

Esempio  $M = \mathbb{R}^2 - \{(0,0)\}$

Addendum  
alle ltr. XVII-XIX

$$\psi: (p, \varphi) \longmapsto (x, y) \quad \psi: \begin{cases} x = p \cos \varphi \\ y = p \sin \varphi \end{cases}$$

$$d\psi: \begin{cases} dx = dp \cos \varphi - p \sin \varphi d\varphi \\ dy = dp \sin \varphi + p \cos \varphi d\varphi \end{cases} \quad \begin{matrix} p > 0 \\ \varphi \in [0, \varphi) \end{matrix}$$

$\psi_*$

$$\begin{pmatrix} \cos \varphi & -p \sin \varphi \\ \sin \varphi & p \cos \varphi \end{pmatrix} \quad \begin{matrix} \frac{\partial}{\partial p} \leftrightarrow (1, 0) \\ \frac{\partial}{\partial \varphi} \leftrightarrow (0, 1) \end{matrix}$$

$$\begin{aligned} \psi_* \left( \frac{\partial}{\partial p} \right) &= \cos \varphi \cdot \frac{\partial}{\partial x} + \sin \varphi \cdot \frac{\partial}{\partial y} \\ &\downarrow \\ &(1, 0) \end{aligned} \quad = \quad \frac{x}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2 + y^2}} \frac{\partial}{\partial y}$$

$$\begin{aligned} \psi_* \left( \frac{\partial}{\partial \varphi} \right) &= -p \sin \varphi \frac{\partial}{\partial x} + p \cos \varphi \frac{\partial}{\partial y} \\ &\downarrow \\ &(0, 1) \end{aligned} \quad = \quad -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$$

ciò è ovviamente  
in accordo con  
la definizione  
generale:

$$X = \frac{\partial}{\partial p}$$

$$\boxed{(\psi_* X)(f) = X(f \circ \psi)} \quad (\text{cf. lezione precedente})$$

$$\psi_* \left( \frac{\partial}{\partial p} \right) (f) = \frac{\partial}{\partial p} (f \circ \psi) = \frac{\partial f}{\partial x} \frac{\partial x}{\partial p} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial p} =$$

"  $f = f(x, y)$ "      "  $f = f(p, \varphi)$ "

$$= \frac{\partial f}{\partial x} \cos \varphi + \frac{\partial f}{\partial y} \sin \varphi$$

$$\psi_* \left( \frac{\partial}{\partial p} \right) = \cos \varphi \frac{\partial}{\partial x} + \sin \varphi \frac{\partial}{\partial y} \quad \checkmark$$

$$\psi_* \left( \frac{\partial}{\partial q} \right) (f) = \frac{\partial}{\partial q} (f \circ \psi) =$$

$$= \frac{\partial f}{\partial x} \frac{\partial x}{\partial q} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial q} = \frac{\partial f}{\partial x} \underbrace{(-p \sin q)}_{-y} + \frac{\partial f}{\partial y} \underbrace{(p \cos q)}_y$$

variazione sul tempo:

$$\frac{\partial f}{\partial p} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial p} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial p}$$

$$\frac{\partial f}{\partial q} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial q} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial q}$$

$$\begin{pmatrix} \frac{\partial x}{\partial p} & \frac{\partial y}{\partial p} \\ \frac{\partial x}{\partial q} & \frac{\partial y}{\partial q} \end{pmatrix}$$

ovvero:

$J^T$

$$\begin{pmatrix} \frac{\partial}{\partial p} \\ \frac{\partial}{\partial q} \end{pmatrix} = \begin{pmatrix} \cos q & \sin q \\ -p \sin q & p \cos q \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

$$\begin{cases} \frac{\partial}{\partial p} = \cos q \frac{\partial}{\partial x} + \sin q \frac{\partial}{\partial y} \\ \frac{\partial}{\partial q} = -p \sin q \frac{\partial}{\partial x} + p \cos q \frac{\partial}{\partial y} \end{cases}$$

$\frac{\partial}{\partial p}$  e  $\frac{\partial}{\partial q}$  sono with come comb. lin. di  $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$