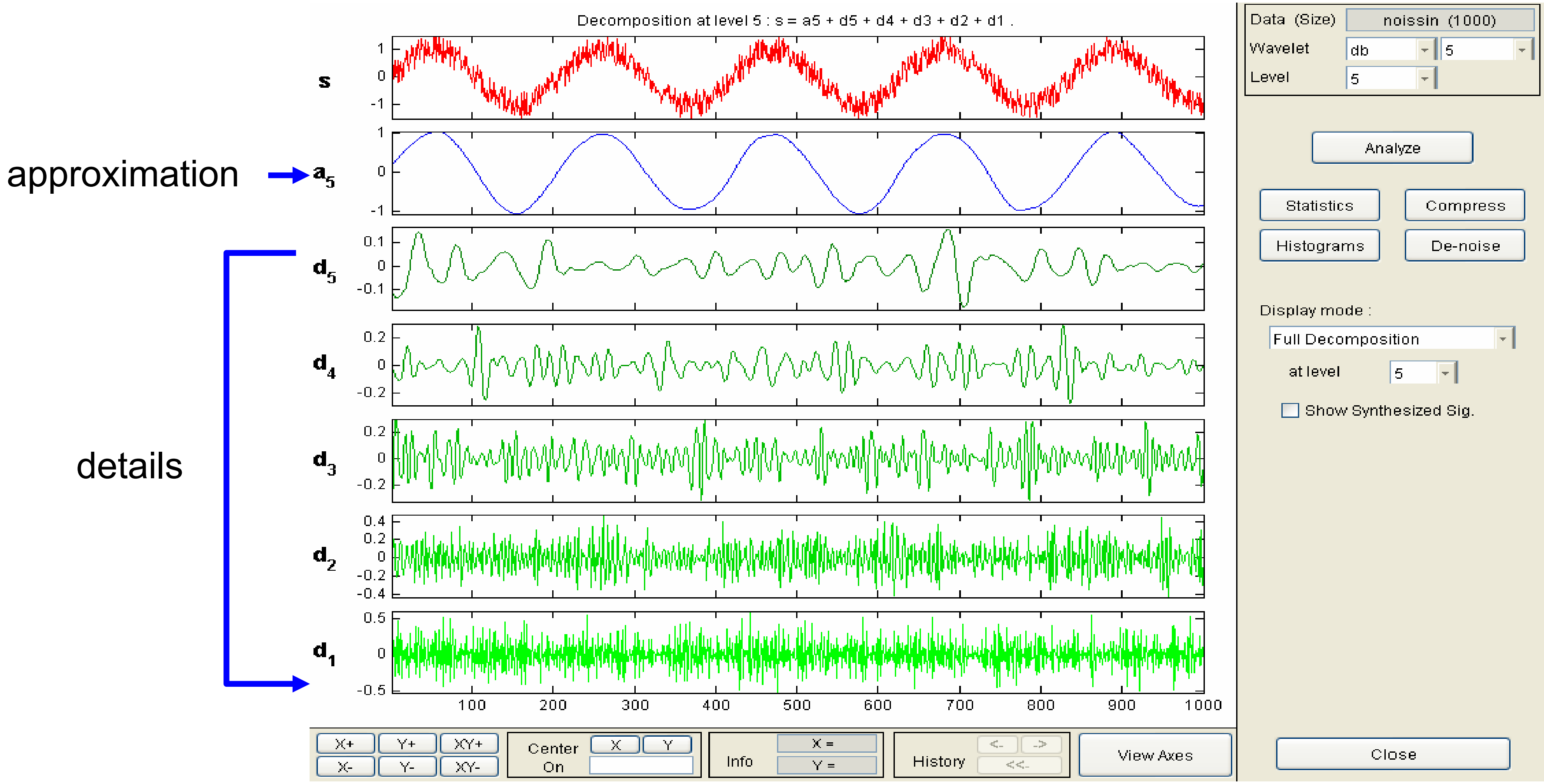
Fig. 4.6. A Wavelet Tour of Signal Processing, 3rd ed. Mexican hat wavelet for $\sigma = 1$ and its Fourier transform.

Wavelet filter (db3)





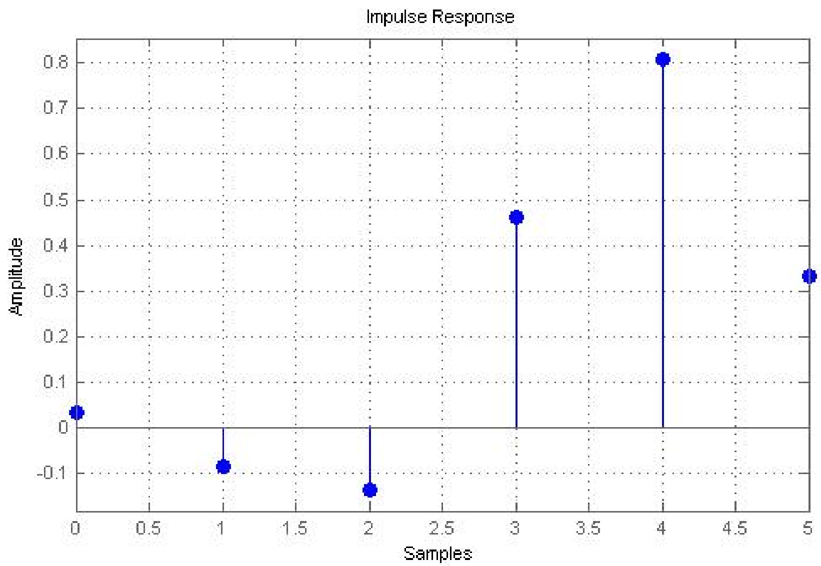
Scaling function

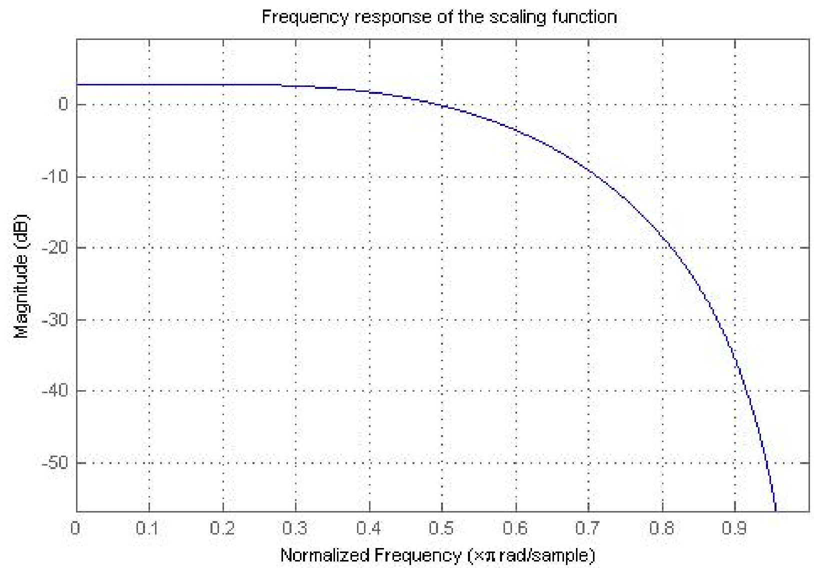


Wavelet representation = approximation + details

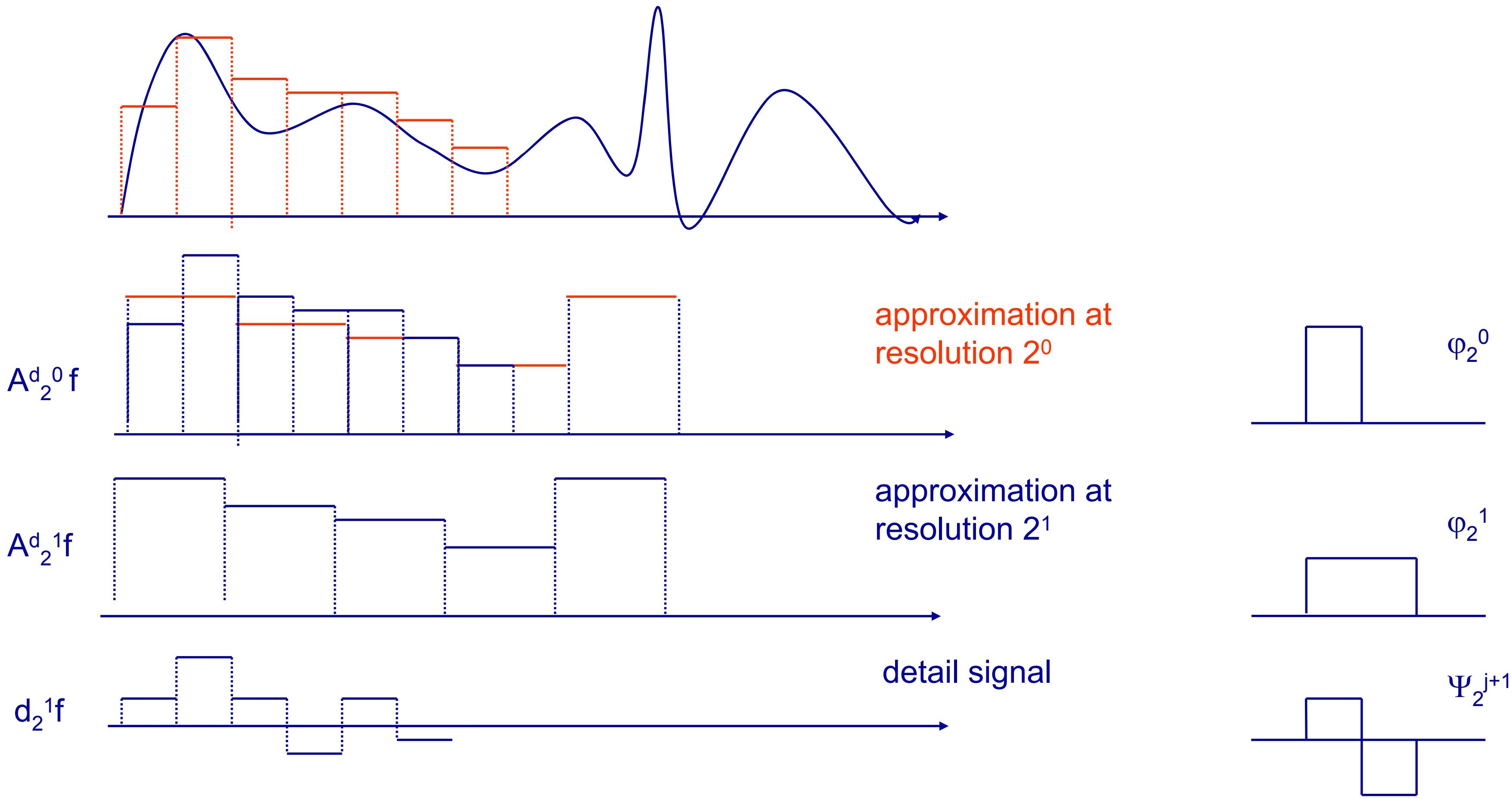
approximation ↔ scaling function
details ↔ wavelets

Scaling function (db3)





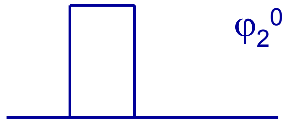
A different perspective



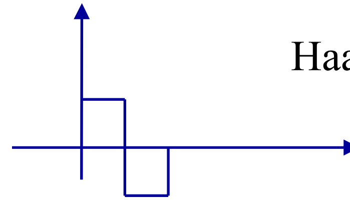
$$A^d_{2^j} f = A^d_{2^{j+1}} f + d_{2^{j+1}} f$$

Haar pyramid [Haar 1910]

Haar basis function



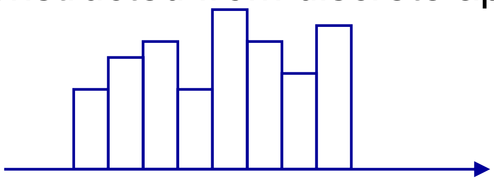
$\varphi_2^{\,0}$



Haar wavelet

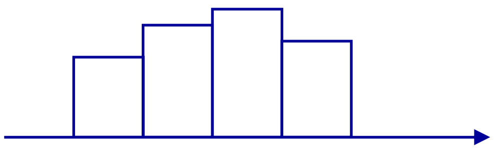
reconstructed from discrete approximations

sig_0

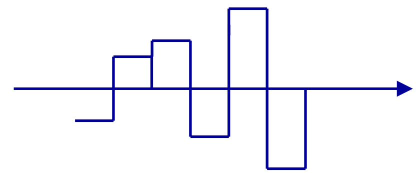


residuals from details

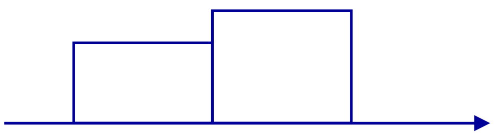
sig_1



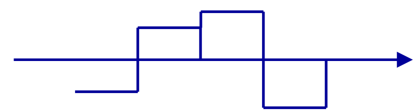
$r_1 = \Sigma_k d_1(k)\ \Psi_{k,1}$



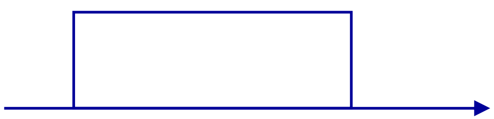
sig_2



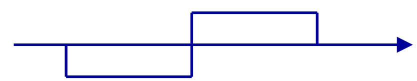
$r_2 = \Sigma_k d_2(k)\ \Psi_{k,2}$



sig_3



$r_3 = \Sigma_k d_3(k)\ \Psi_{k,3}$



$\varphi_{i,k} = 2^{-i/2}\ \varphi(x/2^i - k)$

$sig_i = \Sigma_k a_i(k)\ \varphi_{k,i}$

$s = sig_3 + \Sigma_{i,k} d_{k,i}(k)\ \Psi_{k,i} = \Sigma_k a_3(k)\ \varphi_{k,3} + \Sigma_{i,k} d_{k,i}(k)\ \Psi_{k,i}$

Hints

- Haar wavelet → piece-wise constant functions → far from optimal
- Stronberg 1980 → piece-wise linear functions → better approximation properties
- Meyer 1989 → continuously differentiable functions
- Mallat and Meyer 1989 → Theory for multiresolution signal approximation