Esempio $\quad M = \mathbb{R}^2 \setminus \{(0,0)\}$

Addendum alle lez. XVII - XIX

$$\psi : (\rho, \varphi) \longmapsto (x, y) \qquad \psi : \begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases}$$

$$\underset{\psi_*}{\overset{d\psi}{\equiv}} : \begin{cases} dx = d\rho\cos\varphi - \rho\sin\varphi\, d\varphi \\ dy = d\rho\sin\varphi + \rho\cos\varphi\, d\varphi \end{cases} \qquad \begin{matrix} \rho > 0 \\ \varphi \in [0, \varphi) \end{matrix}$$

$$\begin{pmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{pmatrix} \qquad \frac{\partial}{\partial\rho} \leftrightarrow (1,0) \qquad \frac{\partial}{\partial\varphi} \leftrightarrow (0,1)$$

$$\underset{(1,0)}{\psi_*\left(\frac{\partial}{\partial\rho}\right)} = \cos\varphi \cdot \frac{\partial}{\partial x} + \sin\varphi \cdot \frac{\partial}{\partial y}$$

$$= \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y}$$

$$\underset{(0,1)}{\psi_*\left(\frac{\partial}{\partial\varphi}\right)} = -\rho\sin\varphi\frac{\partial}{\partial x} + \rho\cos\varphi\frac{\partial}{\partial y} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

ciò è ovviamente in accordo con la definizione generale:

$$\boxed{(\psi_* X)(f) = X(f \circ \psi)}$$

(cf. lezione precedente)

$$X = \frac{\partial}{\partial\rho}$$

$$\psi_*\left(\frac{\partial}{\partial\rho}\right)(f) = \frac{\partial}{\partial\rho}(f \circ \psi) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial\rho} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial\rho} =$$

"$f = f(x,y)$" "$f = f(\rho,\varphi)$"

$$= \frac{\partial f}{\partial x}\cos\varphi + \frac{\partial f}{\partial y}\sin\varphi$$

$$\psi_*\left(\frac{\partial}{\partial\rho}\right) = \cos\varphi\frac{\partial}{\partial x} + \sin\varphi\frac{\partial}{\partial y} \qquad \checkmark$$

$$\Psi_*\left(\frac{\partial}{\partial\varphi}\right)(f) = \frac{\partial}{\partial\varphi}(f\circ\Psi) =$$

$$= \frac{\partial f}{\partial x}\frac{\partial x}{\partial\varphi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial\varphi} = \frac{\partial f}{\partial x}(\underbrace{-\rho\sin\varphi}_{-y}) + \frac{\partial f}{\partial y}(\underbrace{\rho\cos\varphi}_{x})$$

variazione sul tema:

$$\frac{\partial f}{\partial\rho} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial\rho} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial\rho}$$

$$\frac{\partial f}{\partial\varphi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial\varphi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial\varphi}$$

$$\begin{pmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial y}{\partial\rho} \\ \frac{\partial x}{\partial\varphi} & \frac{\partial y}{\partial\varphi} \end{pmatrix}$$

$$J^T$$

ovvero:

$$\begin{pmatrix} \frac{\partial}{\partial\rho} \\ \frac{\partial}{\partial\varphi} \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\rho\sin\varphi & \rho\cos\varphi \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

$$\begin{cases} \frac{\partial}{\partial\rho} = \cos\varphi\frac{\partial}{\partial x} + \sin\varphi\frac{\partial}{\partial y} \\ \frac{\partial}{\partial\varphi} = -\rho\sin\varphi\frac{\partial}{\partial x} + \rho\cos\varphi\frac{\partial}{\partial y} \end{cases}$$

$\frac{\partial}{\partial\rho}$ e $\frac{\partial}{\partial\varphi}$ sono visti come comb. lin. di $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$