## Equivalence

EECS 20

Lecture 12 (February 12, 2001)

Tom Henzinger

## Quiz

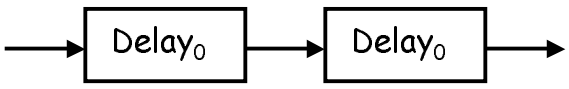
1. Draw the transition diagram of the system

$$Delay_0 : [\ Nats_0 \to Bins\ ] \to [\ Nats_0 \to Bins\ ]$$

$$\forall\ x \in [\ Nats_0 \to Bins\ ],\ \forall\ y \in Nats_0,$$

$$(\ Delay_0\ (x))\ (y) = \begin{cases} 0 & \text{if } y = 0 \\ x\ (y\text{-}1) & \text{if } y > 0 \end{cases}$$

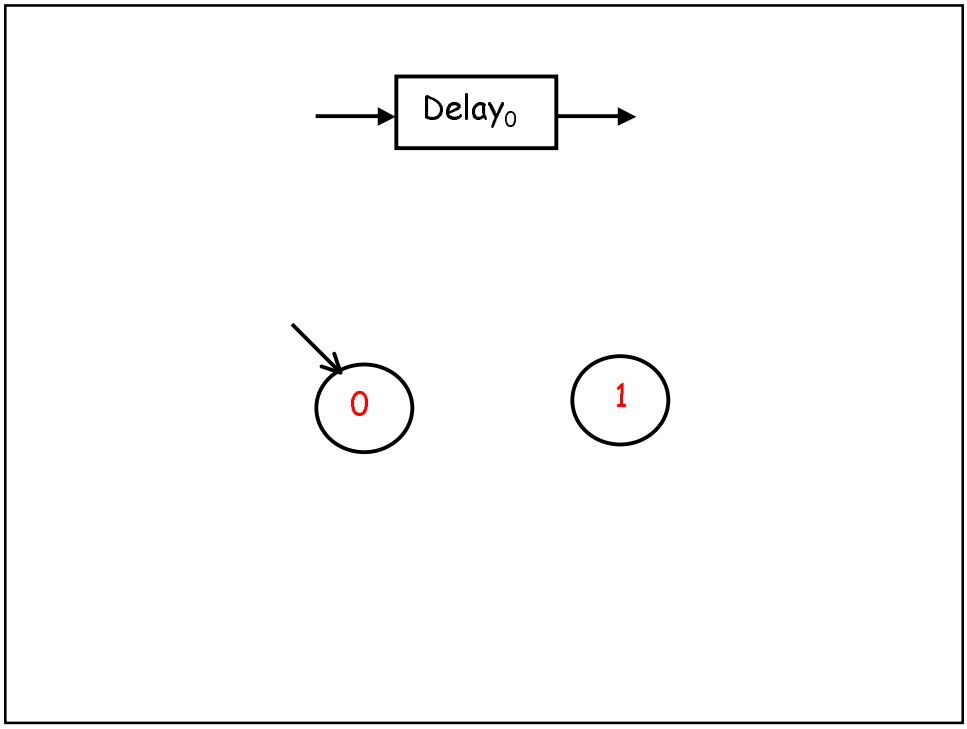
2. Draw the transition diagram of the system

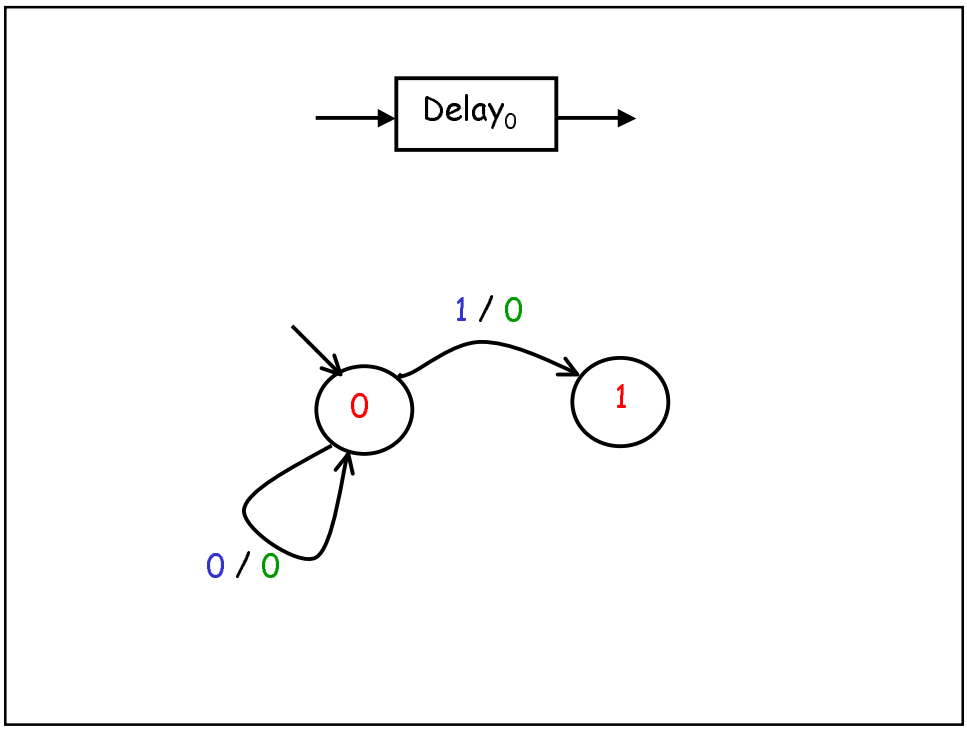


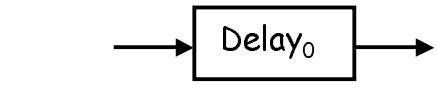
State before time t :

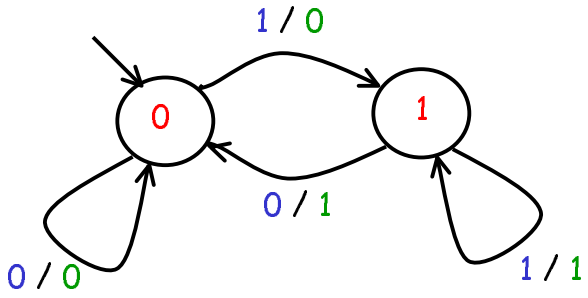
0 if input at time t-1 was 0, or if t = 0

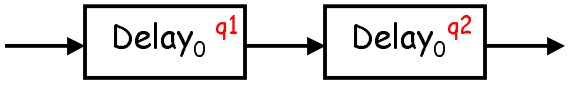
1 if input at time t-1 was 1


Delay0
0
1

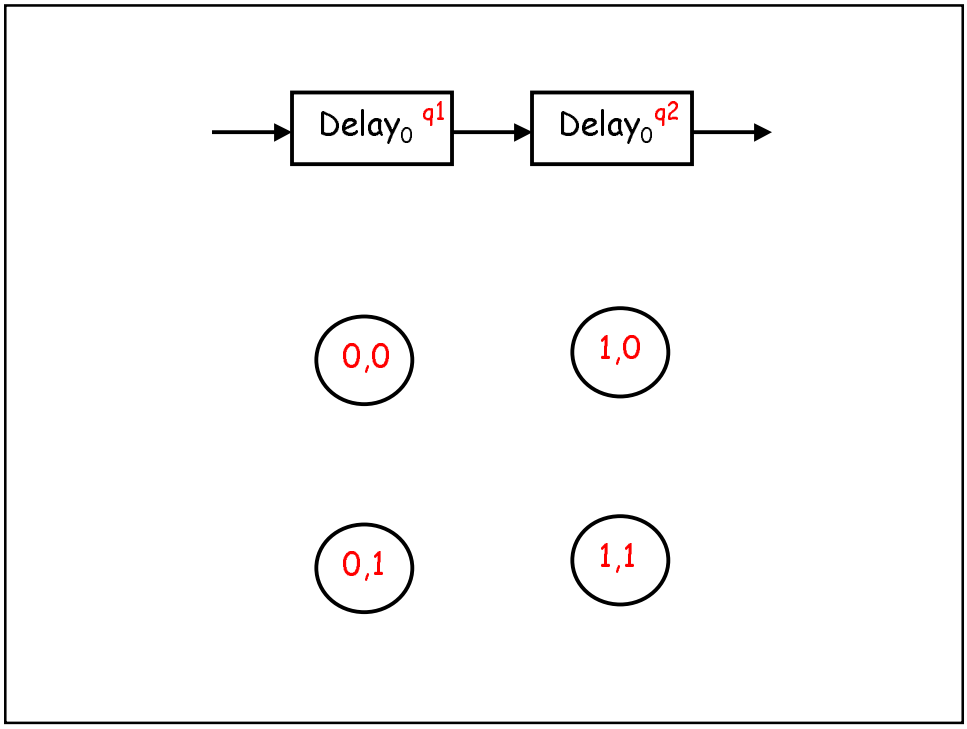

Delay0
1 / 0
0
1
0 / 0

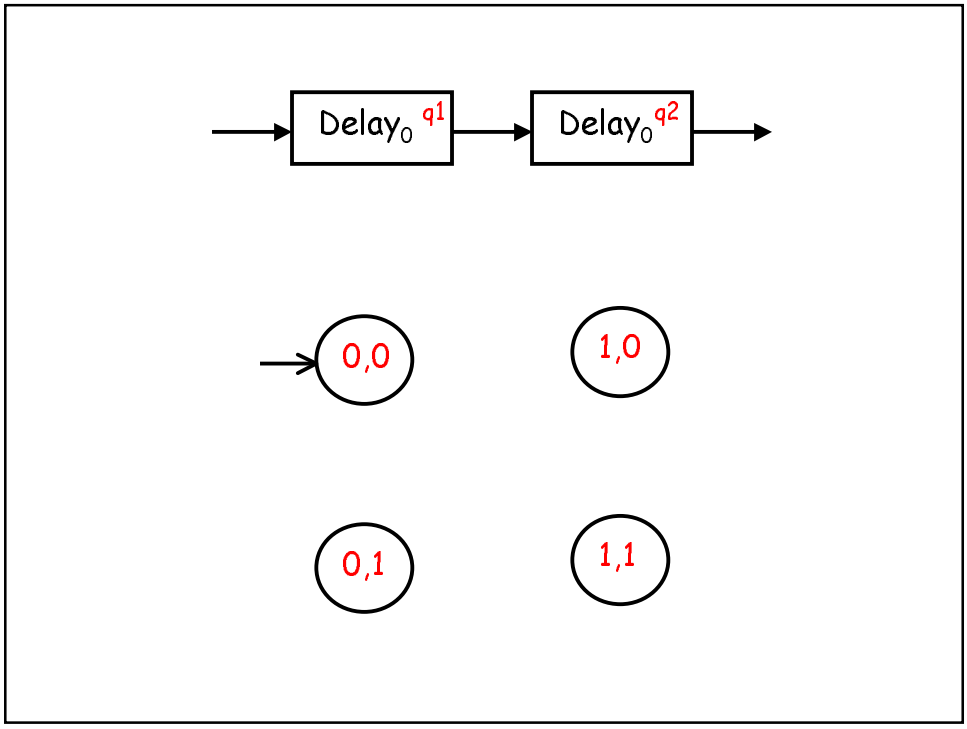


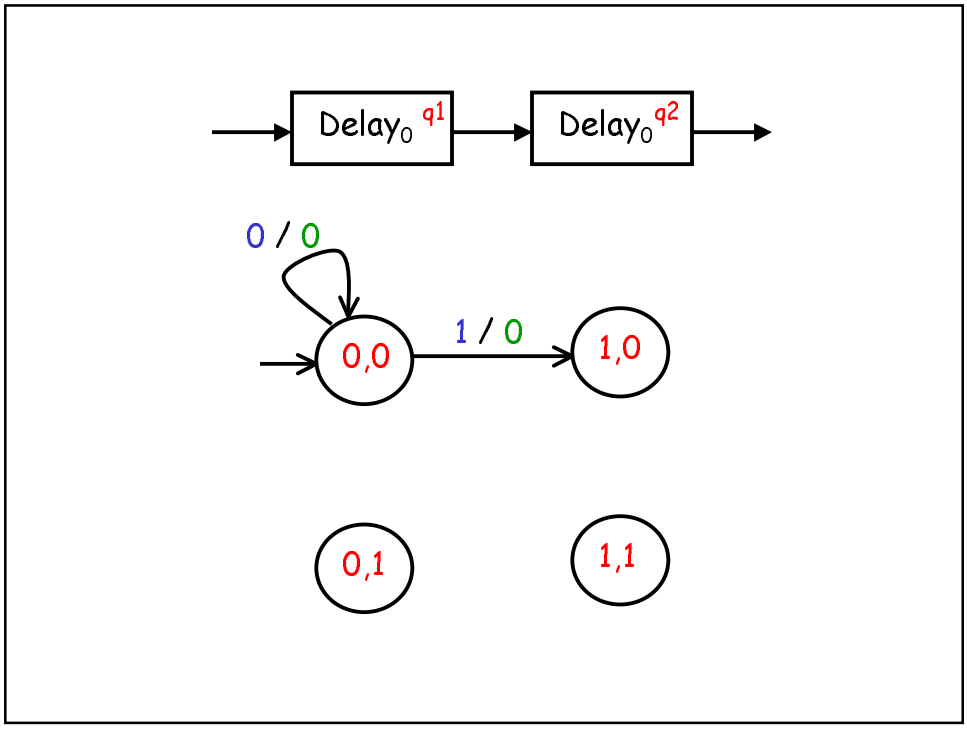



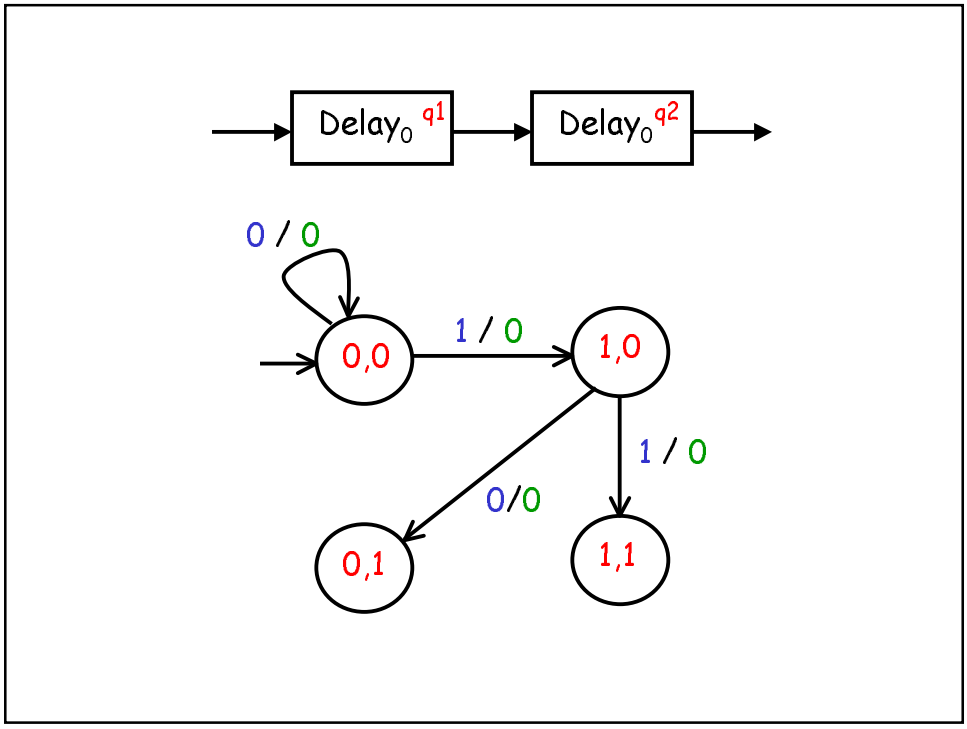



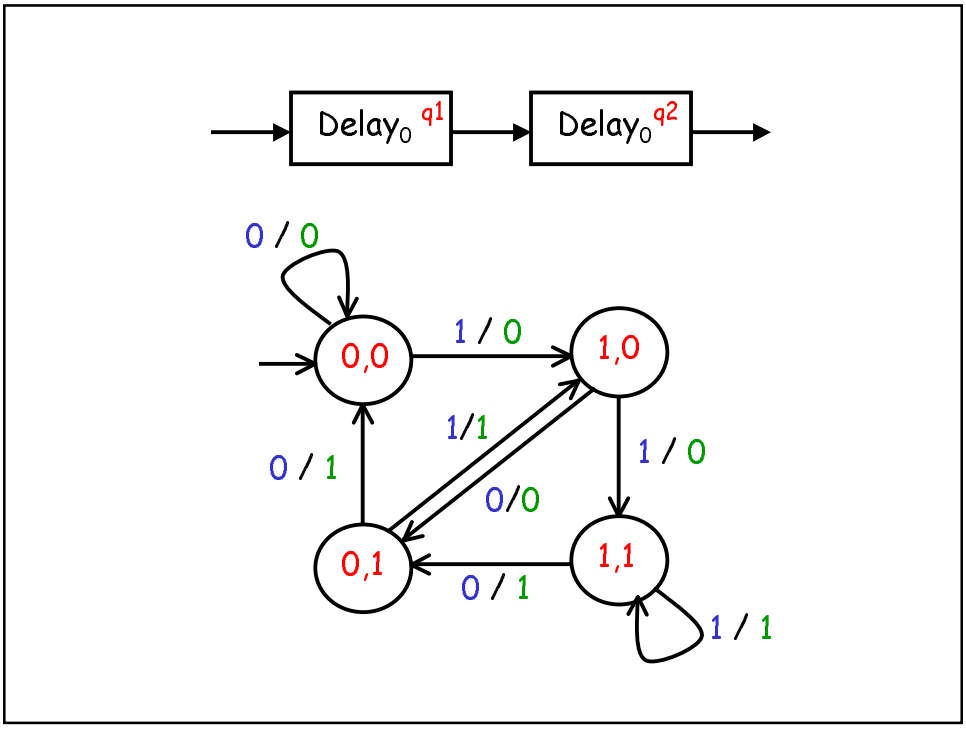

State: ( q1, q2 )

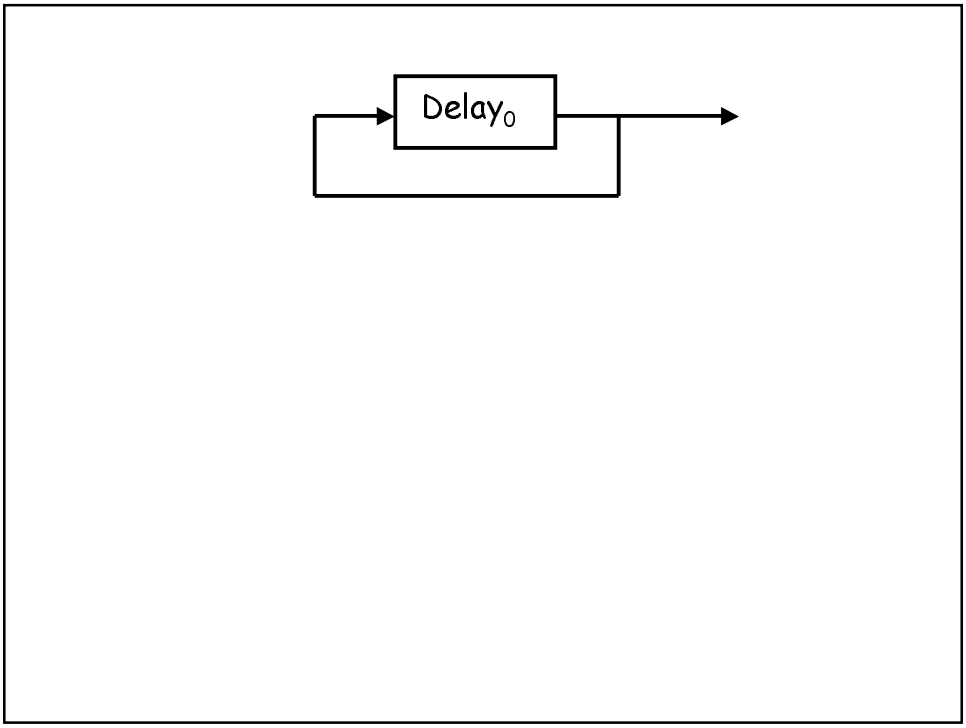

Delay0 q1
Delay0 q2
0,0
1,0
0,1
1,1

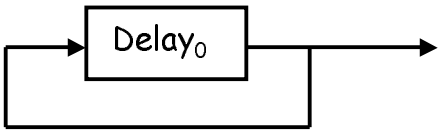

Delay0 q1
Delay0 q2
0,0
1,0
0,1
1,1


Delay0 q1
Delay0 q2
0 / 0
1 / 0
0,0
1,0
0,1
1,1

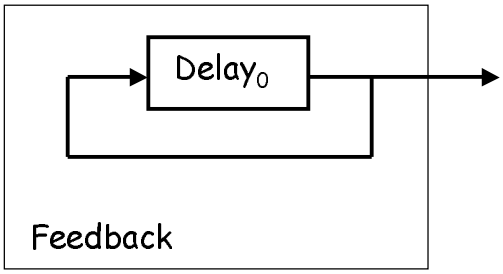

Delay0 q1
Delay0 q2
0 / 0
1 / 0
0,0
1,0
1 / 0
0/0
0,1
1,1


Delay0 q1
Delay0 q2
0 / 0
0,0
1 / 0
1,0
1/1
0 / 1
1 / 0
0/0
0,1
1,1
0 / 1
1 / 1


Delay0

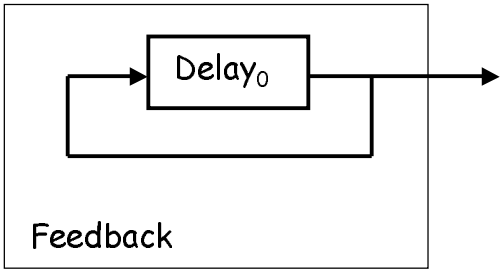



Well-formed !




Feedback : $[\ \mathrm{Nats}_0 \rightarrow \mathrm{Unit}\ ] \rightarrow [\ \mathrm{Nats}_0 \rightarrow \mathrm{Bins}\ ]$

where Unit = { • } .



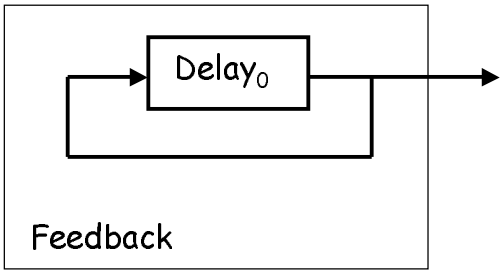

Inputs : Unit

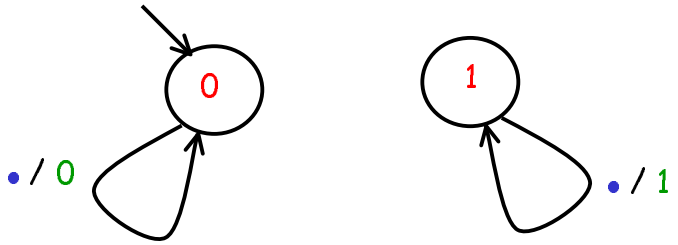
Outputs: Bins

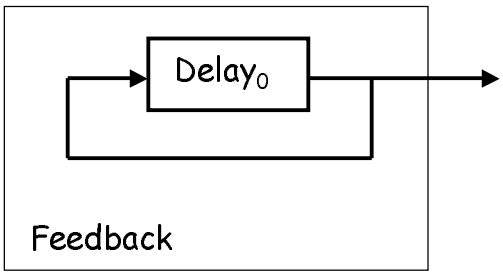
States: { 0, 1 }

initialState = 0

| q | x | output (q,x) | nextState (q,x) |
|---|---|---|---|
| 0 | • | 0 | 0 |
| 1 | • | 1 | 1 |

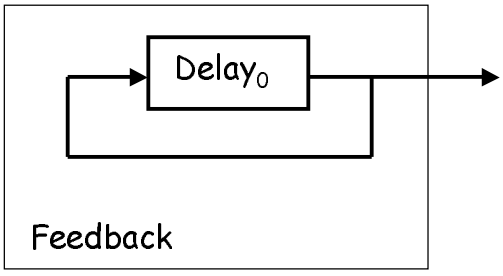


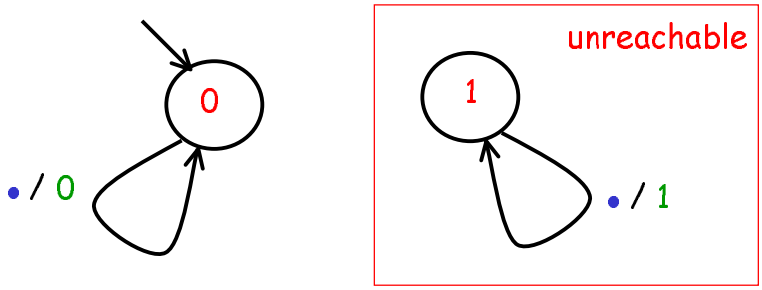






Only one run !

| Time | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| Input | • | • | • | • | • | • | • | • |
| Output | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| State | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |



A state q of a state machine M is unreachable

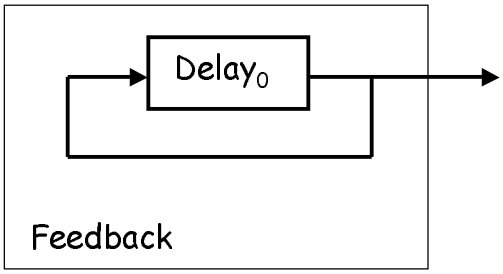
iff

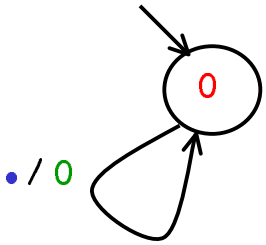
q occurs on no run of M ;

that is, iff

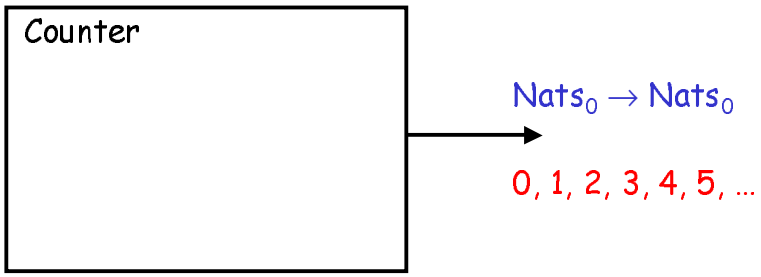
there is no path from the initial state of M to q .

Unreachable states can be removed without changing the system (input/output function) implemented by M .

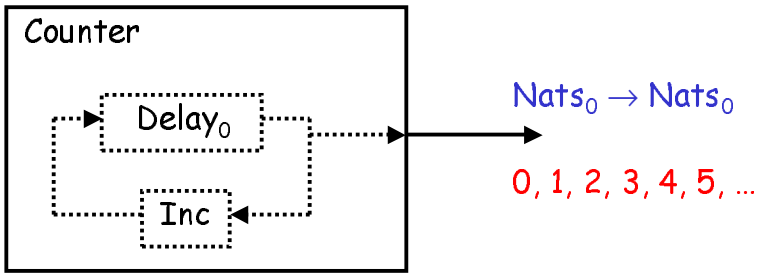


## A more interesting system without inputs




## State-machine implementation of Counter

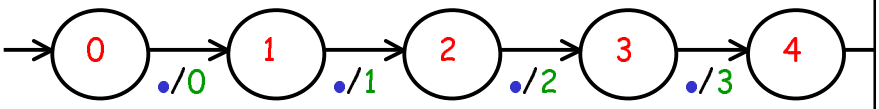



where $\forall\, n \in Nats_0,\ \ inc\,(n) = n + 1$ .

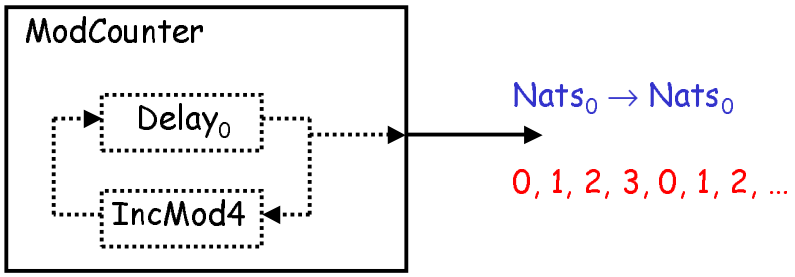
One run

| Time | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| Input | • | • | • | • | • | • | • | • |
| Output | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| State | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

Infinitely many states

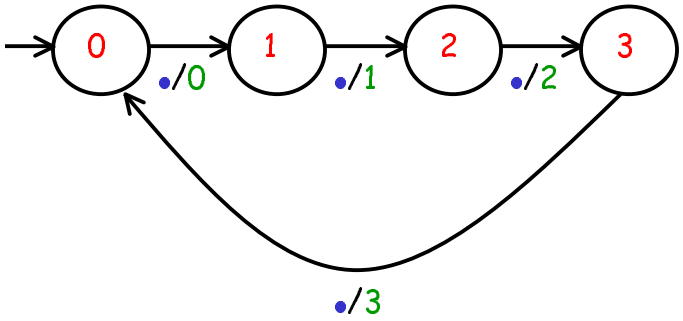



Yet another system without inputs




where $\forall\, n \in Nats_0$, incMod4 (n) = ( n + 1 ) mod 4.

Transition diagram of ModCounter




Two state machines M1 and M2 are equivalent

iff

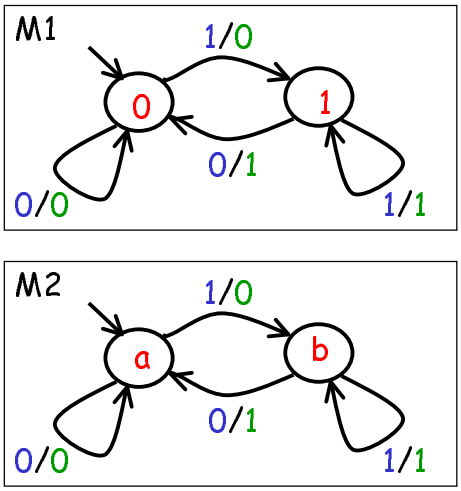
they implement the same system (input/output function);

that is, iff

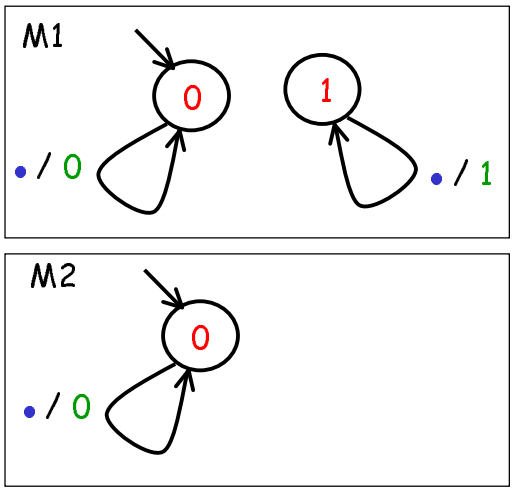
1. Inputs [M1] = Inputs [M2] ,
2. Outputs [M1] = Outputs [M2] , and
3. $\forall x \in [\, Nats_0 \rightarrow Inputs \,],\ M1\,(x) = M2\,(x)$ .

$\in [\, Nats_0 \rightarrow Outputs \,]$

## Equivalent State Machines



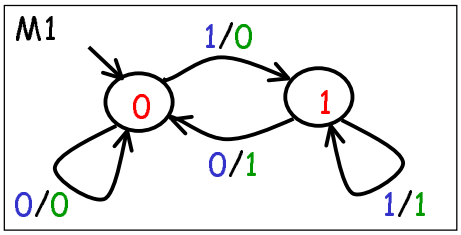

## Equivalent State Machines



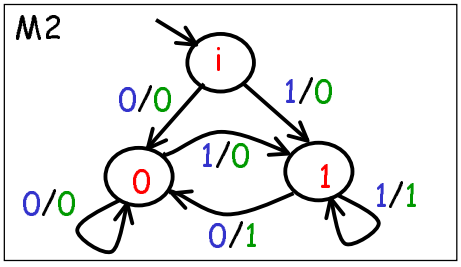

2 states

1 state

## Equivalent State Machines




2 states




3 states

Theorem :

Two state machines M1 and M2 are equivalent

iff

there exists a bisimulation between M1 and M2 .

A binary relation B between States [M1] and States [M2] ;

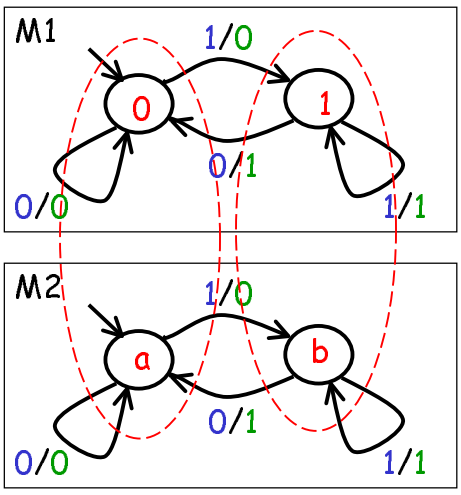
that is, $B \subseteq$ States [M1] $\times$ States [M2] .

A binary relation $B \subseteq States[M1] \times States[M2]$ is a bisimulation

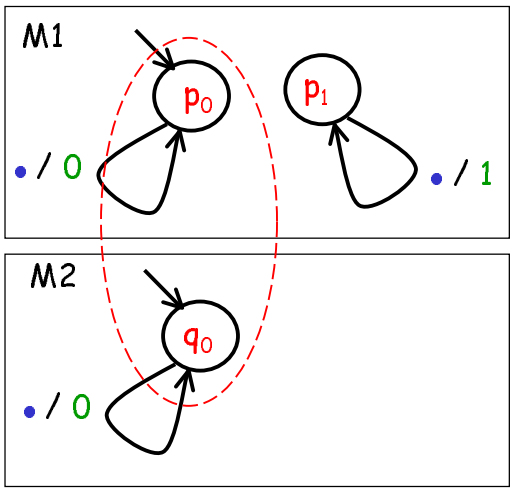
iff

1. $(initialState[M1], initialState[M2]) \in B$ and
2. $\forall p \in States[M1]$, $\forall q \in States[M2]$,
   if $(p, q) \in B$,
   then $\forall x \in Inputs[M1]$,
   $output[M1](p, x) = output[M2](q, x)$ and
   $(nextState[M1](p, x), nextState[M2](q, x)) \in B$.

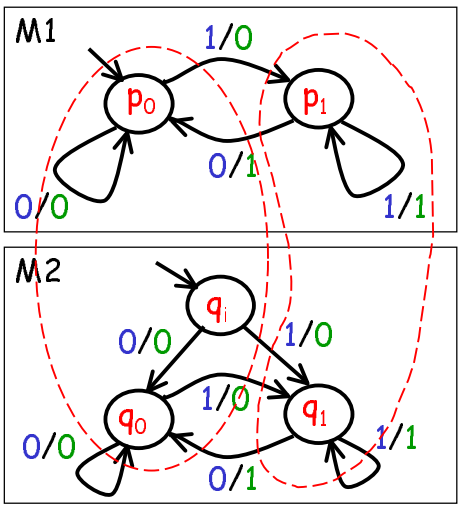
Bisimulation $B = \{ (0, a), (1, b) \}$



Bisimulation $B = \{ ( p_0, q_0 ) \}$




Bisimulation $B = \{ ( p_0, q_i ), ( p_0, q_0 ), ( p_1, q_1 ) \}$



Equivalence between state machines:

refers only to input and output signals.

Bisimulation between state machines:

refers to states.

Why is bisimulation useful ?

"Equivalence" says something about infinitely many possible input signals.

For finite state machines, "bisimulation" says something about finitely many possible relationships between states.

Bisimulation, therefore, is easier to check than equivalence.