GEOMETRIA - Prova scritta del 14/7/2010

(Prof. M. Spera)

(1) Data la curva piana $\mathcal{C}: \begin{cases} x = t \\ y = t^3 \end{cases}$ $\quad t \neq 0$

(è sottinteso l'uso di un riferimento cartesiano)

se ne calcoli in due modi l'evoluta $\mathcal{E}$ (in forma parametrica). Determinare in seguito la lunghezza dell'arco di evoluta tra i punti P, Q corrispondenti ai valori $t = 1$, $t = 2$ rispettivamente.

Per $t \sim 0$ si abbozzi il grafico di $\mathcal{E}$.

(2) Si consideri la superficie Σ data da

$$\underline{r}(u,v) = (u(1-v),\ u^3(1-v),\ v) \qquad (v < 1)$$

Di che tipo di superficie si tratta? Se ne determini (la prima e la seconda forma fondamentale, nonché) la curvatura gaussiana. Considerata la curva

$$\mathcal{C}: \quad \underline{r} = \underline{r}(t) = (t,\ t^3,\ 0),\ t > 0 \quad (\mathcal{C} \subset \Sigma)$$

si calcoli $\dfrac{\nabla}{dt}\left[\underbrace{\dfrac{\dot{r}(t)}{\|\dot{r}(t)\|}}_{\underline{t}(t)}\right]$ $\qquad \mathcal{C}$ è una geodetica di Σ?

(3) Data la superficie Σ dell'es. (2), dimostrare che l'angolo ottenuto dal trasporto parallelo di un vettore qualsiasi lungo un arbitrario circuito chiuso γ su Σ tale che $\gamma = \partial D$ (D dominio $\subset \Sigma$) è nullo.

Tempo a disposizione: 2h $\qquad$ Le risposte vanno adeguatamente giustificate.

Geometria 14/7/10

(1) $\mathcal{C}: \quad y = x^3$ det. l'evoluta in due modi

$$\mathcal{C}: \begin{cases} x = t \\ y = t^3 \end{cases} \qquad t \neq 0$$

evoluta: inviluppo delle normali

$$\dot{x}(t)(x - x(t)) + \dot{y}(t)(y - y(t)) = 0$$

$$\dot{x} = 1 \qquad \dot{y} = 3t^2$$

$F(x,y,t) = 0 \qquad x - t + 3t^2(y - t^3) = 0$

$$\boxed{x + 3t^2 y - t - 3t^5 = 0} \quad \text{normali}$$

$\dfrac{\partial F}{\partial t} = 0 \qquad 6ty - 1 - 15t^4 = 0$

$$y = \frac{1}{6t}(1 + 15t^4) = \frac{1}{6t} + \frac{15}{6}t^3$$

$$= \frac{1}{6t} + \frac{5}{2}t^3$$

$$x + 3t^2 \frac{1}{6t}(1 + 15t^4) - t - 3t^5 = 0$$

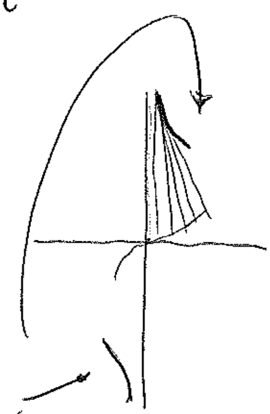
$$x + \frac{1}{2}t(1 + 15t^4) - t - 3t^5 = 0$$

$$x + \frac{t}{2} + \frac{15}{2}t^5 - t - 3t^5 = 0$$

$$x - \frac{t}{2} + \frac{9}{2}t^5 = 0 \qquad x = \frac{t}{2} - \frac{9}{2}t^5$$

Evoluta: $\begin{cases} x = \dfrac{t}{2} - \dfrac{9}{2}t^5 \\ y = \dfrac{1}{6t} + \dfrac{5}{2}t^3 \end{cases}$

Se $t \sim 0$ $\qquad xy = \dfrac{1}{12}$



evoluta (appr)

Altro metodo

$$Q = P + \frac{\|\dot{P}\|^2}{\langle i\dot{P}, \ddot{P}\rangle}\, i\,\dot{P}$$

(formalismo misto)

$P : (t, t^3) \qquad i\dot{P} = (-3t^2, 1)$

$\dot{P} : (1, 3t^2)$

$\ddot{P} : (0, 6t) \qquad \|\dot{P}\|^2 = 1 + 9t^4$

$\langle i\dot{P}, \ddot{P}\rangle = 6t$

$$Q \downarrow \begin{cases} x = t + \dfrac{1+9t^4}{6t}(-3t^2) = t - \dfrac{1}{2}t(1+9t^4) = \underline{\dfrac{t}{2} - \dfrac{9}{2}t^5} \quad \checkmark \\ y = t^3 + \dfrac{1+9t^4}{6t} \cdot 1 \end{cases}$$

$$= t^3 + \frac{1}{6t} + \frac{3}{2}t^3 = \underline{\frac{1}{6t} + \frac{5}{2}t^3} \quad \checkmark$$

Lunghezza $\widehat{PQ}$ $\quad P \leftrightarrow t=1$

$Q \leftrightarrow t=2$

$$|k| = \frac{\|\dot{P}\|^3}{|\langle i\dot{P}, \ddot{P}\rangle|} = \frac{(1+9t^4)^{3/2}}{6|t|}$$

$$|\rho| = \frac{1}{|k|} = \frac{6|t|}{(1+9t^4)^{3/2}}$$

$t=1 \quad |\rho_1| = \dfrac{6}{10^{3/2}}$

$t=2 \quad |\rho_2| = \dfrac{12}{(1+9\cdot 16)^{3/2}}$ $\quad \underbrace{}_{=144}$

$$|\rho_2| = \frac{12}{145^{3/2}}$$

$$l = \frac{6}{10^{3/2}} - \frac{12}{145^{3/2}} = \frac{29^{3/2}\cdot 6 - 2^{3/2}\cdot 12}{290^{3/2}}$$

$$\left[\sim \quad \frac{6}{27} - \frac{12}{12^3} = \frac{2}{3^2} - \frac{1}{3^2\cdot 4^2} = \frac{\overset{16}{4^2}\cdot 2 - 1}{3^2\cdot 4^2} = \frac{31}{144} \right]$$

(2) $\mathscr{C} : \begin{cases} x = u \\ y = u^3 \\ z = 0 \end{cases}$

$V : (0, 0, 1) \qquad \underline{w} = (-u, -u^3, 1)$

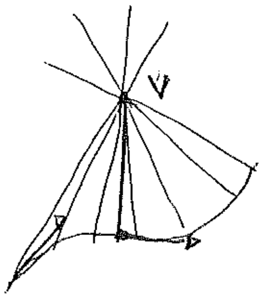
$$\underline{r}(u, v) = \underline{r}_{\mathscr{C}} + v\,\underline{w}$$

$$= \begin{pmatrix} u \\ u^3 \\ 0 \end{pmatrix} + v \begin{pmatrix} -u \\ -u^3 \\ 1 \end{pmatrix}$$

$$\underline{r}(u,v) = \left(u(1-v), \; u^3(1-v), \; v \right)$$

Cono generalizzato
(sviluppabile)

$\Rightarrow$ direttamente, $K = 0$



$$\underline{r}_u = (1-v, \; 3u^2(1-v), \; 0)$$
$$\underline{r}_v = (-u, \; -u^3, \; 1)$$

$$\underline{N} = \frac{1}{\sqrt{\;}} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1-v & 3u^2(1-v) & 0 \\ -u & -u^3 & 1 \end{vmatrix} = \underline{i}\; 3u^2(1-v) - \underline{j}\,(1-v) + \underline{k}\left[-u^3(1-v) + 3u^3(1-v)\right] =$$

$$= \frac{1}{\sqrt{\;}} \left(3u^2(1-v)\,\underline{i} - (1-v)\,\underline{j} + 2u^3(1-v)\,\underline{k} \right)$$

$$\sqrt{} = |1-v| \sqrt{9u^4+4u^6+1}$$

$$\underline{N} = \underbrace{\frac{1-v}{|1-v|}}_{\substack{\| \\ +1 \text{ se } v \leq 1}} \frac{1}{\sqrt{}} (3u^2, -1, 2u^3) \longrightarrow$$

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② $\mathscr{C}: \begin{cases} x = t \\ y = t^3 \\ z = 0 \end{cases}$ $\qquad \underline{r}_{\mathscr{C}}(t) = (t, t^3, 0)$

$$\dot{\underline{r}}(t) = (1, 3t^2, 0)$$

$$\ddot{\underline{r}}_{\mathscr{C}}(t) = (0, 6t, 0)$$

$$\underline{t} = \frac{\dot{\underline{r}}}{\|\dot{\underline{r}}\|} = \frac{1}{\sqrt{1+9t^4}} (1, 3t^2, 0)$$

$$\frac{\nabla}{dt} \underline{t} = \underbrace{\left(\frac{ds}{dt}\right)}_{\substack{\| \\ \|\dot{r}\|}} \frac{\nabla}{ds} \underline{t} = \sqrt{1+9t^4} \left(\frac{\nabla}{ds} \underline{r}''\right)$$

$$= \sqrt{1+9t^4} \left(\kappa_g \cdot \underline{N} \times \underline{t}\right)$$

dimostrato a lezione

$$\kappa_g = \langle \underline{N}, \underline{b} \rangle \kappa$$

$$\kappa = \frac{\|\dot{\underline{r}} \times \ddot{\underline{r}}\|}{\|\dot{\underline{r}}\|^3} \qquad \dot{\underline{r}} \times \ddot{\underline{r}} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 3t^2 & 0 \\ 0 & 6t & 0 \end{vmatrix} = \underline{k} \cdot 6t$$

$$\|\dot{\underline{r}} \times \ddot{\underline{r}}\| = 6|t|$$

$$\kappa = \kappa(t) = \frac{6|t|}{(1+9t^4)^{3/2}}$$

Prima e seconda forma fondamentale

$$E = \|r_u\|^2 = (1-v)^2 [1 + 9u^4]$$

$$F = \langle r_u, r_v \rangle = -u(1-v) - u^3 \cdot 3u^2 (1-v)$$
$$= (1-v)\{-u - 3u^5\} = -(1-v)u(1+3u^4)$$
$$= (v-1)u(1+3u^4)$$

$$G = \|r_v\|^2 = 1 + u^2 + u^6$$

$$\underline{r}_{uu} = (0, 6u(1-v), 0)$$
$$\underline{r}_{uv} = (-1, -3u^2, 0)$$
$$\underline{r}_{vv} = = (0, 0, 0)$$

$$\underline{N} = \frac{1}{\sqrt{9u^4 + 4u^6 + 1}} (3u^2, -1, 2u^3)$$

$$e = \langle \underline{N}, \underline{r}_{uu} \rangle = \frac{1}{\sqrt{\ }} \cdot 6u(v-1)$$

$$f = \frac{1}{\sqrt{\ }} (-3u^2 + 3u^2 + 0) = 0$$

$$g = 0 \Rightarrow K = \frac{e\overset{0}{g} - \underset{=0}{f}^2}{EG - F^2} = 0$$

come già trovato.

$$\underline{b} = \frac{\dot{\underline{r}} \times \ddot{\underline{r}}}{\|\dot{\underline{r}} \times \ddot{\underline{r}}\|} = \frac{6t \cdot \underline{k}}{6|t|} = \frac{t}{|t|}\underline{k}$$ (Chiaro a priori...)

$$= \underline{k} \quad \text{per } t > 0$$

$$\langle \underline{b}, \underline{N} \rangle = \frac{1}{\sqrt{9t^4+4t^6+1}} \cdot 2t^3 \cdot 1 = \frac{2t^3}{\sqrt{9t^4+4t^6+1}}$$

Dunque
$$k_g = \overbrace{\frac{2t^3}{\sqrt{9t^4+4t^6+1}}}^{\langle b, N\rangle} \cdot \frac{\overbrace{6t}^{k}}{(1+9t^4)^{3/2}} \overset{t>0}{=} \frac{12t^4}{\sqrt{\blacklozenge}\,(1+9t^4)^{3/2}}$$

$\underline{N} \times t =$

$$\frac{1}{\sqrt{1+9t^4}} \cdot \frac{1}{\sqrt{1+9t^4+4t^6}} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3t^2 & -1 & 2t^3 \\ 1 & 3t^2 & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{\circ}} \frac{1}{\sqrt{\blacklozenge}} \left[\underline{i}\,(-6t^5) - \underline{j}\,(-2t^3) + \underline{k}\,(9t^4+1) \right]$$

$$= \frac{1}{\sqrt{\circ}} \frac{1}{\sqrt{\blacklozenge}} \left[-6t^5\,\underline{i} + 2t^3\,\underline{j} + (9t^4+1)\,\underline{k} \right]$$

In definitiva

$$\frac{\nabla}{dt}\underline{t} = \overbrace{\sqrt{1+9t^4}}^{\sqrt{\circ}} \cdot k_g \cdot \underline{N} \times \underline{t}$$

$$= \cancel{\sqrt{\circ}} \cdot \overbrace{\left(\frac{1}{\cancel{\sqrt{\circ}}} \frac{12t^4}{(1+9t^4)^{3/2}} \right)}^{k_g} \cdot \overbrace{\frac{1}{\sqrt{\circ}} \frac{1}{\sqrt{\blacklozenge}} \left[-6t^5\,\underline{i} + 2t^3\,\underline{j} + (9t^4+1)\,\underline{k} \right]}^{\underline{N} \times \underline{t}}$$

$$= \frac{1}{(1+9t^4)} \frac{12t^4}{\sqrt{9t^4+4t^6+1}} \left[-6t^5\,\underline{i} + 2t^3\,\underline{j} + (9t^4+1)\,\underline{k} \right]$$

controllo : $\underline{\tau} \perp \underline{N}$?

$$-6t^5 \cdot 3t^2 + 2t^3 \cdot (-1) + (9t^4+1) \cdot 2t^3$$

$$= -18t^7 - 2t^3 + 18t^7 + 2t^3 = 0 \quad \checkmark$$

È chiaro che $\mathcal{C}$ non è una geodetica

③ In base al teorema di Levi-Civita,
e al fatto che $K \equiv 0$, il trasporto
parallelo su un circuito chiuso γ, γ
frontiera di Ω, è nullo:

In dettaglio: $\alpha = \int_{\Omega} K = 0$