ANALISI MATEMATICA I - Bioinformatica a.a. 2007/08

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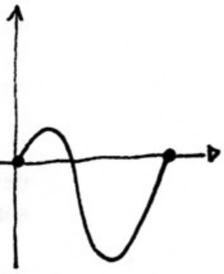
Prologo

(1) Dominio e grafico di $f(x) = \log x$.

Quanto vale $\lim_{x \to 0^+} \log x$

(2) Esiste $\lim_{x \to +\infty} \sin x$? Motivare la risposta

(3) Qual è il segno dell'integrale definito della funzione in figura?



(4) Dominio di $f(x) = (\log x)^e$

(5) Derivata della f del pto 4.

(6) Calcolare $\int \sin(x^2)\, x \, dx$

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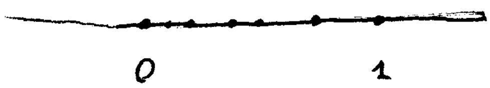
(E1) Per quali valori di $\alpha \in \mathbb{R}$ esiste finito e $\neq 0$

$$L_\alpha = \lim_{x \to 0^+} \frac{[\log(1+x)]^2 - [\sin x]^2}{x^\alpha}$$

→

E2 Sia $\{a_n\}$, $n \geq 0$, con $a_0 = 1$ e $a_n \searrow 0$



Calcolare $$\sum_{n=0}^{\infty} \log\left(\frac{1+a_n}{1+a_{n+1}}\right)$$

E3 Calcolare $$\int x \arctan x \, dx$$

E4 Risolvere il problema di Cauchy

$$\begin{cases} y'' + y' + y = \sin t \\ y(0) = 0 \\ y'(0) = 1 \end{cases} \qquad y = y(t)$$

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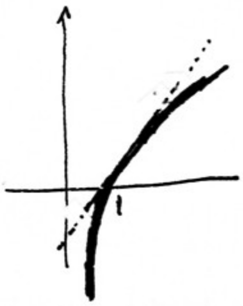
T1 a) Enunciare e dimostrare il teorema di Lagrange

b) calcolare $$\lim_{x \to +\infty} \frac{\log(x + \sqrt{x}) - \log x}{\sqrt{x}}$$

T2 Enunciare e dimostrare il teorema di Fermat

Prologo

(1)



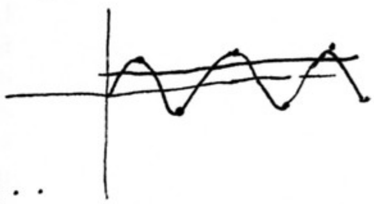
$x > 0$

$$\lim_{x \to 0^+} \log x = -\infty$$

(2) NO $\sin$, al tendere di x a $+\infty$ oscilla tra -1 e 1

In modo più preciso, esistono, per. sotto successioni tendenti a ± 1, rispettivamente [anzi, ad un qualsiasi valore compreso tra -1 e 1]

$$x_k := \frac{\pi}{2} + 2k\pi \qquad k = 0, 1, 2 \dots$$



$$\sin x_k \equiv 1$$

$$x'_k := \frac{3}{2}\pi + 2k\pi \qquad k = 0, 1, 2 \dots$$

$$\sin x'_k \equiv -1$$

(3) È negativo [prop. dell'int. def...]

(4) deve essere $\log x$ definito $\Rightarrow x > 0$

e inoltre $\log x > 0 \Rightarrow x > 1$;

dunque $x > 1$; $f(x) = e^{e \log(\log x)}$ anche direttamente

(5) $f'(x) = f(x)\, e \dfrac{1}{\log x} \dfrac{1}{x} = e (\log x)^{e-1} \cdot \dfrac{1}{x}$

(6) $\int \sin(x^2)\, x\, dx = \int \sin(x^2)\, d\frac{x^2}{2} = \frac{1}{2} \int \sin(x^2)\, dx^2$

$= -\frac{1}{2} \cos(x^2) + c$

$$\lim_{x \to 0^+} \frac{[\log(1+x)] - [\sin x]}{x^\alpha}$$

$$[\log(1+x)]^2 - [\sin x]^2 =$$

$$= [\log(1+x) + \sin x] \cdot [\log(1+x) - \sin x]$$

$$= \left[x - \frac{x^2}{2} + x + o(x^2)\right]\left[\cancel{x} - \frac{x^2}{2} - \cancel{x} + o(x^2)\right]$$

$$\left[2x - \frac{x^2}{2} + o(x^2)\right]\left[-\frac{x^2}{2} + o(x^2)\right]$$

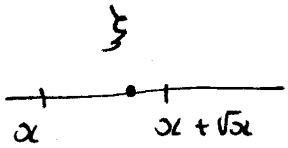
$$= -x^3 + o(x^3)$$

$\Rightarrow$ L_α esiste finito e non nullo

per $\alpha = 3$ e $L_3 = -1$

T1 b)

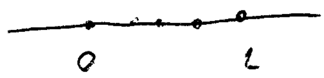
$$\frac{\log(x+\sqrt{x}) - \log x}{\sqrt{x}} = \text{(Lagrange)}$$



$$\frac{1}{\xi} \cdot \frac{(\cancel{x} + \sqrt{x} - \cancel{x})}{\sqrt{x}} = \frac{1}{\xi} \to 0 \quad \text{se } x \to +\infty$$

$(\xi \to +\infty$ se $x \to +\infty)$

(2) $$\sum_{n=0}^{\infty} \log\left(\frac{1+a_n}{1+a_{n+1}}\right) \qquad a_0 = 1, \quad a_n \searrow 0$$



$$\Big\| $$

$$\lim_{N\to\infty} \sum_{n=0}^{N} \qquad \sum_{n=0}^{N} \log\left(\frac{1+a_n}{1+a_{n+1}}\right) =$$

$$= \sum_{n=0}^{N} \left[\log(1+a_n) - \log(1+a_{n+1})\right]$$

(telescopica)

$$= \log(1+a_0) - \cancel{\log(1+a_1)} + \cancel{\log(1+a_1)} - \cancel{\log(1+a_2)} + \dots - \log(1+a_{N+1})$$

$$= \log 2 - \log(1+\underbrace{a_{N+1}}_{\to 0 \quad N\to\infty})$$

$$\Rightarrow \sum_{n=0}^{\infty} \cdot \; = \log 2$$

③

$$\int \overbrace{x}^{g'} \, \overbrace{\arctan x}^{f} \, dx = (p.p.)$$

$$= \overbrace{\arctan x}^{f} \, \overbrace{\frac{x^2}{2}}^{g} - \int \overbrace{\frac{x^2}{2}}^{g} \, \overbrace{\frac{1}{1+x^2}}^{f'} \, dx$$

$$= \frac{1}{2} x^2 \arctan x - \frac{1}{2} \int \frac{x^2}{1+x^2} \, dx$$

$$= \frac{1}{2} \left[x^2 \arctan x - \int \frac{x^2+1-1}{1+x^2} \, dx \right]$$

$$= \frac{1}{2} \left[x^2 \arctan x - \int 1 \, dx + \int \frac{1}{1+x^2} \, dx \right]$$

$$= \frac{1}{2} \left[x^2 \arctan x - x + \arctan x + c \right]$$

$$= \frac{1}{2} \left[(1+x^2) \arctan x - x \right] + c_1 \quad \text{nuova cost.}$$

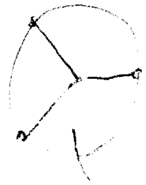
verifica: deriviamo $\int$ si ha:

$$\frac{1}{2} \left[2x \arctan x + \overbrace{(1+x^2) \frac{1}{1+x^2}}^{+1} - 1 \right]$$

$$= x \arctan x \quad \checkmark$$

(4)

$$\begin{cases} y'' + y' + y = \sin t \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$



$$\lambda^2 + \lambda + 1 = 0 \qquad \lambda = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$(\lambda^3 - 1 = (\lambda^2+\lambda+1)(\lambda-1)) \qquad = \frac{-1 \pm i\sqrt{3}}{2}$$

$$y(t) = \alpha\, e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2} t\right) + \beta\, e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}}{2} t\right)$$

sol. gen. equazione omogenea $y'' + y' + y = 0$

* cerchiamo una sol della forma $y = A \cos t + B \sin t$

si ha: $y' = -A \sin t + B \cos t$

$y'' = -A \cos t - B \sin t$

$\Rightarrow \quad -A \sin t + B \cos t = \sin t$

$(-A-1) \sin t + B \cos t = 0 \quad \Rightarrow \quad A = -1, \; B = 0$

$\Rightarrow \quad y = -\cos t$

(controllo: $y' = +\sin t \quad y'' = \cos t \quad y'' + y' + y = \sin t$)

$$y(t) = \alpha\, e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) + \beta\, e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$- \cos t$$

$$y'(t) = \alpha\left[-\frac{1}{2}e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) + e^{-t/2}\left[-\sin\left(\frac{\sqrt{3}}{2}t\right)\right]\cdot\frac{\sqrt{3}}{2}\right.$$

$$+ \beta\left[-\frac{1}{2}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right) + e^{-t/2}\cos\frac{\sqrt{3}}{2}t\cdot\frac{\sqrt{3}}{2}\right]$$

$$+ \sin t$$

$$y(0) = 0 \quad \Rightarrow \quad \alpha - 1 = 0 \quad \Rightarrow \alpha = 1$$

$$y'(0) = 1 \quad \Rightarrow \quad \alpha\left[-\frac{1}{2}\right] + \beta\left[\frac{\sqrt{3}}{2}\right] + 0 = 1$$

$$-\frac{\alpha}{2} + \frac{\sqrt{3}}{2}\beta = 1$$

$$-\frac{1}{2} + \frac{\sqrt{3}}{2}\beta = 1$$

$$\frac{\sqrt{3}}{\cancel{2}}\beta = \frac{3}{\cancel{2}} \qquad \beta = \sqrt{3}$$

in definitiva:

$$y(t) = -\cos t + e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) + \sqrt{3}\, e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$$