2.9 Exercises - Part 1

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Exercise 1. (a) Let ${}_RM$ be a R-module and ${}_RR$ the regular module. Show that the abelian group $\mathrm{Hom}_R(R, M)$ is a left R-module and that the map

$$\varphi : \mathrm{Hom}_R(R, M) \to M,\ f \mapsto f(1)$$

is an isomorphism of R-modules. *(3 points)*

(b) Let $f \in \mathrm{Hom}_R(M, N)$ be a homomorphism of R-modules. Show that f is a monomorphism if and only if $fg = 0$ implies $g = 0$ for any $g \in \mathrm{Hom}_R(L, M)$. Show f is an epimorphism if and only if $gf = 0$ implies $g = 0$ for any $g \in \mathrm{Hom}_R(N, L)$. *(4 points)*

Exercise 2. (a) Let ${}_RL, {}_RN \leq {}_RM$. Show that M is the direct sum of L and N if and only if $L + N = M$ and $L \cap N = 0$. Does the same hold true for more than two summands? *(4 points)*

(b) Given $f \in \mathrm{Hom}_R(L, M)$ and $g \in \mathrm{Hom}_R(M, L)$ such that $gf = \mathrm{id}_L$, show that $M = \mathrm{Im}\, f \oplus \ker g$. *(3 points)*

Exercise 3. Given a field k, consider the ring $R = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix} = \{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in k\}$.

(a) Show that $P_1 = \{\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in k\}$ and $P_2 = \{\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} \mid c \in k\}$ are left ideals of ${}_RR$ and that $I_1 = \{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in k\}$ and $I_2 = \{\begin{pmatrix} 0 & 0 \\ b & c \end{pmatrix} \mid b, c \in k\}$ are right ideals of R_R. *(4 points)*

(b) Recall that R is isomorphic to the path algebra of the quiver $\mathbb{A}_2 \colon \underset{1}{\bullet} \xrightarrow{\alpha} \underset{2}{\bullet}$. Find representations of $\mathbb{A}_2$ corresponding to P_1 and P_2 under the isomorphism $k\mathbb{A}_2 \cong R$. *(4 points)*

Exercise 4. (a) Let $\varphi : S \to R$ a ring homomorphism. Show that any left R-module M is also a left S-module via the map $S \times M \to M$, $(s, m) \mapsto \varphi(s)m$. *(4 points)*

(b) Let ${}_RM$ and define $\mathrm{Ann}_R(M) = \{r \in R \mid rm = 0 \text{ for any } m \in M\}$. M is called *faithful* if $\mathrm{Ann}_R(M) = 0$. Check that $\mathrm{Ann}_R(M)$ is a two-sided ideal of R, and set $S = R/\mathrm{Ann}_R(M)$. Verify that M has a natural structure of S-module, given by the map $S \times M \to M$, $(\overline{r}, m) \mapsto rm$. Show that M is a faithful S-module. *(4 points)*