TOPOLOGIA E GEOMETRIA DIFFERENZIALE
a.a. 2009/10 Prof. M. Spera
Lezione VIII

Il principio dell'argomento
(indicatore logaritmico)
caso particolare

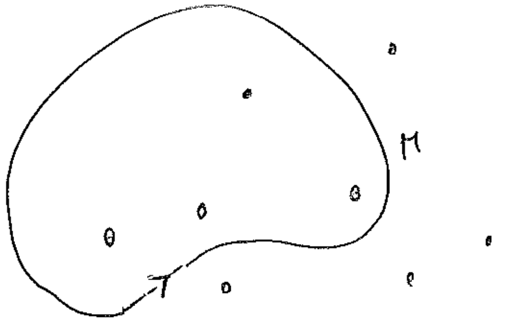
P, Q polinomi $\qquad \deg P = n$, $\deg Q = m \qquad n, m \geq 1$

$$P(z) = A_n \prod (z - a_k)^{n_k}$$ ← molteplicità algebrica

radici distinte

$$\sum n_k = n$$

$$Q(z) = B_m \prod (z - b_j)^{m_j} \qquad \sum m_j = m$$



$$f = \frac{P}{Q}$$

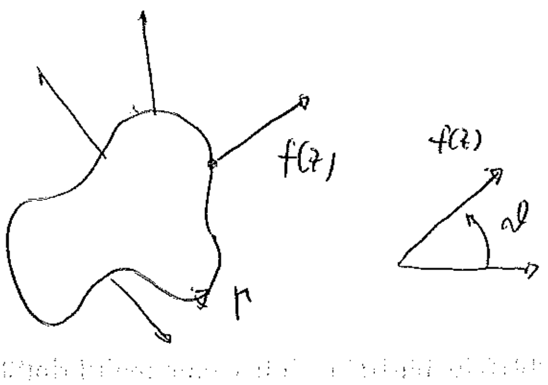
indicatore logaritmico

$$\Delta_\Gamma(f) \equiv \frac{1}{2\pi i}\int_\Gamma \frac{f'}{f}\,dz = n_\Gamma - m_\Gamma$$

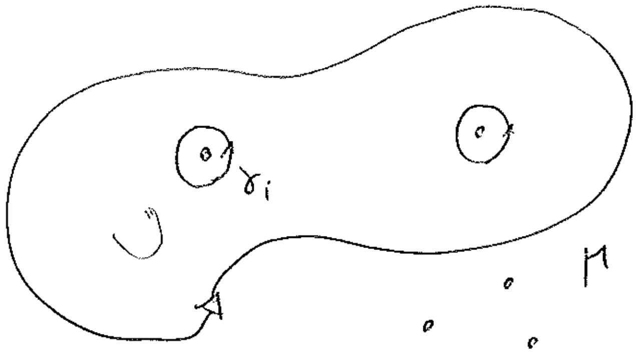
numero degli zeri (rispettivi) racchiusi da Γ

$$\equiv \frac{1}{2\pi i}\int_\Gamma d\log f = d\log|f| + i\,\underset{\sim}{\overset{\varphi}{d\arg(f)}}$$

$$= \frac{1}{2\pi}\left\{\begin{array}{l}\text{variazione totale}\\ \text{dell'argomento di}\\ f \text{ lungo } \Gamma\end{array}\right\}$$



Dimostrazione



$f \in \mathcal{H}(\Omega \setminus \{a\})$

$\mathrm{Res}_a(f) := \int_\gamma f(z)\,dz$

residuo di f in a [residue]

semplice chiusa gir. regolare (più correttamente, residuo della 1-forma $f(z)dz$ in a)

$$\frac{1}{2\pi}\int_\Gamma d\log f \underset{\text{(Green)}}{=\dots=} \frac{1}{2\pi}\sum_i \int_{\gamma_i} d\log f =$$

$$= \frac{1}{2\pi}\sum_i \int_{\gamma_i} \left(\sum n_k\, d\log(z-a_k) - \sum m_j\, d\log(z-b_j)\right)$$

ma $\displaystyle\sum_i \int_{\gamma_i} n_k\, d\log(z-a_k) = \dots = \sum_i n_k\,\delta_{ik} = n_k$

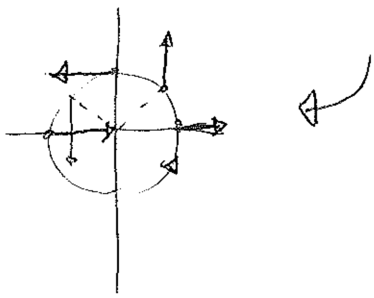
da cui si giunge facilmente alla conclusione

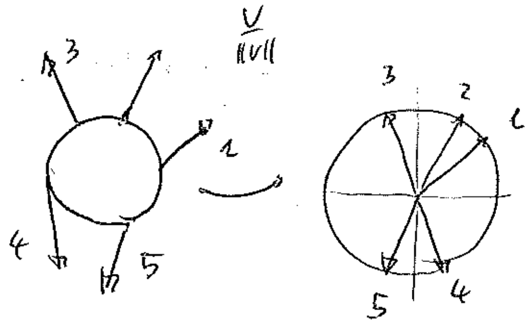
Esempi $\quad f(z) = z^n \quad n \in \mathbb{Z} \qquad \vec{V} = (\mathrm{Re} f, \mathrm{Im} f)$

$$d \log z^n = n \, d\log z$$

$p = 0$ origine $\quad$ pto critico $\quad$ (in generale $\{z : f(z) = 0\}$)

$$\mathrm{ind}_0(\vec{V}) \equiv \mathrm{ind}_0(f) = \frac{1}{2\pi i}\int_{C_1} d\log z^n = n$$



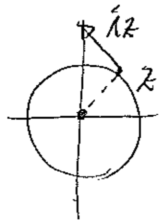


$$f(z) = i \cdot z \qquad \vec{V} = (-\mathrm{Im} z, \mathrm{Re} f)$$

$z = 0$ pto critico

$$i(x + iy) = -y + ix$$

$$d\log(iz) = \ldots = d\log z$$



In generale consideriamo $\vec{v} : (x,y) \to \vec{v}(x,y)$

campo vett. liscio su (un aperto di) $\mathbb{R}^2$ $= (X(x,y), Y(x,y))$

$p \in \mathbb{R}^2$ è detto critico per $\vec{v}$ se $\vec{v}(p) = \underline{0}$

Definiamo $i_p(\vec{v}) = \dfrac{1}{2\pi} \displaystyle\int_{\gamma^+} \frac{Y\,dX - X\,dY}{X^2+Y^2}$

γ^+ omotopa ad una circ. descritta una volta in senso antiorario

indice di $\vec{v}$ relativo a p

(o anche indice di p rel. a $\vec{v}$)

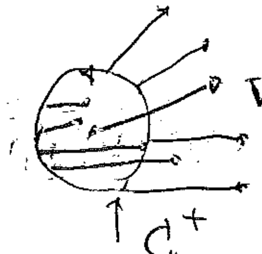
ha senso $\forall\, p$

$$\parallel\!\!\!\,|$$

$\mathrm{ind}_\Gamma(0)$ con $\Gamma = \vec{v}(\gamma)$

(ind è un invariante omotopico (ed omologico))

Se p non è critico, si ha subito $i_p(\vec{v}) = 0$: infatti

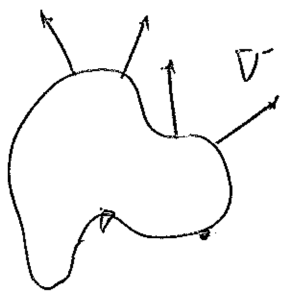


per ragioni di continuità, $\vec{v}$ rimane loc. non nullo e la variazione angolare totale lungo C_r^+ è 0

Se p ha un # finito di pti critici, si pone

$i(\vec{v}) = \sum_p i_p(\vec{v})$

indice di $\vec{v}$



Dato $\vec{v}$ definito lungo γ (semplice, chiusa..)

Si def. $\mathrm{ind}_\gamma(\vec{v}) := \dfrac{1}{2\pi} \displaystyle\int_\gamma \frac{X\,dY - Y\,dX}{X^2+Y^2}$

indice di $\vec{v}$ lungo γ i.e. stessa formula

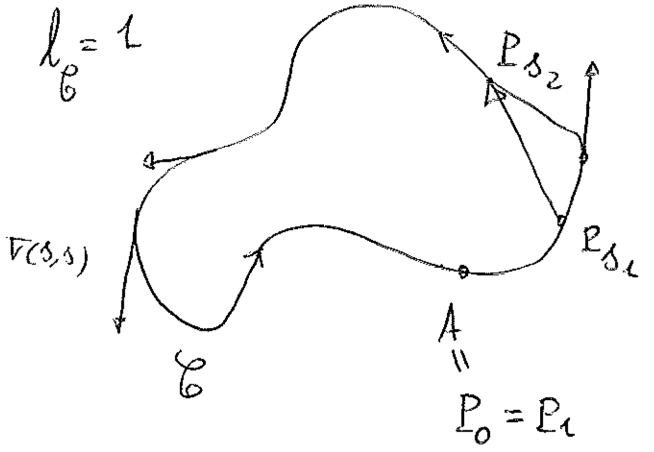
In part. se $\vec{v} = \underline{t}$ (v. tangente), è definito $i(\gamma)$ indice di γ "tout court"

L' "Umlaufsatz" di Hopf : γ semplice, chiusa, gen. regolare $\Rightarrow i(\gamma) = +1$

[eq: $\frac{1}{2\pi}\int k\,ds = +1$ v. corso di Geometria]

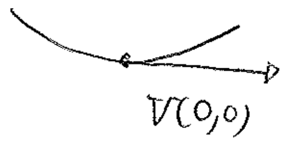
curvatura con segno

$l_{\mathscr{C}} = 1$



$$\bar{V}(s_1, s_2) := \frac{\overrightarrow{P_{s_1}P_{s_2}}}{\|\overrightarrow{P_{s_1}P_{s_2}}\|}$$

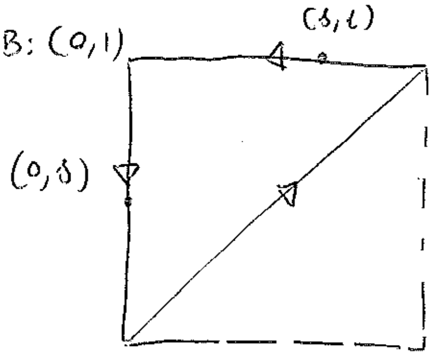
$$\bar{V}(s,s) = \underline{t}(s)$$



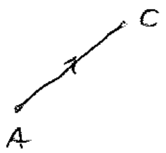
$\bar{V}$ è definito su $\{0 \le s_1 \le s_2 \le 1\} = \nabla$

ed è sempre $\neq \underline{0}$ $\Rightarrow$ è privo di pti critici.

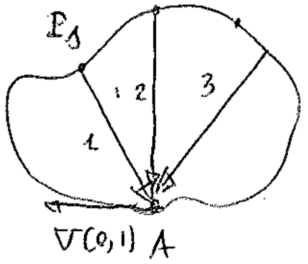
Consideriamo la variazione angolare totale di $\bar{V}$ da (1,1) lungo il circuito in figura



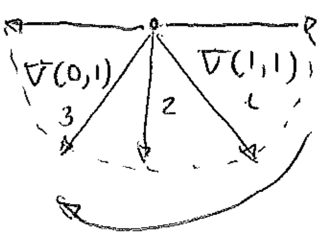
variazione su AC = $2\pi\, i(\gamma)$



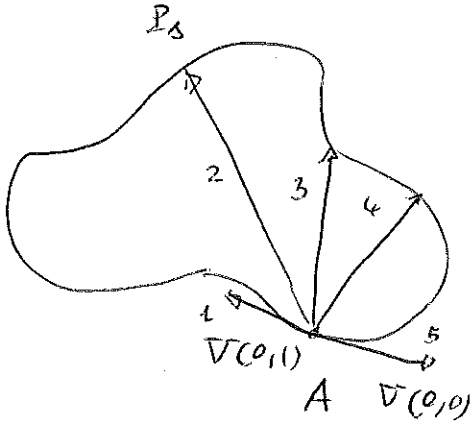
variazione su CB = $-\pi$ (da C a B)

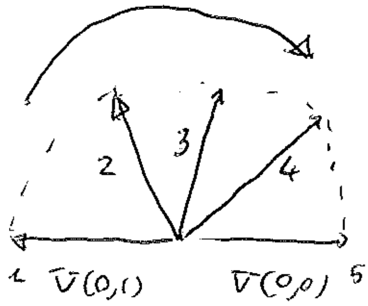


$\bar{V}(s,1) = \overrightarrow{P_sA}$



<u>variazione</u> <u>su</u> <u>BA</u> = $-\pi$ (da B ad A)





Si ha allora

$$0 = 2\pi\, i(\gamma) - \pi - \pi$$

$$= 2\pi\,(i(\gamma) - 1)$$

$$\Rightarrow \quad i(\gamma) = 1$$

★★ Teorema (Poincaré - Bendixon)

$$i(\bar{V}) \neq 0 \quad \text{in} \quad \mathcal{U} \subset \mathbb{R}^2$$

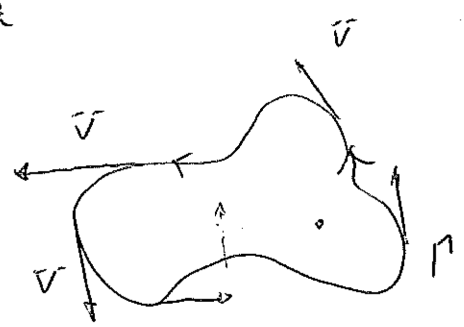
aperto connesso

$$\Rightarrow \quad \exists \ p \in \mathcal{U}$$

critico per $\bar{V}$ $\quad (\bar{V}(p) = 0)$

Dim. Se p non è critico è $i_p(\bar{V}) = 0$ □

Conseguenza (ancora detta teorema di P-B)



Se il sistema dinamico $\begin{cases} \dot{x} = X(x,y) \\ \dot{y} = Y(x,y) \end{cases}$

ammette un'orbita chiusa Γ (Γ semplice), allora $\exists \ p$ critico all'interno.

$$\text{ind} \ \bar{V} = +1 \quad (\text{Umlaufsatz})$$

$\Rightarrow$ (P-B) $\exists \ p$ critico all'interno di Γ.

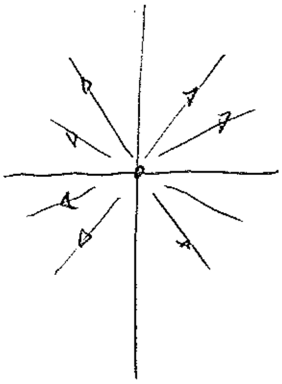
☆ <u>sistemi</u> <u>dinamici</u> <u>planari</u>

esempi

planar dynamical systems

$$\begin{cases} \dot{x} = x \\ \dot{y} = y \end{cases} \qquad x = x_0 e^t \qquad y = y_0 e^t$$

$\bar{V}(x,y) = (x,y)$



"nodo instabile"

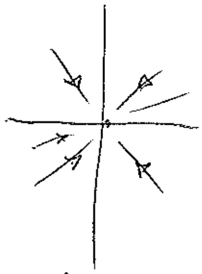
$\mathrm{ind}_0(\bar{V}) = \ldots = +1$

☆☆ ritratto di fase (phase portrait)

$$\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$$

"nodo stabile"

$\mathrm{ind}_0(\bar{V}) = +1$



notare!

$$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$$

$\ddot{x} = -\dot{y} = -x$

$\Rightarrow \ddot{x} + x = 0$

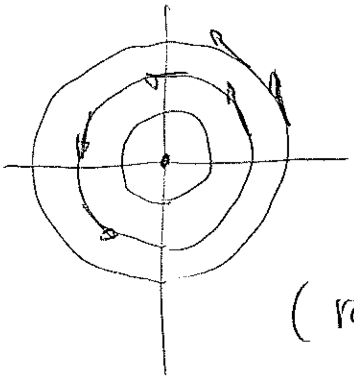
☆ oscillatore armonico

$$\begin{cases} x = A\cos(t+a) \\ y = A\sin(t+a) \end{cases}$$

A: ampiezza

a: fase

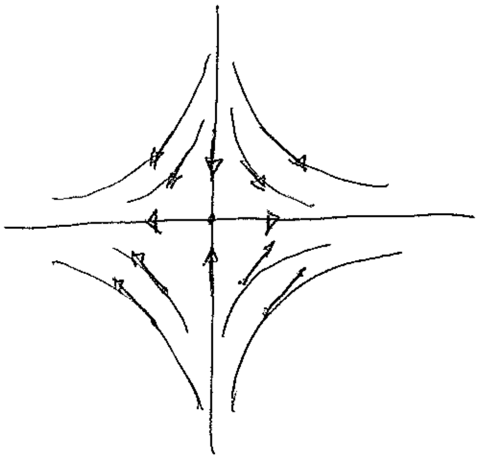
"centro"



ritratto di fase (rotazione uniforme)

$\bar{V} = (-y, x) \qquad \mathrm{ind}_0(\bar{V}) = +1$

$$\begin{cases} \dot{x} = x \\ \dot{y} = -y \end{cases} \qquad \begin{aligned} x &= x_0 e^{t} \\ y &= y_0 e^{-t} \end{aligned}$$

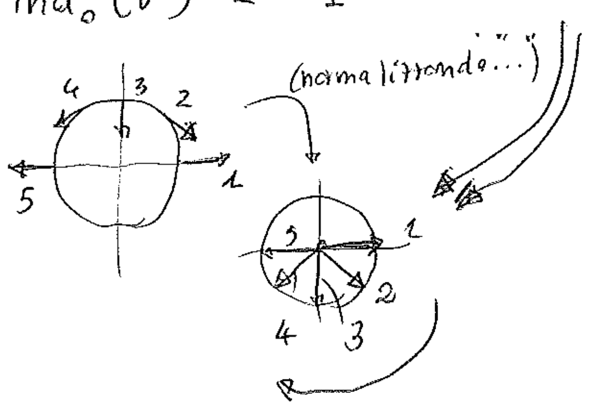


origine: sella

$$\bar{V} = (\overset{X}{x}, \overset{Y}{-y})$$

$$\mathrm{Ind}_0(\bar{V}) = -1$$

è chiaro!

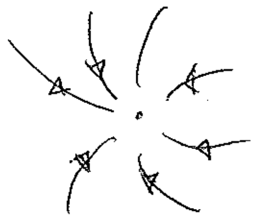


vediamolo analiticamente
check this analytically

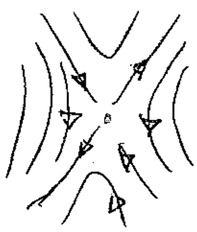
$$X = x$$
$$Y = -y$$

$$\frac{X\,dY - Y\,dX}{X^2+Y^2} = \frac{-x\,dy + y\,dx}{x^2+y^2} = -\frac{x\,dy - y\,dx}{x^2+y^2}$$

$\Rightarrow \ldots \quad \mathrm{Ind}_0(\bar{V}) = -1$



nodo



Sella