Appunti del corso di GEOMETRIA
(Prof. M. Spera)

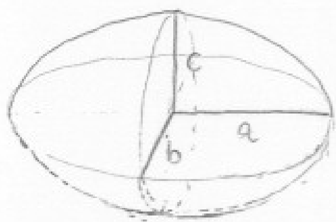
Lezione XII

ESEMPI, ESERCIZI e COMPLEMENTI VARI

◇ Sulle quadriche (sup. algebriche del 2° ordine)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (a \geq b \geq c > 0)$$

✯ ellissoide



coordinate ellissoidali

$$\begin{cases} x = a \cos u \cos v & u \in [0,\pi] \\ y = b \cos u \sin v & v \in [0,2\pi) \\ z = c \sin u \end{cases}$$

pti ellittici

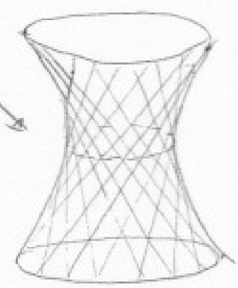
$K > 0$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

✯ iperboloide ad una falda (iperboloide iperbolico)

(o rigato)

ellisse di gola (è la linea di stringimento)



$$\begin{cases} x = a \cosh u \cos v & u \in \mathbb{R} \\ y = b \cosh u \sin v & v \in [0,2\pi) \\ z = c \sinh u \end{cases}$$

pti iperbolici $K < 0$

l'iperboloide ad una falda è doppiamente rigato;
le generatrici appartenenti ad una stessa schiera sono sghembe;
una generatrice qualsiasi incontra invece tutte quelle dell'altra schiera. Descrizione:

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2}$$

$$\left(\frac{y}{b} + \frac{z}{c}\right)\left(\frac{y}{b} - \frac{z}{c}\right) = \left(1 + \frac{x}{a}\right)\left(1 - \frac{x}{a}\right)$$

che si ottiene da

$$\begin{cases} 1 + \frac{x}{a} = t\left(\frac{y}{b} + \frac{z}{c}\right) \\ 1 - \frac{x}{a} = \frac{1}{t}\left(\frac{y}{b} - \frac{z}{c}\right) \end{cases}$$

fasci di piani

1ª schiera

eliminando t $(t \neq 0 ..)$

oppure da

$$\begin{cases} 1 + \frac{x}{a} = t'\left(\frac{y}{b} - \frac{z}{c}\right) \\ 1 - \frac{x}{a} = \frac{1}{t'}\left(\frac{y}{b} + \frac{z}{c}\right) \end{cases}$$

2ª schiera

eliminando t'

assi del fascio:

$t = 0$ $\quad 1 + \frac{x}{a} = 0$ $\quad \begin{cases} 1 + \frac{x}{a} = 0 \\ \frac{y}{b} + \frac{z}{c} = 0 \end{cases}$ è una retta della 2ª schiera $(t' = 0,\ t' = \infty)$

$t = \infty$ $\quad \frac{y}{b} + \frac{z}{c} = 0$

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

iperboloide ellittico

(o a due falde)

oppure $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

$$\begin{cases} x = a \sinh u \cos v & u \in \mathbb{R} \\ y = b \sinh u \sin v & v \in (0, 2\pi) \\ z = c \cosh u \end{cases}$$

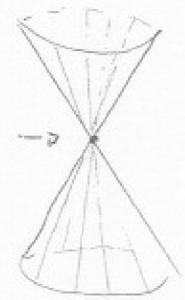
$K > 0$

pti ellittici

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

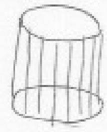
cono

linea di stringimento ridotta ad un punto



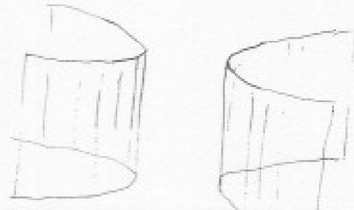
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

cilindro ellittico



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

cilindro iperbolico



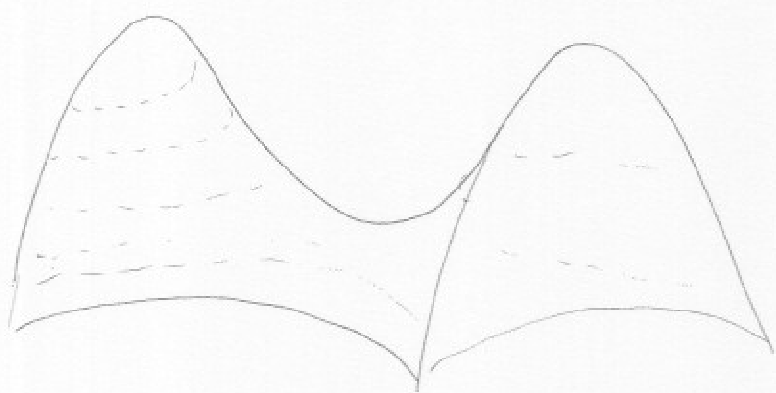
$$2z = \frac{x^2}{p} + \frac{y^2}{q} \qquad p, q > 0$$

paraboloide ellittico

$$2z = \frac{x^2}{p} - \frac{y^2}{q}$$

paraboloide iperbolico

(o a sella)



è una superficie rigata

(doppiamente)

$$\begin{cases} \dfrac{x}{\sqrt{p}} + \dfrac{y}{\sqrt{q}} = 2tz \\[2ex] \dfrac{x}{\sqrt{p}} - \dfrac{y}{\sqrt{q}} = \dfrac{1}{t} \end{cases}$$

direzioni appartenenti ad una stessa giacitura

sono // a $\dfrac{x}{\sqrt{p}} - \dfrac{y}{\sqrt{q}} = 0$

$$\begin{cases} \dfrac{x}{\sqrt{p}} + \dfrac{y}{\sqrt{q}} = \dfrac{1}{t'} \\[2ex] \dfrac{x}{\sqrt{p}} - \dfrac{y}{\sqrt{q}} = 2t'z \end{cases}$$

sono // a $\dfrac{x}{\sqrt{p}} + \dfrac{y}{\sqrt{q}} = 0$

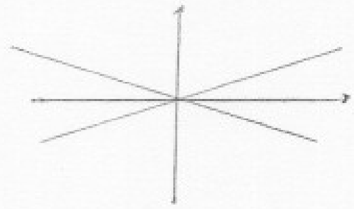
☆ Curve asintotiche

$$\boxed{e\, u'^2 + 2f\, u'v' + g\, v'^2 = 0}$$

(nel piano dei parametri..)

$$(u'\ v')\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix} = 0$$

pto per punto danno le direzioni asintotiche ($k_n = 0$)
cf. Meusnier



☆ Linee di curvatura

Rodrigues: $\quad dN = \lambda\, \alpha'(t) \qquad N' = \lambda \alpha'$

$S = -dN$

$$m_{\mathcal{B}\mathcal{B}}(dN) = \begin{pmatrix} \underbrace{\dfrac{fF - eG}{EG - F^2}}_{a_{11}} & \underbrace{\dfrac{gF - fG}{EG - F^2}}_{a_{12}} \\ \underbrace{\dfrac{eF - fE}{EG - F^2}}_{a_{21}} & \underbrace{\dfrac{fF - gE}{EG - F^2}}_{a_{22}} \end{pmatrix}$$

$\mathcal{B} = (r_u, r_v)$

* Weingarten

$$\boxed{(fE - eF)(u')^2 + (gE - eG)\, u'v' + (gF - fG)(v')^2 = 0}$$

in forma compatta

$$\begin{vmatrix} v'^2 & -u'v' & u'^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0$$

in un intorno di un pto non ombelicale, le curve C_u e C_v sono linee di curvatura $\Leftrightarrow$ $F = f = 0$

(risulta $u'v' = 0 \Rightarrow \begin{cases} u = u_0 \\ v = v \end{cases} \quad \begin{cases} u = u \\ v = v_0 \end{cases}$

Se $F = 0$, le formule di Weingarten divengono

$$m_{BB}(dN) = \begin{pmatrix} -\dfrac{e\cancel{G}}{E\cancel{G}} & -\dfrac{f\cancel{G}}{E\cancel{G}} \\ -\dfrac{f\cancel{E}}{\cancel{E}G} & -\dfrac{g\cancel{E}}{\cancel{E}G} \end{pmatrix}$$

$$\parallel$$

$$\begin{pmatrix} -\dfrac{e}{E} & -\dfrac{f}{E} \\ -\dfrac{f}{G} & -\dfrac{g}{G} \end{pmatrix}$$

$$\underline{\alpha} = \underline{\alpha}(s) : \begin{cases} x = \cos s \\ y = \sin s \\ z = 0 \end{cases} \qquad s = \varphi = \text{l. d'arco} \qquad \varphi \in [0, 2\pi)$$

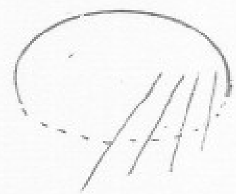
$$\underline{\alpha} = (\cos s, \ \sin s, \ 0)$$

$$\underline{\alpha}' = (-\sin s, \ \cos s, \ 0) = -\sin s \, \underline{i} + \cos s \, \underline{j}$$

$$\underline{w}(s) := (\underline{\alpha}' + \underline{k}) = -\sin s \, \underline{i} + \cos s \, \underline{j} + \underline{k}$$

consideriamo la <u>superficie rigata</u> $\qquad \|\underline{w}\| = 2$

$$\underline{r}(s,t) = \underline{\alpha}(s) + t \, \underline{w}(s)$$



$$= (\underbrace{\cos s - t \sin s}_{x}, \ \underbrace{\sin s + t \cos s}_{y}, \ \underbrace{t}_{z})$$

$$x^2 + y^2 - z^2 = \cos^2 s \cancel{- 2t \sin s \cos s} + t^2 \sin^2 s + \sin^2 s + t^2 \cos^2 s \cancel{+ 2t \sin s \cos s} + t^2 - t^2 = 1$$

* iperboloide iperbolico di rivoluzione

$$x^2 + y^2 - z^2 = 1$$

Sia $P = (1, 0, 0) \in \mathcal{C}$

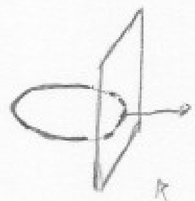
Determiniamo, in $T_P \Sigma$, le dir. <u>asintotiche</u> e le direzioni principali, col calcolo implicito. [per esercizio, si calcoli il tutto tramite la parametrizzazione precedente]

$$x^2 + y^2 - z^2 = 1$$

$$f(x, y, z) = 0$$

si osservi che, in P $\frac{\partial f}{\partial x} = 2x = 2 \neq 0$

$\Rightarrow$ (Dini) $x = x(y, z) \equiv \varphi(y, z)$



in P: $(1, 0, 0)$

$x = \varphi(0, 0) = +1$

$$\varphi^2 + y^2 - z^2 = 1$$

$\frac{\partial}{\partial y}$ $\quad \cancel{2}\varphi\varphi_y + \cancel{2}y = 0 \qquad \varphi\varphi_y + y = 0$

$\frac{\partial}{\partial z}$ $\quad +\cancel{2}\varphi\varphi_z - \cancel{2}z = 0 \qquad \varphi\varphi_z - z = 0$

$\varphi_y(P) = 0$

$\varphi_z(P) = 0$

(Chiaro...)

$\underline{N}(P) = (1, 0, 0)$

$\frac{\partial^2}{\partial y^2}$ $\quad \varphi_y^2 + \varphi\varphi_{yy} + 1 = 0 \qquad \varphi_{yy}(P) = -1$

$\frac{\partial^2}{\partial y \partial z}$ $\quad \varphi_z\varphi_y + \varphi\varphi_{yz} = 0 \qquad \varphi_{yz}(P) = 0$

$\frac{\partial^2}{\partial z^2}$ $\quad \varphi_z^2 + \varphi\varphi_{zz} - 1 = 0 \qquad \varphi_{zz}(P) = +1$

$$\underline{I}^a \begin{cases} E = 1 + \varphi_y^2 & \text{in } P: \ E = 1 \\ F = \varphi_y \varphi_z & \text{in } P \ \ F = 0 \\ G = 1 + \varphi_z^2 & \text{in } P \ \ G = 1 \end{cases}$$

$$\begin{cases} \underline{r} = (\varphi, y, z) \\ \underline{r}_y = (\varphi_y, 1, 0) \\ \underline{r}_z = (\varphi_z, 0, 1) \end{cases}$$

$$\begin{cases} \underline{r}_{yy} = (\varphi_{yy}, 0, 0) \\ \underline{r}_{yz} = (\varphi_{yz}, 0, 0) \\ \underline{r}_{zz} = (\varphi_{zz}, 0, 0) \end{cases}$$

$e = \varphi_{yy}$ in P: $e = -1$

$f = \varphi_{yz}$ in P $f = 0$

$g = \varphi_{zz}$ in P $g = +1$

I^a $E = G = 1$ $F = 0$ (chiaro a priori...)

$\underline{\mathrm{II}}^a$ $e = -1$ $g = 1$ $f = 0$

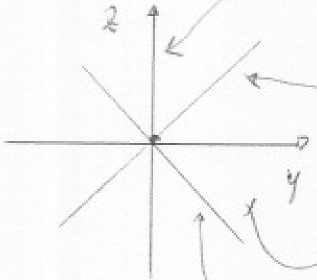
$$K = -1 \quad (\underline{\underline{\text{in}}}\ P)$$

$$H = \frac{1}{2}\,\frac{Ge - 2Ff + Eg}{EG - F^2} = \frac{1}{2}\,\frac{-1+1}{1} = 0 \quad (\text{in } \underline{\underline{P}})$$

$$S(P) = -dN(P) \quad -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

direzioni principali $u = y$, $v = z$  $k_1 = -1$, $k_2 = +1$

in una sup. di rivoluzione, i meridiani e i paralleli <u>sono</u> <u>linee di curvatura</u> !!



$z = 0$ $\qquad y = 0$

<u>direzioni asintotiche</u> $y'^2 - z'^2 = 0$

$$(y' + z')(y' - z') = 0$$

già trovata

la retta $P + t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

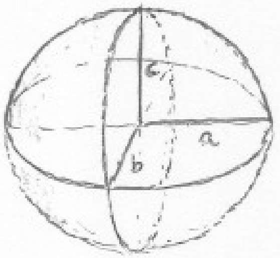
$= \begin{pmatrix} 1 \\ -t \\ t \end{pmatrix}$ è <u>contenuta</u> in Σ: $x^2 + y^2 - z^2 = 1$

(Σ è doppiamente rigato)

4 Curvatura dell'ellissoide

$$\underset{\overset{\|}{0}}{f(x,y,z)} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad\qquad a \geq b \geq c$$

utilizziamo il calcolo diff. implicito
(cf. l'esercizio sulla finestra di Viviani)



Sia $P : (x_0, y_0, z_0)$ tale che $\frac{\partial f}{\partial z}(P) \neq 0$

$\Rightarrow$ (Dini) localmente $z = \varphi(x,y)$ ..

Per una superficie cartesiana: $\quad z = \varphi(x,y) \quad \begin{pmatrix} x = x \\ y = y \\ z = \varphi(x,y) \end{pmatrix}$

è $\quad \underline{r} = (x, \; y, \; \varphi) \qquad\qquad \underline{r} = x\underline{i} + y\underline{j} + \varphi\underline{k}$

$\underline{r}_x = (1, \; 0, \; \varphi_x)$

$\underline{r}_y = (0 \; 1, \; \varphi_y) \qquad\qquad \underline{N} = \begin{vmatrix} i & j & \underline{k} \\ 1 & 0 & \varphi_x \\ 0 & 1 & \varphi_y \end{vmatrix} \frac{1}{\| \;\; \|}$

$$\Downarrow$$

$\underline{r}_{xx} = (0 \quad 0 \quad \varphi_{xx}) \qquad \underline{N} = \frac{1}{\sqrt{1+\varphi_x^2+\varphi_y^2}}(-\varphi_x, -\varphi_y, 1)$

$\underline{r}_{yx} = \underline{r}_{xy} = (0 \quad 0 \quad \varphi_{xy})$

$\underline{r}_{yy} = (0 \quad 0 \quad \varphi_{yy})$

$$\Rightarrow \quad E = 1 + \varphi_x^2 \qquad e = \frac{\varphi_{xx}}{\sqrt{1+\varphi_x^2+\varphi_y^2}}$$

$$F = \varphi_x \varphi_y \qquad f = \frac{\varphi_{yy}}{\sqrt{}}$$

$$G = 1 + \varphi_y^2 \qquad g = \frac{\varphi_{yy}}{\sqrt{}}$$

$$EG - F^2 = 1 + \varphi_x^2 + \varphi_y^2 + \varphi_x^2\varphi_y^2 - \varphi_x^2\varphi_y^2$$
$$= 1 + \varphi_x^2 + \varphi_y^2$$

$$K = \frac{\overbrace{\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2}^{\text{Hessiano}}}{(1+\varphi_x^2+\varphi_y^2)^2}$$

$H = \ldots$

Produciamo ciò che ci serve da

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{\varphi^2}{c^2} = 1 \qquad (\star) \qquad \varphi \neq 0.$$

$\frac{\partial}{\partial x}$:
$$\cancel{2}\frac{x}{a^2} + \frac{\cancel{2}\varphi\varphi_x}{c^2} = 0 \quad (*) \qquad \boxed{\varphi_x = -\frac{c^2 x}{a^2 \cdot \varphi}}$$

$\frac{\partial}{\partial y}$!
$$\cancel{2}\frac{y}{b^2} + \frac{2\varphi\varphi_y}{c^2} = 0 \quad (**) \qquad \boxed{\varphi_y = -\frac{c^2 y}{b^2 \varphi}}$$

$\frac{\partial}{\partial x}$ $(*)$
$$\frac{1}{a^2} + \frac{1}{c^2}\left(\varphi_x^2 + \varphi\varphi_{xx}\right) = 0$$

$$\varphi_x^2 + \varphi \cdot \varphi_{xx} = -\frac{c^2}{a^2}$$

$$\varphi_{xx} = -\left(\frac{c^2}{a^2} + \varphi_x^2\right)\frac{1}{\varphi} = -\left(\frac{c^2}{a^2} + \frac{c^4}{a^4}\frac{x^2}{\varphi^2}\right)\frac{1}{\varphi}$$

$$= -\frac{c^2}{a^2}\left(1 + \frac{c^2}{a^2}\frac{x^2}{\varphi^2}\right)\frac{1}{\varphi}$$

$$= -\frac{c^2}{a^2}\frac{a^2\varphi^2 + c^2x^2}{a^2\varphi^3} = -\frac{c^2}{a^4}\frac{a^2\varphi^2 + c^2x^2}{\varphi^3}$$

$$\varphi_{yy} = -\frac{c^2}{b^4}\frac{b^2\varphi^2 + c^2y^2}{\varphi^3} \qquad \boxed{\varphi_{xy} = \frac{c^4}{a^2b^2}\frac{xy}{\varphi^3}}$$

$$\shortparallel$$

$\frac{\partial}{\partial y}$ $(*)$
$$\varphi_y\varphi_x + \varphi\,\varphi_{xy} = 0 \qquad \varphi_{xy} = -\frac{\varphi_x \cdot \varphi_y}{\varphi}$$

$$\begin{cases} \varphi_{xx} = -\dfrac{c^2}{a^4}\,\dfrac{a^2\varphi^2 + c^2x^2}{\varphi^3} \\[2ex] \varphi_{xy} = -\dfrac{c^4}{a^2b^2}\,\dfrac{xy}{\varphi^3} \\[2ex] \varphi_{yy} = -\dfrac{c^2}{b^4}\,\dfrac{b^2\varphi^2 + c^2y^2}{\varphi^3} \end{cases}$$

calcoliamo $\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2$. Si ha

$$\frac{c^4}{a^4b^4}\,\frac{1}{\varphi^6}\,(a^2\varphi^2 + c^2x^2)(b^2\varphi^2 + c^2y^2) - \frac{c^8}{a^4b^4}\,\frac{x^2y^2}{\varphi^6}$$

$$= \frac{c^4}{a^4b^4\varphi^6}\left[(a^2\varphi^2 + c^2x^2)(b^2\varphi^2 + c^2y^2) - c^4x^2y^2\right]$$

$$= \frac{c^4\varphi^2}{a^4b^4\varphi^6}\Big(\underbrace{a^2b^2\underset{z}{\varphi^2} + c^2b^2x^2 + a^2c^2y^2}_{a^2b^2c^2}\Big)$$

per $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$

$$a^2b^2z^2 + c^2a^2y^2 + b^2c^2x^2 = a^2b^2c^2$$

$$= \frac{1}{\varphi^4}\,\frac{c^4}{a^4b^4}\,a^2b^2c^2 = \frac{c^6}{\varphi^4a^2b^2}$$

$$EF - G^2 = 1 + \varphi_x^2 + \varphi_y^2 = 1 + \frac{c^4}{a^4}\frac{x^2}{\varphi^2} + \frac{c^4}{b^4}\frac{y^2}{\varphi^2} =$$

$$= \frac{a^4 b^4 \varphi^2 + b^4 c^4 x^2 + a^4 c^4 y^2}{a^4 b^4 \varphi^2}$$

$\Rightarrow$

$$K = \frac{c^6}{\varphi^4 a^2 b^2} \cdot \left[\frac{a^4 b^4 \varphi^2}{a^4 b^4 \varphi^2 + b^4 c^4 x^2 + a^4 c^4 y^2} \right]^2$$

$$= \frac{a^6 b^6 c^6}{\left(b^4 c^4 x^2 + c^4 a^4 y^2 + a^4 b^4 z^2 \right)^2}$$

(controllo: sfera: $a = b = c = R$ $\quad K = \dfrac{R^{18}}{(R^2 R^4 R^4)^2} = \dfrac{R^{18}}{R^{20}} = \dfrac{1}{R^2}$ ok!

in definitiva:

$$\boxed{K = \frac{a^6 b^6 c^6}{\left(b^4 c^4 x^2 + c^4 a^4 y^2 + a^4 b^4 z^2 \right)^2}}$$

in un pto di E: $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$

il denom. non è mai nullo...

4 Controesempio all' inverso del "theorema egregium"

$$(K_1 = K_2 \not\Rightarrow \Sigma_1 \text{ e } \Sigma_2 \text{ loc isometriche})$$

$$(\Leftarrow \text{ è l'egregium})$$


1


2

$$\underline{x}(u,v) = (u\cos v,\ u\sin v,\ \log u) \qquad u>0,\ v\in[0,2\pi) \qquad \text{sup. di rot di log}$$

$$\underline{y}(u,v) = (u\cos v,\ u\sin v,\ v) \qquad \text{elicoide}$$

$$\underline{x}_u = \left(\cos v,\ \sin v,\ \frac{1}{u}\right) \qquad \underline{y}_u = (\cos v,\ \sin v,\ 0)$$

$$\underline{x}_v = (-u\sin v,\ u\cos v,\ 0) \qquad \underline{y}_v = (-u\sin v,\ u\cos v,\ 1)$$

$$E_1 = 1 + \frac{1}{u^2} \qquad E_2 = 1$$

$$F_1 = 0 \qquad F_2 = 0$$

$$G_1 = u^2 \qquad G_2 = u^2 + 1$$

$$\underline{N}_1 = \frac{\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos v & \sin v & \frac{1}{u} \\ -u\sin v & u\cos v & 0 \end{vmatrix}}{\|\ \cdot\ \|} = \frac{\underline{i}\,(-\cos v) - \underline{j}\sin v + \underline{k}\,u}{\|\qquad\|} =$$

$$= \frac{(-\cos v)\underline{i} + (-\sin v)\underline{j} + u^2\underline{k}}{\sqrt{1+u^2}}$$

$$\underline{N}_2 = \frac{\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix}}{\|\ \cdot\ \|} = \frac{\underline{i}\,(\sin v) - \underline{j}\cos v + \underline{k}\,u}{\sqrt{1+u^2}}$$

$\underline{x}_{uu} = \left(0 \quad 0 \quad -\frac{1}{u^2} \right)$ $\qquad e_1 = -\frac{1}{u} \cdot \frac{1}{\sqrt{1+u^2}}$

$\underline{x}_{uv} = \left(-\sin v \;\; \cos v, \; 0 \right)$ $\qquad f_1 = \sin v \cos v - \sin v \cos v + 0 = 0$

$\underline{x}_{vv} = \left(-u\cos v \;\; -u \sin v, \; 0 \right)$ $\qquad g_1 = +u(\cos^2 v + \sin^2 v) = \frac{u}{\sqrt{1+u^2}}$

$\underline{y}_{uu} = \left(0 \quad 0 \quad 0 \right)$ $\qquad e_2 = 0$

$\underline{y}_{uv} = \left(-\sin v \;\; \cos v \;\; 0 \right)$ $\qquad f_2 = (-\sin^2 v - \cos^2 v)\frac{1}{\sqrt{\ }} = \frac{-1}{\sqrt{1+u^2}}$

$\underline{y}_{vv} = \left(-u\cos v \;\; -u\sin v \;\; 0 \right)$ $\qquad g_2 = \ldots = 0$

$$K_1 = \frac{-\frac{1}{1+u^2}}{\left(1+\frac{1}{u^2}\right)u^2} = \frac{-\frac{1}{1+u^2}}{(1+u^2)} = -\frac{1}{(1+u^2)^2}$$

$$\Rightarrow \boxed{K_1 = K_2}$$

$$K_2 = \frac{-\frac{1}{1+u^2}}{u^2+1} = -\frac{1}{(1+u^2)^2}$$

Sia ora $u = \text{cost} = u_0$ $\qquad \underline{x}(u_0, v) = (u_0 \cos v, u_0 \sin v, \log u_0)$

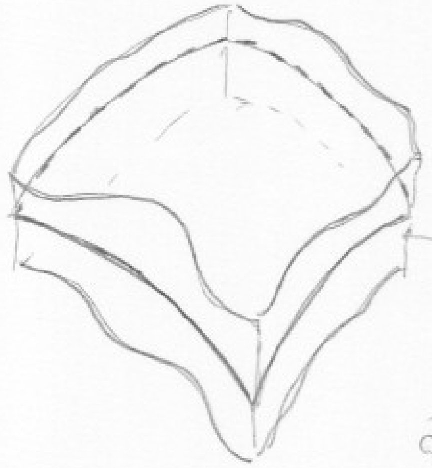
$\underline{y}(u_0, v) = (u_0 \cos v, u_0 \sin v, v)$

tratti corrispondenti di tali curve non hanno la stessa lunghezza.

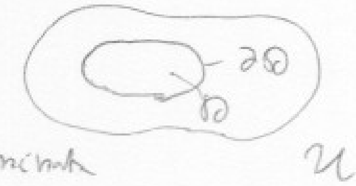
$$\boxed{\frac{ds_1^2}{ds_2^2} = \frac{u_0^2}{u_0^2+1}} \Leftarrow \left[\begin{aligned} ds_1^2 &= u_0^2 \, dv^2 \\ ds_2^2 &= (u_0^2+1)\, dv^2 \end{aligned} \right]$$

★ Superficie minime $(H = 0)$ [Lagrange, 1760]

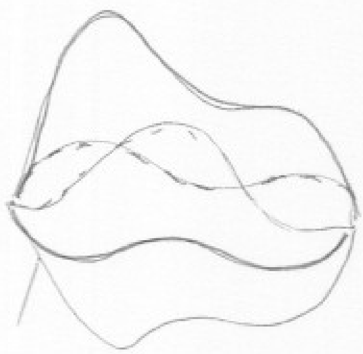
↑ curvatura media



$\underline{r} = \underline{r}(u,v) \qquad (u,v) \in \mathcal{U}$



Variazione normale determinata da $h : \bar{D} \to \mathbb{R}$ (liscia)



$$\underline{\varphi}(\underbrace{u, v}_{\in\, \bar{D}}, \underbrace{t}_{\in\,(-\varepsilon,\varepsilon)}) := \underline{r}(u,v) + t\, h(u,v)\, \underline{N}(u,v)$$

per t fissato ho $\underline{r}^t = \underline{r}^t(u,v)$

si ha:

$$\begin{cases} \underline{r}^t_u = \underline{r}_u + t h \underline{N}_u + t h_u \underline{N} \\ \underline{r}^t_v = \underline{r}_v + t h \underline{N}_v + t h_v \underline{N} \end{cases}$$

e, pertanto, per la relativa metrica, si ha:

profilo fissato

ricerca di una superficie di area minima:

★ problema di Plateau

[o della bolla di sapone]

$$E^t = E + t h \Big[\underbrace{\langle \underline{r}_u, \underline{N}_u \rangle}_{-e} + \underbrace{\langle \underline{r}_u, \underline{N}_u \rangle}_{-e} \Big] + \ldots$$

$$F^t = F + t h \big[-2f \big] + \ldots$$

$$G^t = G + t h \big[-2g \big] + \ldots$$

Sicché $\quad E^t G^t - F^{t^2} = EG - F^2 +$

$$+ (-2th)\left[Eg - 2Ff + Ge \right] + \dots$$

$$= (EG - F^2)\left[1 - 4th H \right] + o(t)$$

elemento d'area $\longrightarrow \qquad d\sigma^t = d\sigma \cdot (1 - 2tHh)$

$$A(t) = \int_{\Omega} d\sigma^t \qquad\qquad \frac{d\sigma^t}{dt} = -2tHh\,d\sigma$$

$$\mathring{A}(0) = \int_{\Omega} \frac{d\sigma^t}{dt} = -\int_{\Omega} 2hH\,d\sigma$$

Si ha $\mathring{A}(0) = 0 \quad \forall\, h \quad$ (<u>stazionarietà</u>)

variazione normale

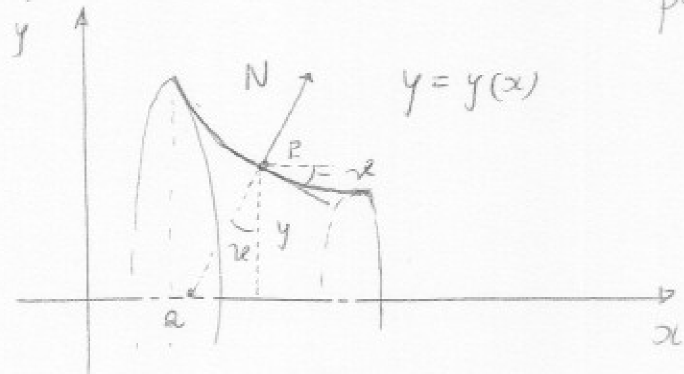
$\Longleftrightarrow \quad H = 0$

(interpretazione variazionale della condizione $H = 0$)

È un'equazione di E-L.

★ <u>La catenoide</u> è l'unica superficie di rivoluzione <u>minima</u>

Una dimostrazione tramite le eq. di Eulero-Lagrange è stata già fornita prima (v. cap. XI). Qui procediamo per via geometrica.



Dalla teoria generale delle sup. di rivoluzione le curvature principali sono quella del (generico) meridiano e l'inverso della <u>gran normale</u>

$$\overline{PQ} = \frac{y}{\cos\vartheta} = y \cdot \sqrt{1+\tan^2\vartheta} = y\sqrt{1+y'^2}$$

$H = \frac{1}{2}(k_1 + k_2) = 0$ diventa

$$\frac{y''}{(1+y'^2)^{\frac{3}{2}}} = \frac{1}{y}\,\frac{1}{(1+y'^2)^{\frac{1}{2}}}$$

⚠ I segni sono corretti...

$\Rightarrow$ (assumendo $y' \neq 0$...) $\quad \dfrac{2y''y'}{1+y'^2} = \dfrac{2y'}{y} \Rightarrow$

$$d\log(1+y'^2) = d\log y^2 \Rightarrow$$

$$1 + y'^2 = k^2 y^2 \quad (k \text{ costante})$$

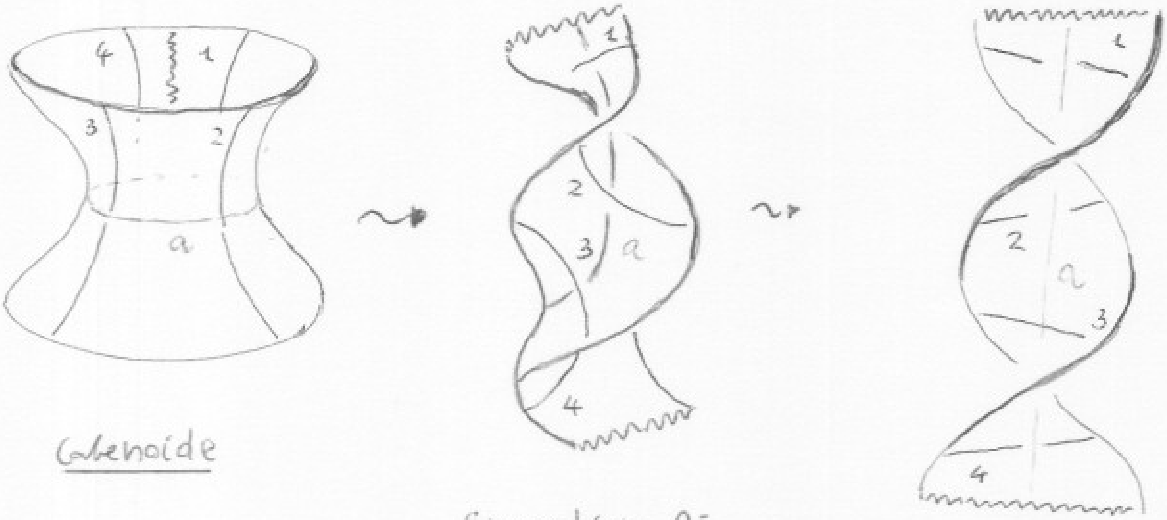
$$y'^2 = k^2y^2 - 1 \qquad y' = \pm\sqrt{(ky)^2-1}$$

$$\frac{dy}{\sqrt{(ky)^2-1}} = \pm\, dx \quad \rightsquigarrow \text{ ricordando}$$

$\cosh^2\xi - \sinh^2\xi = 1$ si arriva a

$$y = \frac{1}{k}\cosh(kx + c)$$

i.e. si ha una <u>catenaria</u>



Catenoide   superficie di Scherk   elicoide

★ Sup. di Scherk $\underline{r}(x,y) = \left(x,\ y,\ \log\left(\frac{\cos y}{\cos x}\right)\right)$

definita sui "quadrati neri" della "scacchiera" (v. fig.)

$H = 0$