GEOMETRIA   Prova scritta del 9/9/2008

Prof. M. Spera

Tempo a disposizione 2h
Le risposte vanno adeguatamente giustificate

① Sia data la superficie cartesiana Σ

$$z = x^2 - 2xy - 2y^2 + x^3 + 1$$

Determinare la curvatura gaussiana, media e le curvature principali in $P_0 : (0,0,1)$.

Calcolare la curvatura normale delle curve su Σ aventi in P_0 la direzione di $v = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

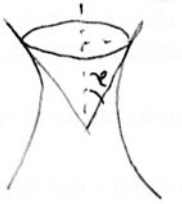
Determinare le direzioni asintotiche e abbozzare il grafico dell'indicatrice di Dupin (in $T_{P_0} \Sigma$)

② Sia dato l'iperboloide di rotazione (a una falda)

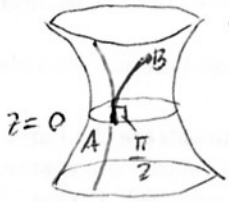
$\Sigma : \quad x^2 + y^2 - z^2 - 1 = 0$. Sia $\mathcal{C} = \Sigma \cap \{z = 1\}$

Calcolare la curvatura geodetica di $\mathcal{C}$ e determinare l'angolo di rotazione determinato dal trasporto // del versore tangente a $\mathcal{C}$ (per un dato orientamento di $\mathcal{C}$).

Determinare la curvatura normale di $\mathcal{C}$.

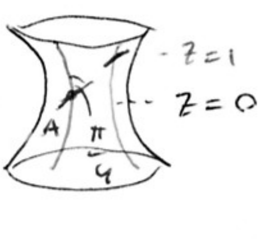


③ Data Σ come in ②,



a) dire se $\widehat{AB}$ può essere una geodetica.

b) Detta $\mathcal{C}$ la geodetica uscente da A con azimut $= \frac{\pi}{4}$, assumendo che incontri un meridiano a quota $z=1$, quale azimut formerà con quest'ultimo?



①

$$z = \overbrace{x^2 - 2xy - 2y^2 + x^3}^{f(x,y)} + 1$$

$$z(x,y) = 1 + (x \quad y)\begin{pmatrix} 1 & -1 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + x^3$$

$$= 1 + \frac{1}{2}(x \quad y)\underbrace{\begin{pmatrix} 2 & -2 \\ -2 & -4 \end{pmatrix}}_{\mathcal{H}_f(0,0) = \mathrm{II}_{P_0}}\begin{pmatrix} x \\ y \end{pmatrix} + x^3$$

P_0: $(0,0,1)$

$(0,0)$ pto critico di f

in questo caso

$$K(0,0) = \det \mathcal{H}_f(0,0) = -8 - 4 = -12$$

↑ curvatura gaussiana

$$(1 \quad 2)\begin{pmatrix} \overset{-2}{\widehat{2-4}} \\ \underset{-10}{\underbrace{-2-8}} \end{pmatrix} = -2 - 20 = -22$$

Sia $\tilde{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$k_n = \frac{\tilde{v}^t \, \mathrm{II}_{P_0} \, \tilde{v}}{\|v\|^2} = \frac{(1 \quad 2)\begin{pmatrix} 2 & -2 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{5}$$

$$= \frac{2(1 - \overbrace{2 \cdot 1 \cdot 2}^{4} - \overbrace{2 \cdot 4}^{8})}{5} = \frac{2(-11)}{5}$$

$$= -\frac{22}{5} \quad V$$

Curvature principali: autovalori di II_{P_0}

$$\begin{vmatrix} 2-\lambda & -2 \\ -2 & -4-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(-4-\lambda) - 4 = 0$$

$$(2-\lambda)(4+\lambda) + 4 = 0$$

$$8 - 4\lambda + 2\lambda - \lambda^2 + 4 = 0$$

$$12 - 2\lambda - \lambda^2 = 0$$

$$\lambda^2 + 2\lambda - 12 = 0$$

$$\lambda = -1 \pm \sqrt{1+12} = -1 \pm \sqrt{13}$$

di segni opposti, come deve essere.

(controllo $k_1 k_2 = (-1-\sqrt{13})(-1+\sqrt{13}) = 1 - 13 = -12$)

curvatura media $H = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}(-2) = -1$

autospazi V_+

$[V_- = V_+^\perp]$

$$[2 - \overbrace{(-1+\sqrt{13})}^{k_2}]x - 2y = 0$$

$$(3 - \sqrt{13})x - 2y = 0$$

$$y = \frac{3-\sqrt{13}}{2}x$$

$$\sqrt{13} = \sqrt{16-3} = 4\sqrt{1-\frac{3}{16}} \sim 4\left(1 - \frac{3}{2 \cdot 16}\right)$$

$$= 4 - \frac{3 \cdot 2}{16} \approx 4 - \frac{3}{8}$$

$$= \frac{29}{8} = 3{,}675$$

$$\frac{3-3{,}6}{2} = \frac{-0{,}6}{2} \sim -0{,}3$$

29 | 8
50 3,625
20
40

[Altro modo: diametri coniugati $\perp$ (assi)]

$$(-m \quad 1)\begin{pmatrix}2 & -2\\ -2 & -4\end{pmatrix}\begin{pmatrix}1\\ m\end{pmatrix} = 0$$

$$(-m \quad 1)\begin{pmatrix}2 \ -2m\\ -2 \ -4m\end{pmatrix} = 0$$

$$-m(2-2m) - 2 - 4m = 0$$

$$-2m + 2m^2 - 2 - 4m = 0$$

$$2m^2 - 6m - 2 = 0$$

$$m^2 - 3m - 1 = 0$$

$$m = \frac{3 \pm \sqrt{9+4}}{2} = \frac{3 \pm \sqrt{13}}{2}$$

Dupin [risp. alle Curv. principali]

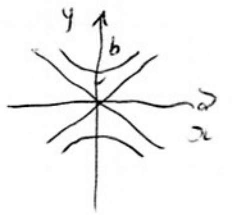
$$k_1 \xi^2 + k_2 \eta^2 = \pm 1$$

$$(-1 - \sqrt{13})\xi^2 + (-1 + \sqrt{13})\eta^2 = \pm 1$$

$$\frac{-\xi^2}{\underbrace{\left(\sqrt{\frac{1}{+1+\sqrt{13}}}\right)}_{a}^2} + \frac{\eta^2}{\underbrace{\left(\sqrt{\frac{1}{-1+\sqrt{13}}}\right)}_{b}^2} = \pm 1$$

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$a = \sqrt{\frac{1}{1+\sqrt{13}}} \qquad b = \sqrt{\frac{1}{\sqrt{13}-1}}$$



as: $y = \pm \frac{b}{a} x$

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$$

$$b \sim \sqrt{\frac{\sqrt{13}+1}{12}} \sim \sqrt{\frac{4,6}{12}}$$

$$\sim \sqrt{3,8} \sim 1,9$$

$46 \, \lfloor 12$
$100 \quad 3,8$
40

$$\sqrt{3,8} = \sqrt{4-0,2} = 2\sqrt{1-\frac{0,2}{4}}$$

$$= 2\left(1 - \frac{0,2}{8}\right)$$

$$\sqrt{\frac{\sqrt{13}-1}{12}} \sim \sqrt{\frac{7,6}{12}} \sim \sqrt{0,2} \sim 0,45$$

$$2 - \frac{0,2}{4}$$

39 L

$$= 2 - \frac{2}{40} = 2 - \frac{1}{20} = \frac{39}{20}$$

$$= 1,95$$

rezioni asintotiche

$$(1 \quad m) \begin{pmatrix} 2 & -2 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ m \end{pmatrix} = 0$$

$$(1 \quad m) \begin{pmatrix} 2-2m \\ -2-4m \end{pmatrix} = 0$$

$[2x^2 - 4xy - 4y^2 = 0]$

$$2 - 4m - 4m^2 = 0$$

$$2-2m + m(-2-4m)$$

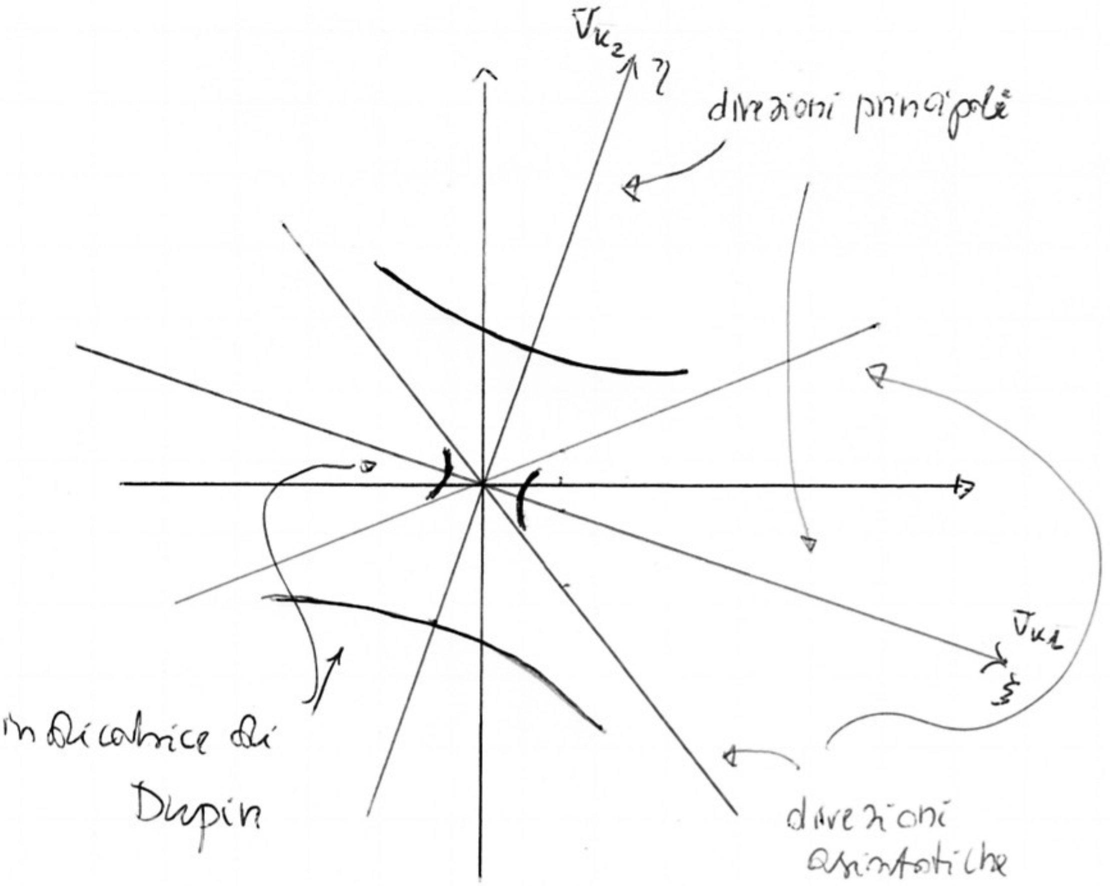
$$-4m^2 - 4m + 2 =$$

$$4m^2 + 4m - 2 = 0$$

$$2m^2 + 2m - 1 =$$

$$2m^2 + 2m - 1 = 0$$

$$m = \frac{-1 \pm \sqrt{1+2}}{2} = \frac{-1 \pm \sqrt{\overset{1,7}{3}}}{2} \begin{cases} \sim -1,35 \\ \sim +0,4 \end{cases}$$

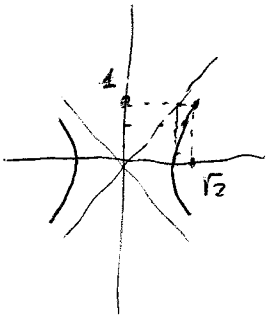


②

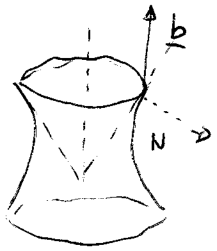
$$x^2+y^2-z^2-1=0$$

$$z^2 = \rho^2-1$$

$$z = \pm\sqrt{\rho^2-1}$$



$$\mathcal{C}: \begin{cases} z = 1 \\ z = \sqrt{-1+x^2+y^2} = \sqrt{-1+\rho^2} \end{cases}$$



$$\begin{cases} z = 1 \\ 1 = -1+\rho^2 \end{cases} \qquad 2 = \rho^2 \quad \Rightarrow \quad \rho = \sqrt{2}$$

$$\mathcal{C}: \begin{cases} z=1 \\ x^2+y^2=2 \end{cases} \qquad k_{\mathcal{C}} = \frac{1}{\rho} = \frac{1}{\sqrt{2}}$$

curvatura geodetica: $$k_g = k_{\mathcal{C}} \cdot \overbrace{\langle \underline{b}, \underline{N} \rangle}^{\text{costante lungo } \mathcal{C}}$$

$$\underline{b} = (0,0,1)$$

basta calcolare $\underline{N}$ in un pto di $\mathcal{C}$.

poniamo $y=0$ sia $P_0: (\sqrt{2}, 0, 1)$

$$z^2 = x^2-1 \qquad ' = \frac{d}{dx}$$

$$2zz' = 2x$$

$$zz' = x \qquad z' = \frac{x}{z} \qquad z' = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$m_\perp = -\frac{1}{\sqrt{2}} \qquad \underline{N} = \frac{\left(1 \quad 0 \quad -\frac{1}{\sqrt{2}}\right)}{\| \cdot \|}$$

$$= \frac{(+\sqrt{2}\ ,\ 0\ ,\ -1)}{\|\qquad\|} = \frac{(+\sqrt{2}\ \ 0\ -1)}{\sqrt{3}}$$

$$\underline{N}_{P_0} = \left(+\sqrt{\frac{2}{3}}\ ,\ 0\ ,\ -\frac{1}{\sqrt{3}}\right)$$

$$\langle \underline{b}, \underline{N} \rangle = -\frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad k_g = -\frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{6}}$$

Da $\dfrac{d\nu}{ds} = k_g$ si ha

successivamente $d\nu = -\dfrac{1}{\sqrt{6}}\, ds$

$$\nu = -\frac{1}{\sqrt{6}}\, l \qquad (\nu_0 = 0 \dots)$$

$$= -\frac{1}{\sqrt{6}}\, 2\pi \cdot \sqrt{2}$$

$$= -\frac{2\pi}{\sqrt{3}} \quad (+\, 2k\pi)$$

↑ opportuno

* variante:

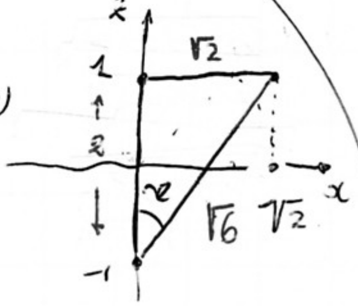
$z - 1 = z' \cdot (x - \sqrt{2})$

$z' = \sqrt{2}$

int. con l'asse z $(x=0)$

$z = 1 + \sqrt{2}\,(-\sqrt{2})$

$= 1 - 2 = -1$



$$\text{trasp. } \| = 2\pi - \beta = 2\pi(1 - \sin\nu) = 2\pi\left(1 - \frac{\sqrt{2}}{\sqrt{6}}\right) =$$

$$= 2\pi\left(1 - \frac{1}{\sqrt{3}}\right) = \boxed{2\pi} - \frac{2\pi}{\sqrt{3}}$$

R trovato prima

Curvatura normale: da $k^2 = k_g^2 + k_n^2$ si ha

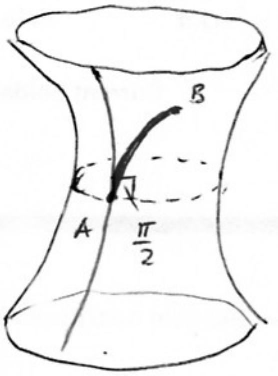
$$k_n = \pm\sqrt{k^2 - k_g^2} = \pm\sqrt{\frac{1}{2} - \frac{1}{6}} = \pm\sqrt{\frac{3-1}{6}}$$

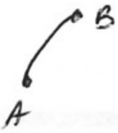
$$= \pm\sqrt{\frac{2}{6}} = \pm\frac{1}{\sqrt{3}}$$

va scelto il segno −

$$\boxed{k_n = -\frac{1}{\sqrt{3}}}$$

③





non può essere una geo-detica

infatti) lo è (meridiano)

e pertanto è l'unica geodetica uscente da A con la data direzione.

In altro modo, da Clairaut si ha

$$\rho \sin \vartheta = \text{cost.}$$

par ↑ azimut

↑

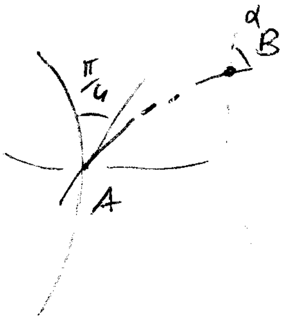
raggio del parallelo

nel nostro caso $\sin \vartheta_{in} = \sin 0 = 0$

$$\rho \sin \vartheta = 0$$

$\Rightarrow$ dato che $\rho > 0$, è $\sin \vartheta \equiv 0$

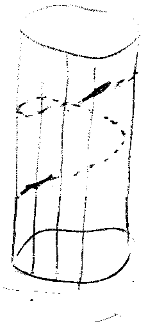
$\Rightarrow$ si trova)



⚠ la geodetica può fare "più giri"

cf. il caso dell'elica cilindrica

non è detto che sia la geodetica minima congiungente A e B



da Clairaut è

$$\underbrace{\rho_0 \sin\frac{\pi}{4}}_{\overset{\shortparallel}{\frac{\sqrt{2}}{2}}} = \rho \sin\alpha$$

$$\Rightarrow \sin\alpha = \frac{1}{\rho\sqrt{2}}$$

se $z=1$ e $\rho=\sqrt{2}$ $\Rightarrow$

$$\sin\alpha = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$

$$\Rightarrow \alpha = \arcsin\frac{1}{2} = \frac{\pi}{6}$$

$$\boxed{\alpha = \frac{\pi}{6}}$$