Università degli Studi di Verona
Corso di Laurea Magistrale in Matematica Applicata

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Some exercises of functional analysis - A.A. 2013/14 - N.2

Pb 1. Let μ be an outer measure on $\mathbb{R}^n$, (f_n) a sequence of summable functions from $\mathbb{R}^n$ to $\bar{\mathbb{R}}$ and (g_n) a sequence of summable functions from $\mathbb{R}^n$ to $\mathbb{R}^+ \cup \{+\infty\}$ such that $|f_n| \leq g_n$ for all $n \in \mathbb{N}$. Assume that (f_n) and (g_n) converge pointwise to $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $g : \mathbb{R}^n \to \mathbb{R}^+ \cup \{+\infty\}$ respectively with g summable and that

$$\lim_n \int g_n d\mu = \int g\, d\mu.$$

Prove that

$$\lim_n \int f_n d\mu = \int f d\mu.$$

Pb 2. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{n\sqrt{x}}{1+n^2x^2}$. Compute

$$\lim_n \int_0^1 f_n(x)dx.$$

Pb 3. Compute

$$\lim_n \frac{1}{n} \int_{\frac{1}{n}}^{+\infty} \frac{\sin x}{x^2} dx.$$

Pb 4. Does the following equality holds?

$$\int_0^{+\infty} \sum_{n=1}^{\infty} \frac{\sin(x^3+n^3)}{x^3+n^3} dx = \sum_{n=1}^{\infty} \int_0^{+\infty} \frac{\sin(x^3+n^3)}{x^3+n^3} dx.$$

Pb 5. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = n^3(x-n)^2 \chi_{[n-\frac{1}{n}, n+\frac{1}{n}]}(x).$$

Prove that (f_n) converges uniformly to zero over compact sets, but

$$\lim_n \int_{\mathbb{R}} f_n(x)dx \neq \int_{\mathbb{R}} \lim_n f_n(x)dx.$$

Pb 6. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = nxe^{-\sqrt{n}x}.$$

Study the pointwise and uniform convergence of (f_n) over subsets of $[0, +\infty)$ and compute

$$\lim_n \int_0^{+\infty} f_n(x)dx, \qquad \lim_n \int_\varepsilon^{+\infty} f_n(x)dx, \quad \varepsilon > 0.$$

Pb 7. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \frac{1}{\pi} \frac{n}{1 + n^2 x^2}.$$

After checking that $\int_{\mathbb{R}} f_n(x)dx = 1$ for all $n \in \mathbb{N}$, study the pointwise and uniform convergence of (f_n) over subsets of $[0, +\infty)$ bounded away from zero ($|x| > \varepsilon$, with $\varepsilon > 0$) prove that

$$\lim_n \int_{\mathbb{R}} f_n(x)\varphi(x)dx = \varphi(0),$$

for every choice of continuous and bounded function φ on $\mathbb{R}$.

Pb 8. Prove that the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) = \int_0^\infty x^2 e^{-x} \sin(xt) dx$$

is continuous. Check if it is also of class C^1.

Pb 9. Construct a sequence of continuous functions f_n on $[0, 1]$ such that $0 \le f_n \le 1$ and

$$\lim_n \int_0^1 f_n(x)dx = 0$$

but such that the sequence (f_n) converges for no $x \in [0, 1]$.

Pb 10. Prove or disprove that

$$\lim_n \int_0^n \big(1 - \frac{n}{x}\big)^n e^{x/2} dx = 2, \qquad \lim_n \int_0^n \big(1 + \frac{n}{x}\big)^n e^{-2x} dx = 1.$$

Pb 11. Compute the following limit

$$\lim_n n^2 \int_{\mathbb{R}^3} e^{-x^2-y^2-z^2} \frac{\cos(x/n) - 1}{x^2}.$$

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