Wavelets bases in higher dimensions

Topics

Basic issues

- Separable spaces and bases
- Separable wavelet bases (2D DWT)
- Fast 2D DWT
- *Lifting steps* scheme
- JPEG2000

Advanced concepts

- Overcomplete bases
  - Discrete wavelet frames (DWF)
    - Algorithme à trous
  - Discrete dyadic wavelet frames (DDWF)
- Overview on edge sensitive wavelets
  - Contourlets

Wavelets in vision

- Human Visual System

Separable Wavelet bases

- In general, to any wavelet orthonormal basis $\{\psi_{j,n}\}_{(j,n)\in Z^2}$ of $\mathbf{L^2}(R)$, one can associate a separable wavelet orthonormal basis of $\mathbf{L^2}(R^2)$:

$$\left\{\psi_{j_1,n_1}(x_1)\,\psi_{j_2,n_2}(x_2)\right\}_{(j_1,j_2,n_1,n_2)\in\mathbb{Z}^4}$$

- The functions $\psi_{j1,n1}(x_1)$ and $\psi_{j2,n2}(x_2)$ mix information at two different scales along x_1 and x_2, which is something that we could want to avoid
- *Separable multiresolutions* lead to another construction of separable wavelet bases with wavelets that are products of functions dilated at the same scale.

Separable multiresolutions

- The notion of resolution is formalized with orthogonal projections in spaces of various sizes.
- The approximation of an image $f(x_1,x_2)$ at the resolution 2^{-j} is defined as the orthogonal projection of f on a space $\mathbf{V}_2^j$ that is included in $\mathbf{L^2}(\mathrm{R}^2)$
- The space $\mathbf{V}_2^j$ is the set of all approximations at the resolution 2^{-j}.
  - When the resolution decreases, the size of $\mathbf{V}_2^j$ decreases as well.
- The formal definition of a multiresolution approximation $\{\mathbf{V}_2^j\}_{j\in Z}$ of $\mathbf{L^2}(\mathrm{R}^2)$ is a straightforward extension of Definition 7.1 that specifies multiresolutions of $\mathbf{L^2}(\mathrm{R})$.
  - The same causality, completeness, and scaling properties must be satisfied.

Separable spaces and bases

- Tensor product
  - Used to extend spaces of 1D signals to spaces of multi-dimensional signals
  - A tensor product $x_1 \otimes x_2$ between vectors of two Hilbert spaces H_1 and H_2 satisfies the following properties

$$\textit{Linearity}$$

$$\forall \lambda \in C, \lambda(x_1 \otimes x_2) = (\lambda x_1) \otimes x_2 = x_1 \otimes (\lambda x_2)$$

$$\textit{Distributivity}$$

$$(x_1 + y_1) \otimes (x_2 + y_2) = (x_1 \otimes x_2) + (x_1 \otimes y_2) + (y_1 \otimes x_2) + (y_1 \otimes y_2) +$$

  - This tensor product yields a new Hilbert space $H = H_1 \otimes H_2$ including all the vectors of the form $x_1 \otimes x_2$ where $x_1 \in H_1$ and $x_2 \in H_2$ as well as a linear combination of such vectors
  - An inner product for H is derived as $\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle = \langle x_1, y_1 \rangle_{H_1} \langle x_2, y_2 \rangle_{H_2}$

Separable bases

- Theorem A.3 Let $H = H_1 \otimes H_2$. If $\left\{e^1{}_n\right\}_{n\in N}$ and $\left\{e^2{}_n\right\}_{n\in N}$ are Riesz bases of H_1 and H_2, respectively, then $\left\{e^1{}_n \otimes e^2{}_m\right\}_{n,m\in N^2}$ is a Riesz basis for H. If the two bases are orthonormal then the tensor product basis is also orthonormal.

→ To any wavelet orthonormal basis one can associate a separable wavelet orthonormal basis of $L^2(R^2)$ $\{\psi_{j,n}(x), \psi_{l,m}(y)\}_{(j,n,l,m)\in Z^4}$

However, wavelets $\psi_{j,n}(x)$ and $\psi_{l,m}(x)$ mix the information at *two different scales* along x and y, which often we want to avoid.

Separable Wavelet bases

- Separable multiresolutions lead to another construction of separable wavelet bases whose elements are products of functions dilated at the same scale.
- We consider the particular case of separable multiresolutions
- A **separable 2D multiresolution** is composed of the tensor product spaces

$$V_j^2 = V_j \otimes V_j$$

- V^2_j is the space of finite energy functions f(x,y) that are linear expansions of separable functions

$$f(x,y) = \sum_n a[n] f_n(x) g_n(y) \qquad f_n \in V_j \quad g_n \in V_j$$

- If $\{V_j\}_{j\in Z}$ is a multiresolution approximation of $L^2(R)$, then $\{V^2{}_j\}_{j\in Z}$ is a multiresolution approximation of $L^2(R^2)$.

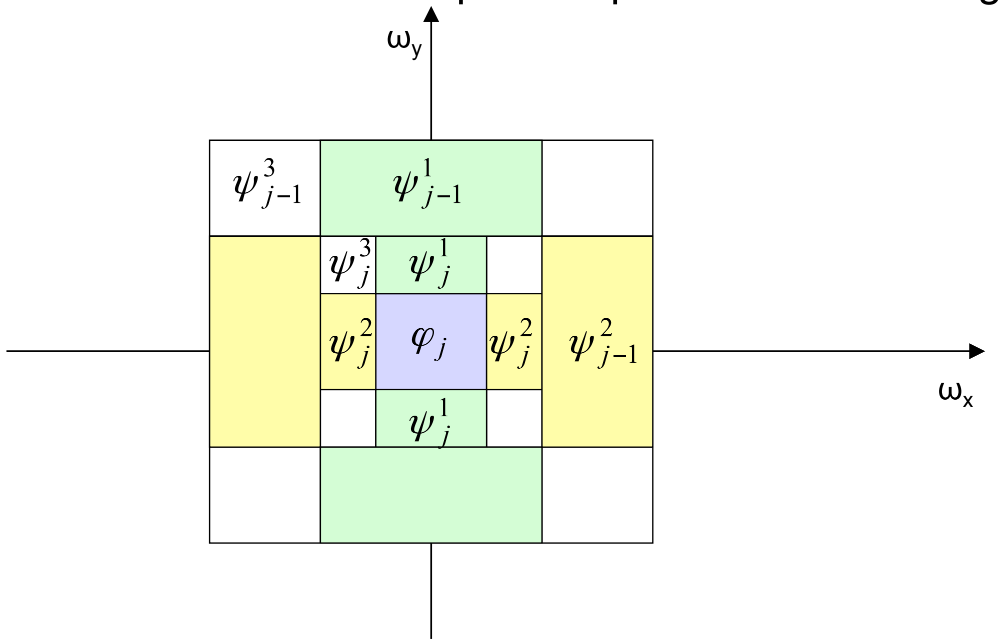
Separable bases

It is possible to prove (Theorem A.3) that

$$\left\{ \varphi_{j,n,m}(x,y) = \varphi_{j,n}(x)\varphi_{j,m}(y) = \frac{1}{2^j}\varphi\left(\frac{x-2^j n}{2^j}\right)\varphi\left(\frac{y-2^j m}{2^j}\right) \right\}_{(n,m)\in Z^2}$$

is an orthonormal basis of V^2_j.

A 2D wavelet basis is constructed with separable products of a scaling function and a wavelet



Examples

EXAMPLE 7.13: Piecewise Constant Approximation

Let $\mathbf{V}_j$ be the approximation space of functions that are constant on $[2^j m, 2^j(m+1)]$ for any $m \in \mathbb{Z}$. The tensor product defines a two-dimensional piecewise constant approximation. The space $\mathbf{V}_j^2$ is the set of functions that are constant on any square $[2^j n_1, 2^j(n_1+1)] \times [2^j n_2, 2^j(n_2+1)]$, for $(n_1, n_2) \in \mathbb{Z}^2$. The two-dimensional scaling function is

$$\phi^2(x) = \phi(x_1)\,\phi(x_2) = \begin{cases} 1 & \text{if } 0 \leqslant x_1 \leqslant 1 \text{ and } 0 \leqslant x_2 \leqslant 1 \\ 0 & \text{otherwise.} \end{cases}$$

EXAMPLE 7.14: Shannon Approximation

Let $\mathbf{V}_j$ be the space of functions with Fourier transforms that have a support included in $[-2^{-j}\pi, 2^{-j}\pi]$. Space $\mathbf{V}_j^2$ is the set of functions the two-dimensional Fourier transforms of which have a support included in the low-frequency square $[-2^{-j}\pi, 2^{-j}\pi] \times [-2^{-j}\pi, 2^{-j}\pi]$. The two-dimensional scaling function is a perfect two-dimensional low-pass filter the Fourier transform of which is

$$\hat{\phi}(\omega_1)\,\hat{\phi}(\omega_2) = \begin{cases} 1 & \text{if } |\omega_1| \leqslant 2^{-j}\pi \text{ and } |\omega_2| \leqslant 2^{-j}\pi \\ 0 & \text{otherwise.} \end{cases}$$

Separable wavelet bases

- A separable wavelet orthonormal basis of **L^2**(R^2) is constructed with separable products of a scaling function and a wavelet .
- The scaling function is associated to a one-dimensional multiresolution approximation {**V_j**}$_{j\in Z}$.
- Let {**V_2^j**}$_{j\in Z}$ be the separable two-dimensional multiresolution defined by

$$V_j^2 = V_j \otimes V_j$$

- Let **W_2^j** be the detail space equal to the orthogonal complement of the lower-resolution approximation space **V_j^2** in **V_{j-1}^2**:

$$V_{j-1}^2 = V_j^2 \oplus W_j^2$$

- To construct a wavelet orthonormal basis of **L^2**(R^2),Theorem 7.25 builds a wavelet basis of each detail space **W^2_j** .

Separable wavelet bases

Theorem 7.25

Let φ be a scaling function and ψ be the corresponding wavelet generating an orthonormal basis of $L^2(R)$. We define three wavelets

$$\psi^1(x,y) = \varphi(x)\psi(y)$$
$$\psi^2(x,y) = \psi(x)\varphi(y)$$
$$\psi^3(x,y) = \psi(x)\psi(y)$$

and denote for 1<=k<=3

$$\psi^k_{j,n,m}(x,y) = \frac{1}{2^j}\psi^k\left(\frac{x-2^j n}{2^j}, \frac{y-2^j m}{2^j}\right)$$

The wavelet family

$$\left\{\psi^1_{j,n,m}(x,y), \psi^2_{j,n,m}(x,y), \psi^3_{j,n,m}(x,y)\right\}_{(n,m)\in Z^2}$$

is an orthonormal basis of W^2_j and

$$\left\{\psi^1_{j,n,m}(x,y), \psi^2_{j,n,m}(x,y), \psi^3_{j,n,m}(x,y)\right\}_{(j,n,m)\in Z^3}$$

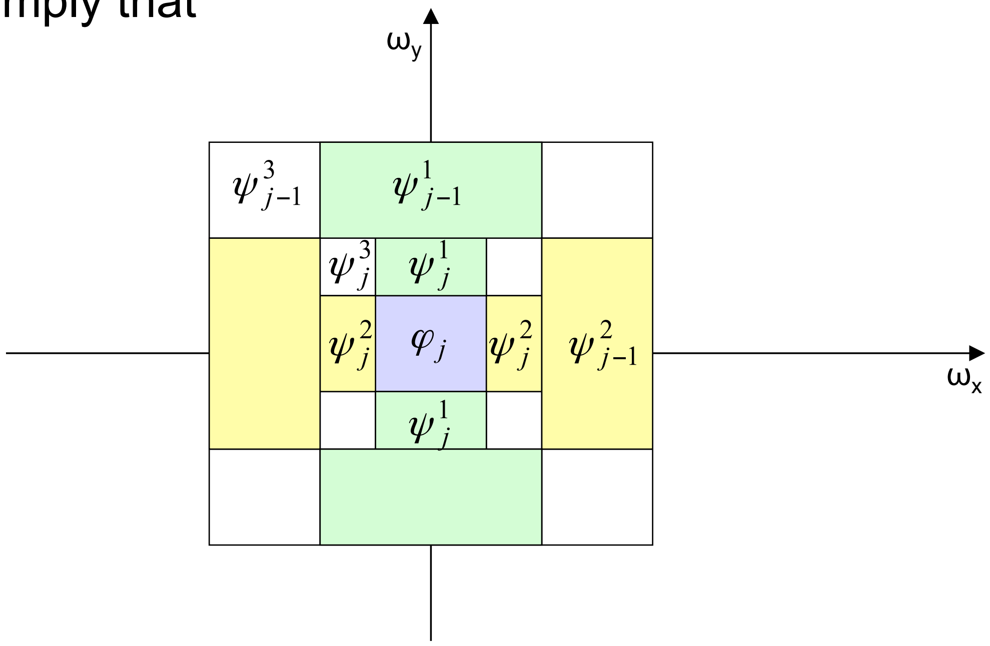
is an orthonormal basis of $L^2(R^2)$

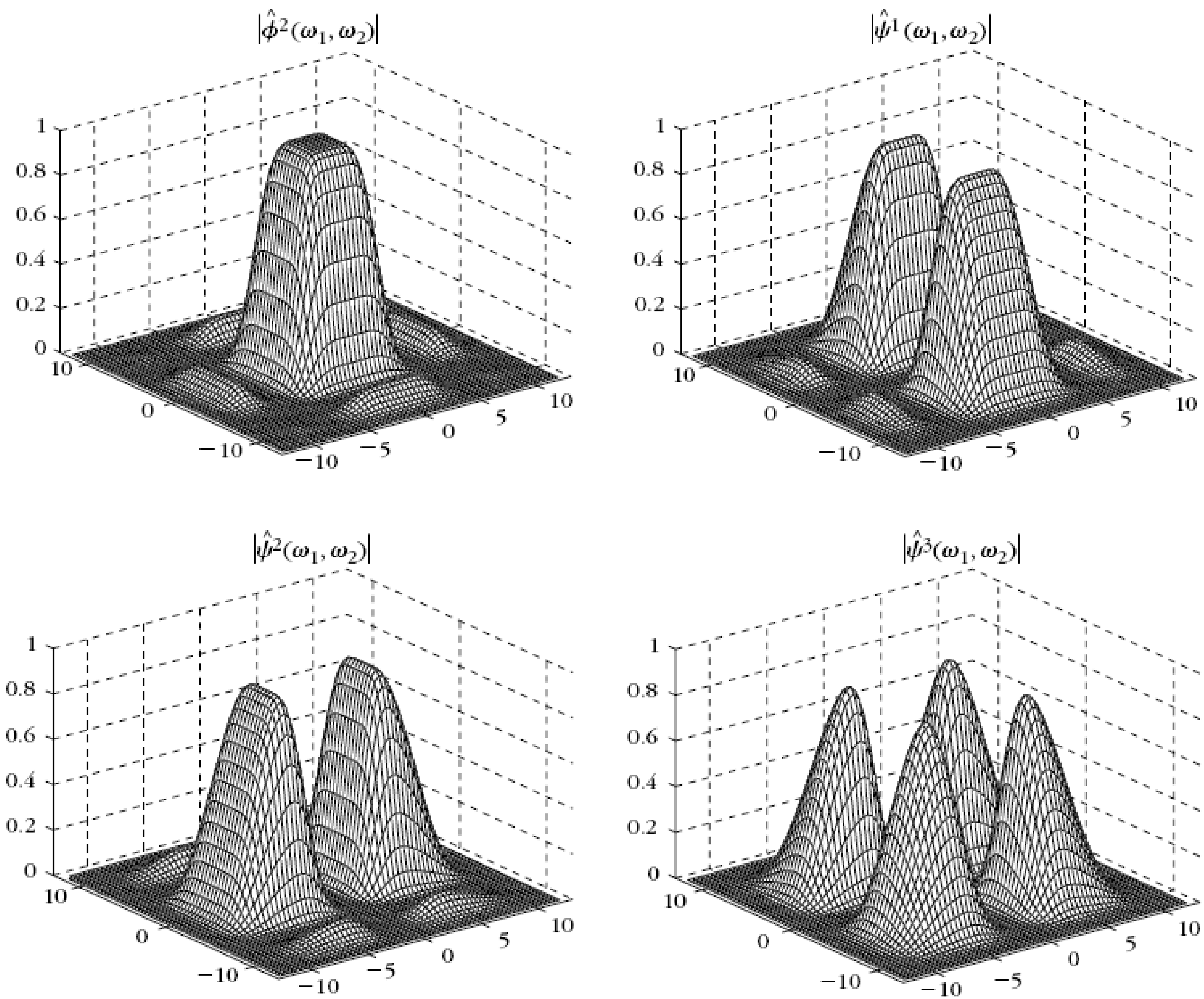
On the same line, one can define **biorthogonal** 2D bases.

Separable wavelet bases

- The three wavelets extract image details at different scales and in different directions.
- Over positive frequencies, $\hat{\varphi}(\omega)$ and $\hat{\psi}(\omega)$ have an energy mainly concentrated, respectively,on $[0,\pi]$ and $[\pi,2\pi]$.
- The separable wavelet expressions imply that

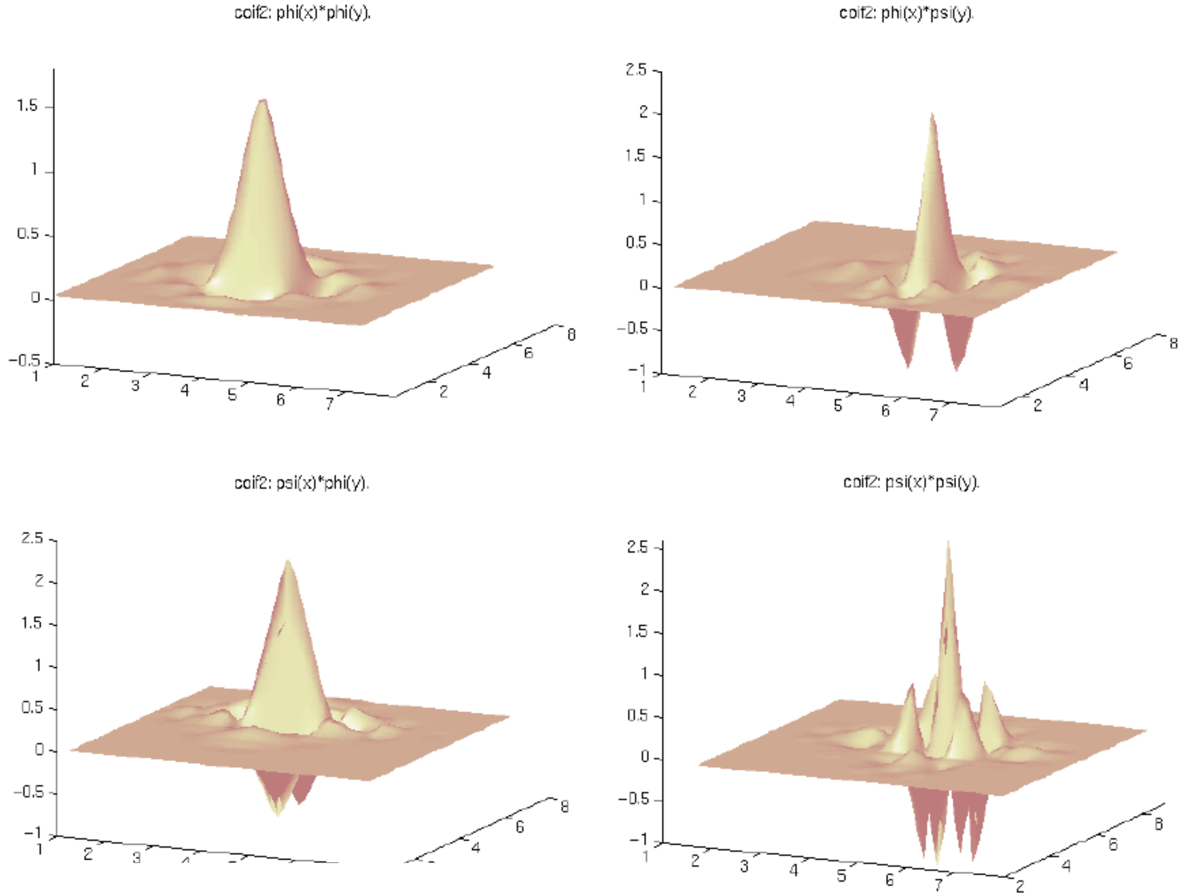
$$\hat{\psi}^1(\omega_x,\omega_y) = \hat{\varphi}(\omega_x)\hat{\psi}(\omega_y)$$
$$\hat{\psi}^2(\omega_x,\omega_y) = \hat{\psi}(\omega_x)\hat{\varphi}(\omega_y)$$
$$\hat{\psi}^3(\omega_x,\omega_y) = \hat{\psi}(\omega_x)\hat{\psi}(\omega_y)$$




$|\hat{\phi}^2(\omega_1,\omega_2)|$
$|\hat{\psi}^1(\omega_1,\omega_2)|$
$|\hat{\psi}^2(\omega_1,\omega_2)|$
$|\hat{\psi}^3(\omega_1,\omega_2)|$

Bi-dimensional wavelets

$$\varphi(x,y) = \varphi(x)\varphi(y)$$
$$\psi^1(x,y) = \varphi(x)\psi(y)$$
$$\psi^2(x,y) = \psi(x)\varphi(y)$$
$$\psi^3(x,y) = \psi(x)\psi(y)$$



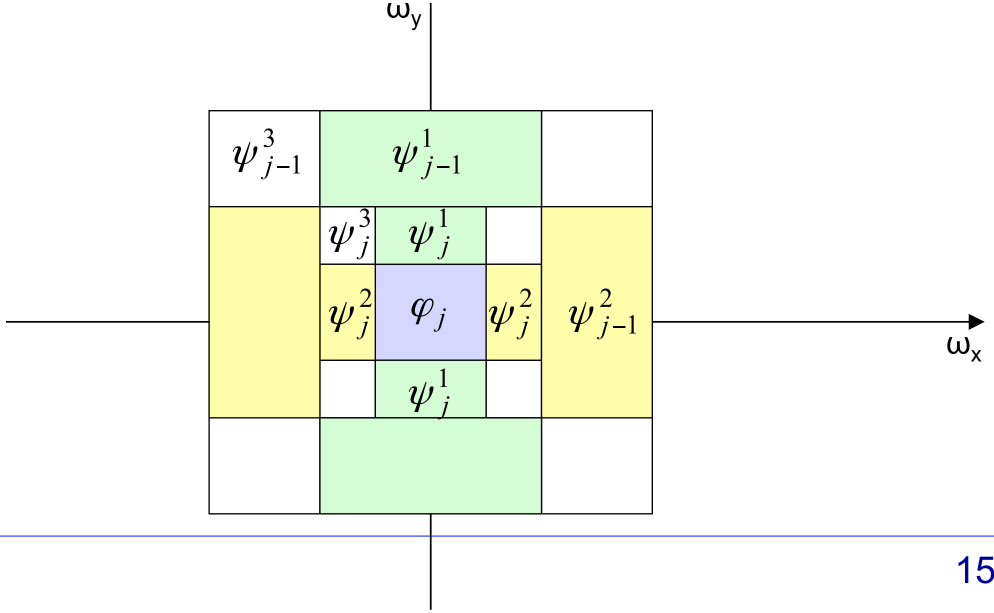
$$\frac{1}{\sqrt{a_1 a_2}}\psi\left(\frac{x_1-b_1}{a_1}, \frac{x_2-b_2}{a_2}\right) \text{ where } (x = (x_1,x_2) \in R^2)$$

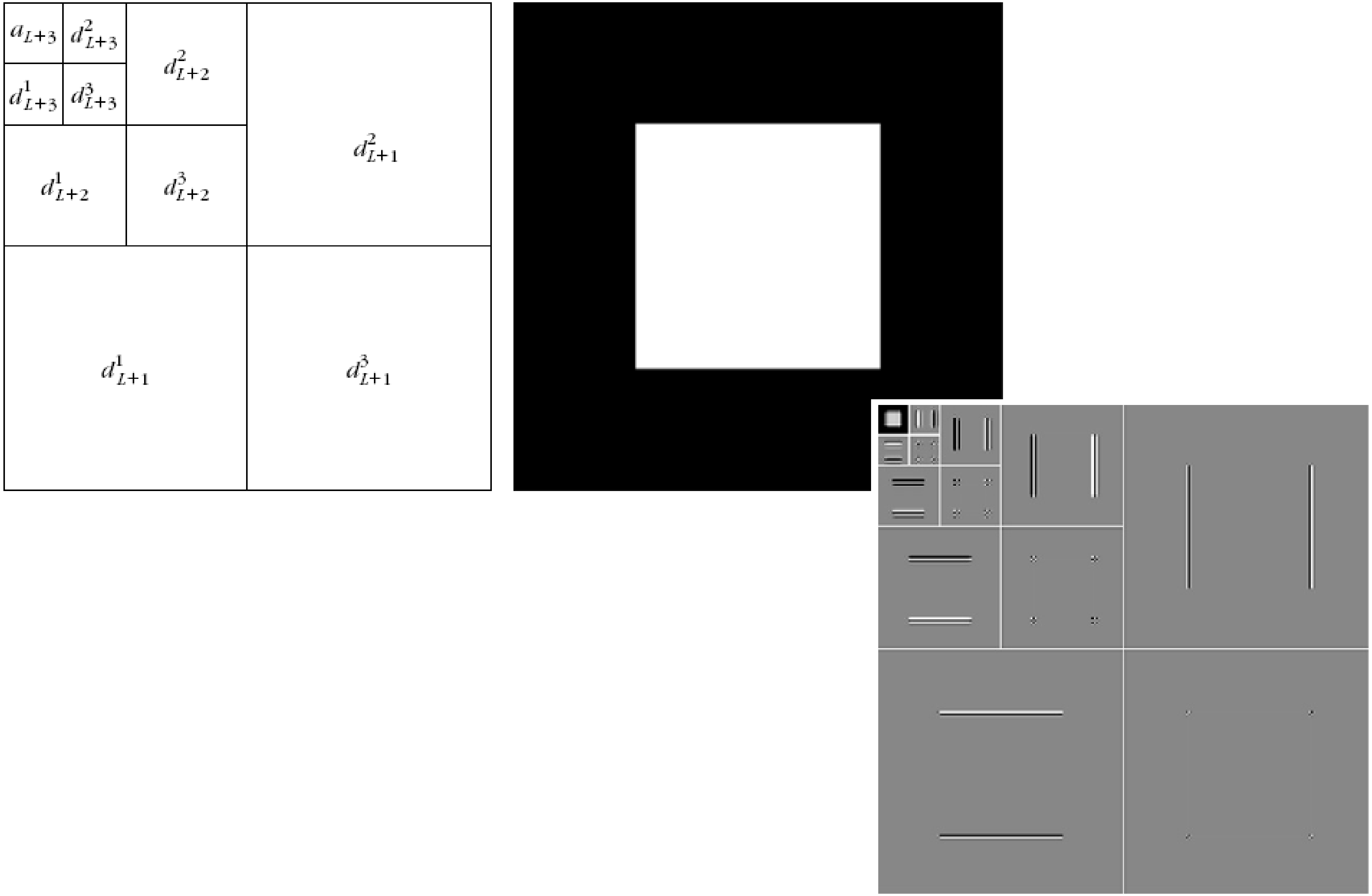
Example: Shannon wavelets

EXAMPLE 7.16

For a Shannon multiresolution approximation, the resulting two-dimensional wavelet basis paves the two-dimensional Fourier plane (ω_1, ω_2) with dilated rectangles. The Fourier transforms $\hat{\phi}$ and $\hat{\psi}$ are the indicator functions of $[-\pi, \pi]$ and $[-2\pi, -\pi] \cup [\pi, 2\pi]$, respectively. The separable space $\mathbf{V}_j^2$ contains functions with a two-dimensional Fourier transform support included in the low-frequency square $[-2^{-j}\pi, 2^{-j}\pi] \times [-2^{-j}\pi, 2^{-j}\pi]$. This corresponds to the support of $\hat{\phi}_{j,n}^2$ indicated in Figure 7.23.

The detail space $\mathbf{W}_j^2$ is the orthogonal complement of $\mathbf{V}_j^2$ in $\mathbf{V}_{j-1}^2$ and thus includes functions with Fourier transforms supported in the frequency annulus between the two squares $[-2^{-j}\pi, 2^{-j}\pi] \times [-2^{-j}\pi, 2^{-j}\pi]$ and $[-2^{-j+1}\pi, 2^{-j+1}\pi] \times [-2^{-j+1}\pi, 2^{-j+1}\pi]$.




a_{L+3}
d^2_{L+3}
d^1_{L+3}
d^3_{L+3}
d^2_{L+2}
d^1_{L+2}
d^3_{L+2}
d^2_{L+1}
d^1_{L+1}
d^3_{L+1}

Biorthogonal separable wavelets

Let $\varphi, \psi, \tilde{\varphi}$ and $\tilde{\psi}$ be a two dual pairs of scaling functions and wavelets that generate a biorthogonal wavelet basis of $\mathrm{L}^2(\mathbb{R})$.

The dual wavelets of ψ^1, ψ^2 and ψ^3 are

$$\tilde{\psi}^1(x,y) = \tilde{\varphi}(x)\tilde{\psi}(y)$$

$$\tilde{\psi}^2(x,y) = \tilde{\psi}(x)\tilde{\varphi}(y)$$

$$\tilde{\psi}^3(x,y) = \tilde{\psi}(x)\tilde{\psi}(y)$$

$$\psi^k_{j,n,m}(x,y) = \frac{1}{2^j}\psi^k\left(\frac{x-2^j n}{2^j}, \frac{y-2^j m}{2^j}\right)$$

One can verify that

$$\left\{\psi^1_{j,n}, \psi^2_{j,n}, \psi^3_{j,n}\right\}_{j,n\in \mathrm{Z}^3}$$

and

$$\left\{\tilde{\psi}^1_{j,n,m}, \tilde{\psi}^2_{j,n,m}, \tilde{\psi}^3_{j,n,m}\right\}_{j,n,m\in \mathrm{Z}^3}$$

are biorthogonal Riesz basis of $\mathrm{L}^2(R^2)$

Fast 2D Wavelet Transform

$$a_j[n,m] = \langle f, \varphi_{j,n,m} \rangle$$

Approximation at scale j

$$d^k{}_j[n,m] = \langle f, \psi^k{}_{j,n,m} \rangle$$

Details at scale j

$$k = 1,2,3$$

$$[a_J, \{d_j^1, d_j^2, d_j^3\}_{1 \le j \le J}]$$

Wavelet representation

Analysis

$$a_{j+1}[n,m] = a_j * \bar{h}\bar{h}[2n,2m]$$

$$d^1_{j+1}[n,m] = a_j * \bar{h}\bar{g}[2n,2m]$$

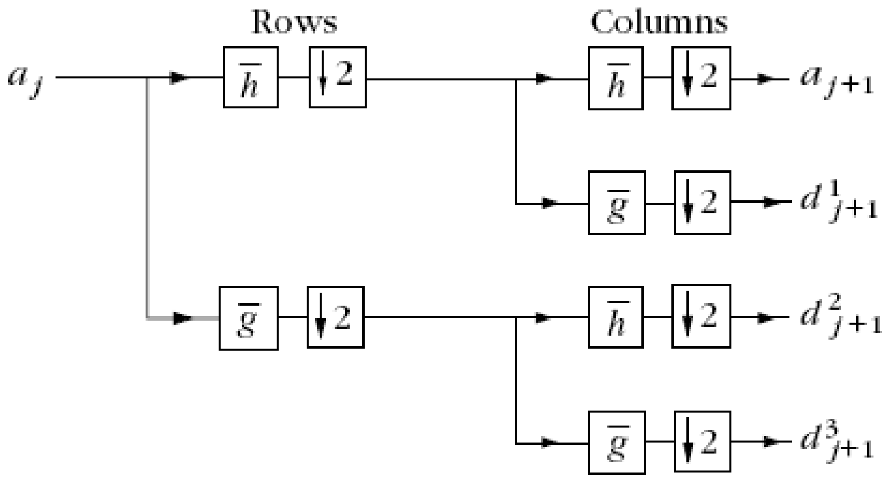
$$d^2_{j+1}[n,m] = a_j * \bar{g}\bar{h}[2n,2m]$$

$$d^3_{j+1}[n,m] = a_j * \bar{g}\bar{g}[2n,2m]$$

Synthesis

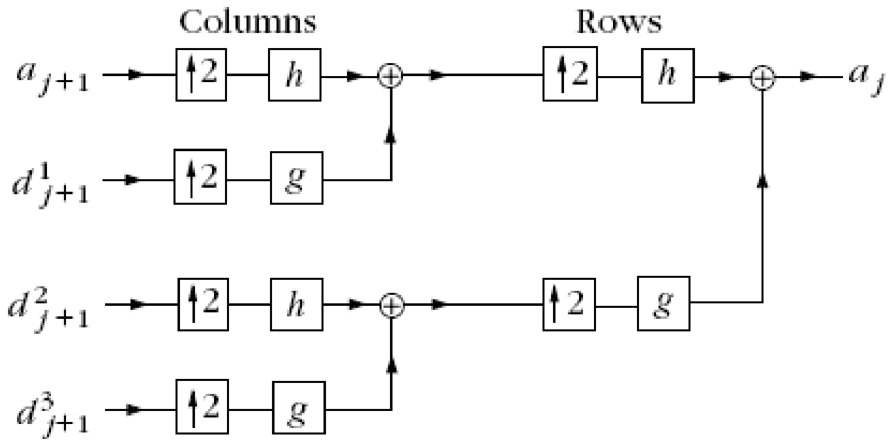
$$a_j[n,m] = \breve{a}_{j+1} * hh[n,m] + \breve{d}^1_{j+1} * hg[n,m] + \breve{d}^2_{j+1} * gh[n,m] + \breve{d}^3_{j+1} * gg[n,m]$$

Fast 2D DWT



Rows
Columns
a_j
$\bar{h}$
$\downarrow 2$
$\bar{g}$
a_{j+1}
d^1_{j+1}
d^2_{j+1}
d^3_{j+1}

(a)



Columns
Rows
a_{j+1}
d^1_{j+1}
d^2_{j+1}
d^3_{j+1}
$\uparrow 2$
h
g
a_j

Finite images and complexity

- When a_L is a finite image of $N=N_1xN_2$ pixels, we face boundary problems when computing the convolutions
  - A suitable processing at boundaries must be chosen

- For square images with N_1N_2, the resulting images a_j and $d_{k,j}$ have $N_1N_2/2^{2j}$ samples. Thus, *the images of the wavelet representation include a total of N samples*.
  - If h and g have size K, one can verify that $2K2^{-2(j-1)}$ multiplications and additions are needed to compute the four convolutions
  - Thus, the wavelet representation is calculated with fewer than $8/3\ KN^2$ operations.
  - The reconstruction of a_L by factoring the reconstruction equation requires the same number of operations.

Separable biorthogonal bases

- One-dimensional biorthogonal wavelet bases are extended to separable biorthogonal bases of $L^2(R^2)$ following the same approach used for orthogonal bases
- Let $\varphi, \psi, \tilde{\varphi}, \tilde{\psi}$ be two dual pairs of scaling functions and wavelets that generate biorthogonal wavelet bases of $L^2(R)$. The dual wavelets of

$$\psi^1(x,y), \psi^2(x,y), \psi^3(x,y)$$

are

$$\tilde{\psi}^1(x,y) = \tilde{\varphi}(x)\tilde{\psi}(y)$$
$$\tilde{\psi}^2(x,y) = \tilde{\varphi}(y)\tilde{\psi}(x)$$
$$\tilde{\psi}^1(x,y) = \tilde{\psi}(x)\tilde{\psi}(x)$$

Separable biorthogonal bases

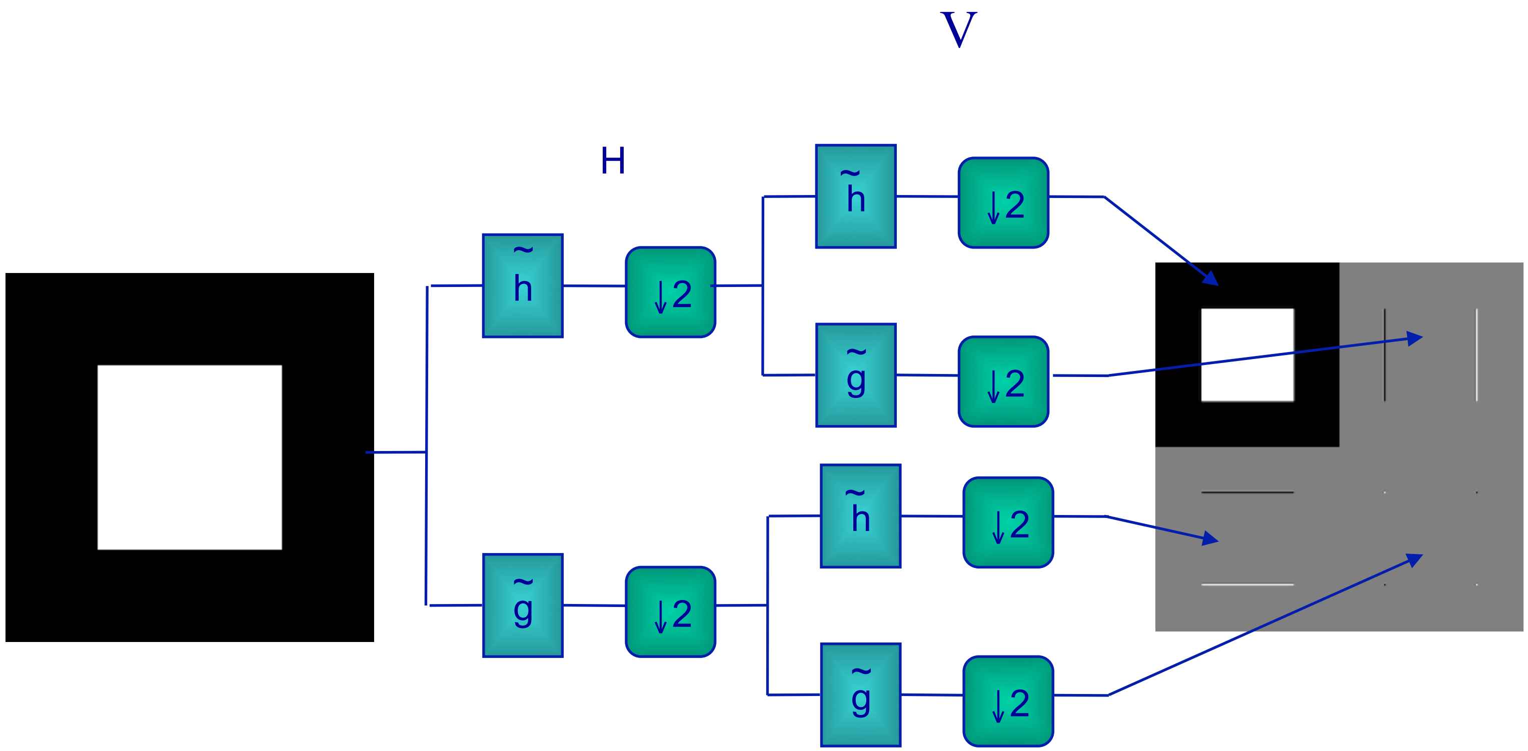
- One can verify that

$$\left\{\psi^1_{j,n,m}, \psi^2_{j,n,m}, \psi^3_{j,n,m}\right\}_{j,n,m \in Z^3}$$

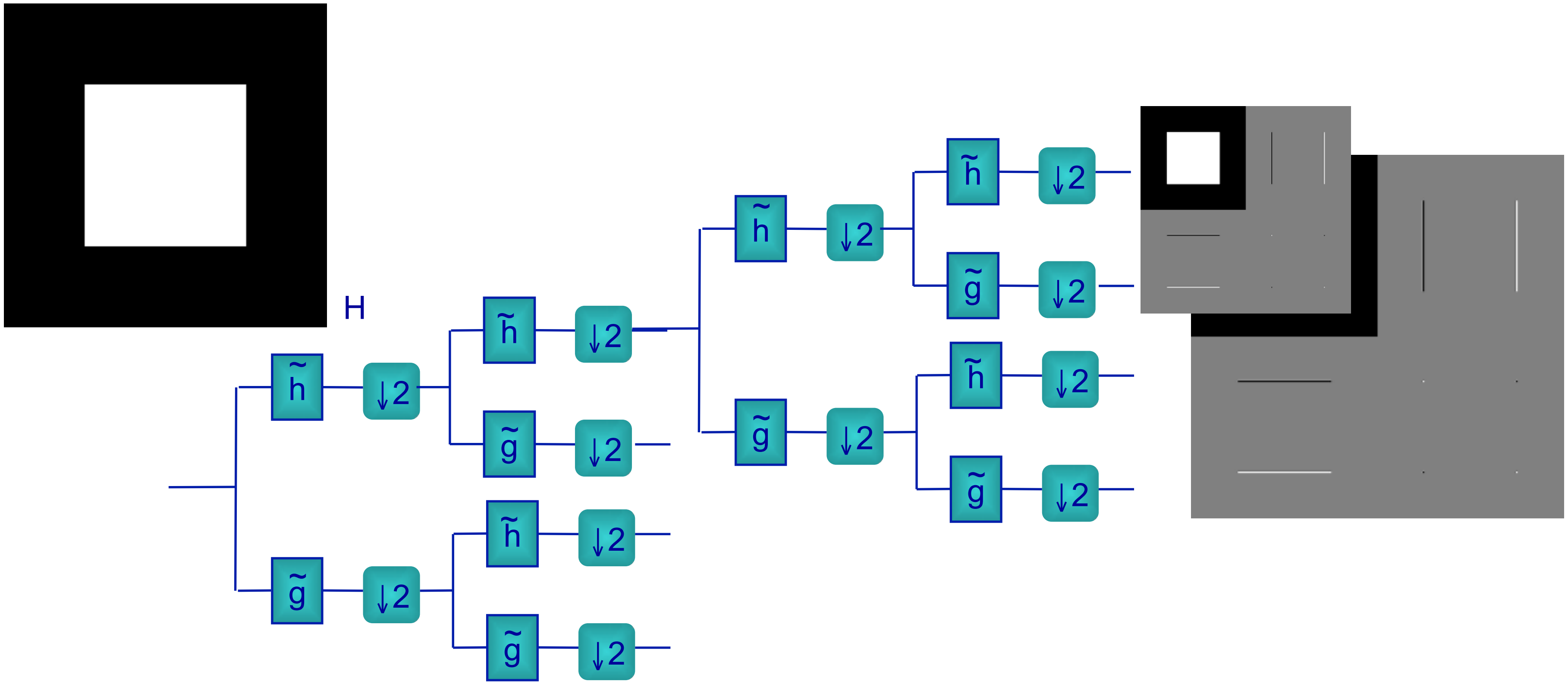
$$\left\{\tilde{\psi}^1_{j,n,m}, \tilde{\psi}^2_{j,n,m}, \tilde{\psi}^3_{j,n,m}\right\}_{j,n,m \in Z^3}$$

- are Riesz basis of $L^2(R^2)$

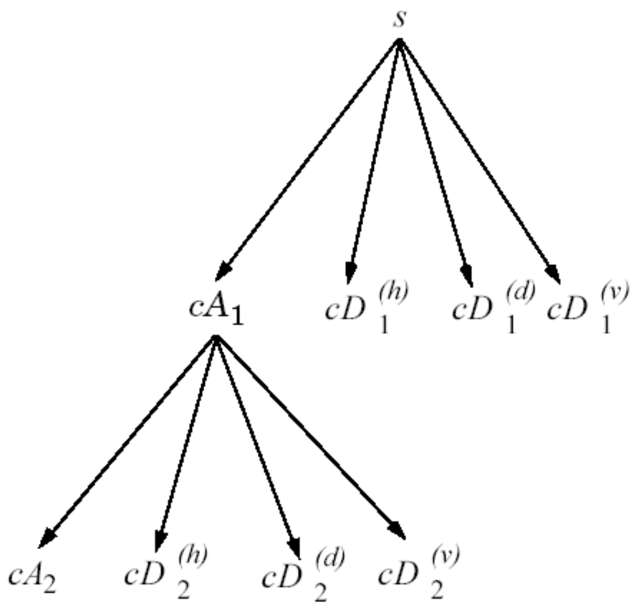
Example

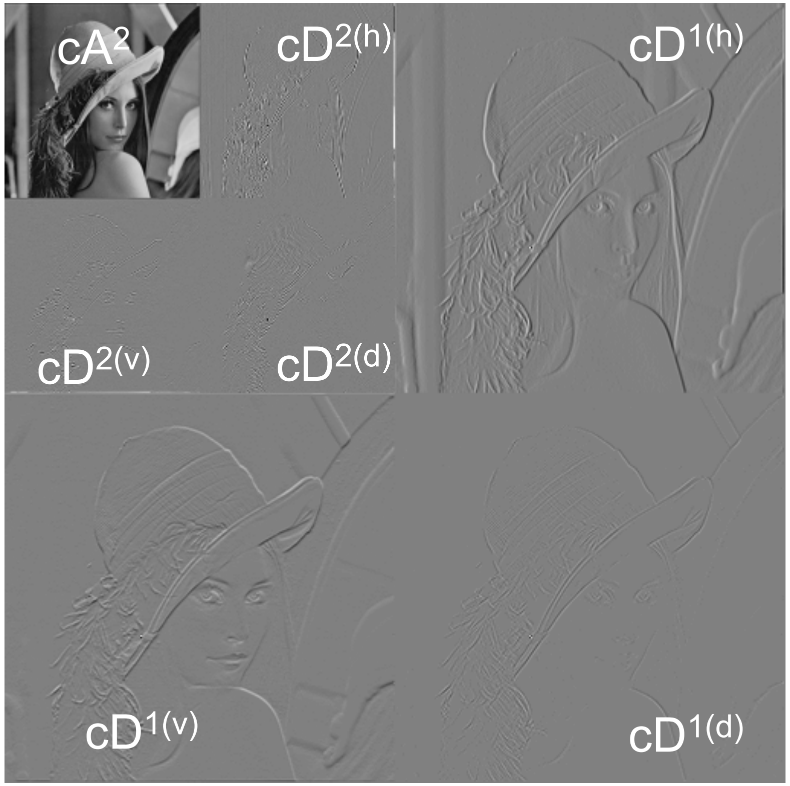


Example



Subband structure for images





Wavelet bases in higher dimensions

- Separable wavelet orthonormal bases of $L^2(R^p)$ are constructed for any $p \geq 2$ with a procedure similar to the two-dimensional extension. Let φ be a scaling function and ψ a wavelet that yields an orthogonal basis of $L^2(R)$.
- We denote θ0=φ and θ1 = ψ. To any integer $0 \leq \varepsilon < 2^p$ written in binary form $\varepsilon = \varepsilon_1, .. \varepsilon_p$ we associate the p-dimensional functions defined in $x = (x_1. . . , x_p)$ by

$$\psi^{\varepsilon}(x) = \vartheta^{\varepsilon_1}(x_1)\ldots\vartheta^{\varepsilon_n}(x_p)$$

- For ε=0 we obtain the p-dimensional scaling function

$$\psi^{0}(x) = \varphi(x_1)\ldots\varphi(x_p)$$

- Non-zero indexes ε correspond to 2^p-1 wavelets. At any scale 2^j and for $n=(n_1, . . . , n_p)$ we denote

$$\psi^{\varepsilon}_{j,n}(x) = 2^{-pj/2}\psi^{\varepsilon}\left(\frac{x_1 - 2^j n_1}{2^j}, \ldots, \frac{x_p - 2^j n_p}{2^j}\right)$$

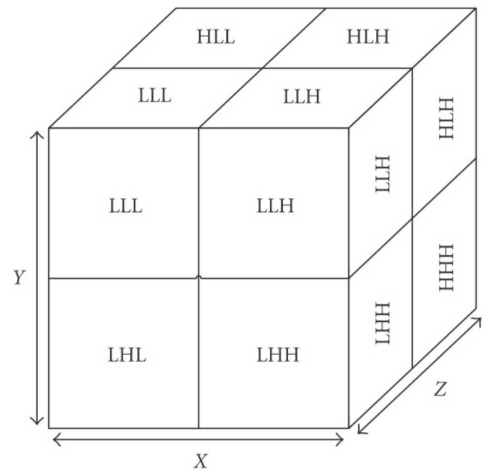
Wavelet bases in higher dimensions

- Theorem 7.25 The family obtained by dilating and translating the 2^p-1 wavelets for ε different from zero

$$\left\{\psi_{j,n}^{\varepsilon}(x)\right\}_{1\le\varepsilon<2^p,(j,n)\in Z^{p+1}}$$

is an orthonormal basis for $L^2(R^p)$.

- 3D DWT



3D DWT

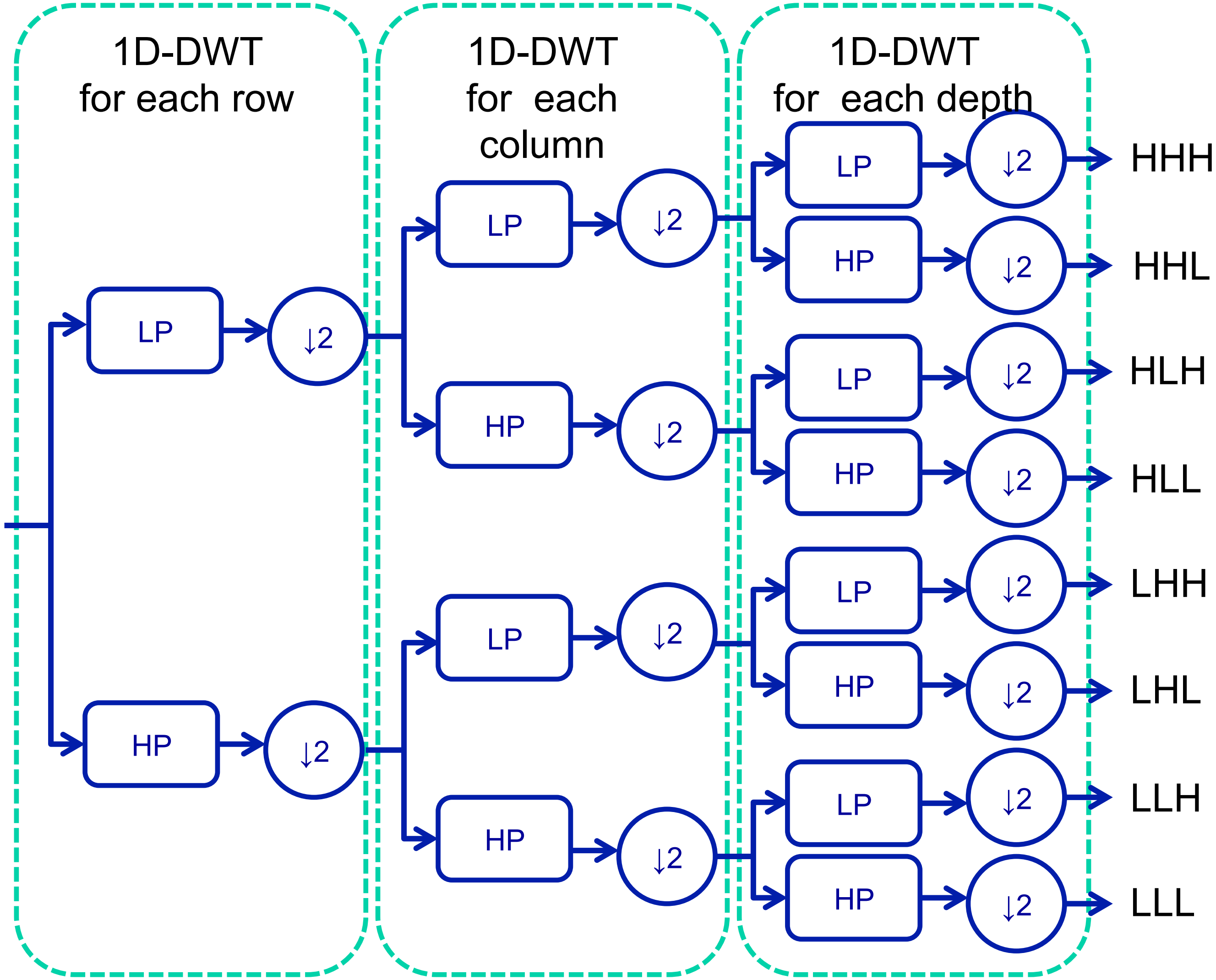
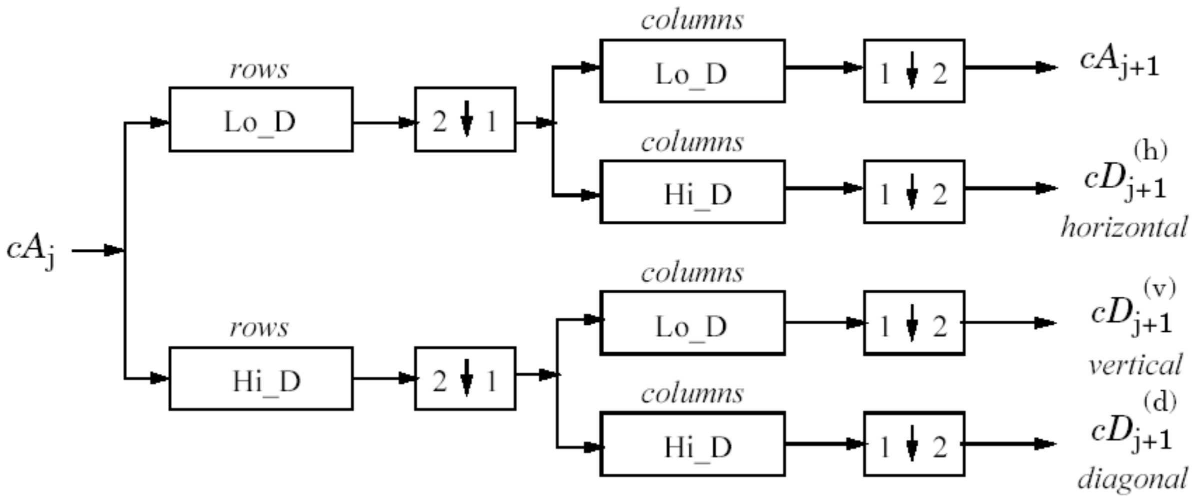


Fig. The filter architecture for 3D wavelet transform

Matlab notations

Decomposition Step



where

2 ↓ 1	Downsample columns: keep the even indexed columns.
1 ↓ 2	Downsample rows: keep the even indexed rows.
rows X	Convolve with filter X the rows of the entry.
columns X	Convolve with filter X the columns of the entry.

Initialization $CA_0 = s$ for the decomposition initialization.

Matlab notations