Finitistic dimensions for commutative noetherian rings

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For the category of modules over a ring R, the homological dimension is a very coarse invariant. So to have a more accurate measure of the complexity of such categories one introduces the finitistic dimensions. The most popular ones are

$$\operatorname{findim} R = \sup\{\operatorname{pdim}(M) \mid M \text{ (strongly) finitely presented } R\text{-module of finite projective dimension}\}$$

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The first one is called the *small finitistic dimension* and the second one is the *big finitistic dimension.*

There is plenty of research done around these dimensions trying to elucidate their meaning in suitable classes of rings. For example, in the case of finite dimensional algebras it has originated the so called Finitistic Dimension Conjectures and there is still a lot to be understood.

The aim of these lectures will be to explain the classical case of commutative noetherian rings, in which both invariants are well determined in terms of other invariants of the ring.

A tentative program could be the following which is distributed in four session of two hours:

Session 1. *Small finitistic dimension for noetherian rings. Auslander-Buchsbaum formula.*

The main result we want to show is that for a local commutative noetherian ring R we have the equality

$$\operatorname{findim} R = \operatorname{depth} R$$

Here we will learn also that the prototype of finitely generated module of finite projective dimension is of the form $R/(x_1, \dots, x_r)$ where $(x_1, \dots, x_r)$ is a regular sequence. The projective resolution of such modules is given by the *Koszul complex.*

Session 2. *The big finitistic dimension is at least the Krull dimension. Localization. Completions, Artin-Rees Lemma and Krull's intersection theorem*

Bass proved, in a quite explicit way, how suitable localizations produce infinitely generated modules of finite projective dimension equal to the Krull dimension. So we will also need to understand well the localization of a modules at a multiplicative set.

In order to get prepared for the next topic we will have an overview of the construction of the completion of a local noetherian ring.

Session 3. *The big finitistic dimension equals de Krull dimension: Mittag-Leffler conditions*

Here we will show that the big finitistic dimension of a commutative noetherian ring is at most the Krull dimension. So combined with the results from section 2 we will conclude that the big finitistic dimension coincides with the Krull dimension. Our aim here is to follow the original proof by Raynaud and Gruson.

Session 4. *The big finitistic dimension equals de Krull dimension: functor categories*

Gruson and Jensen gave an alternative and (apparently) simpler proof of the previous inequality interpreting the Mittag-Leffler conditions in terms of functor categories. We will do our best in explaining also this approach.