GEOMETRIA

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$\boxed{\text{Prova scritta del 1° luglio 2011}}$

a.a. 2010/11 - Tempo a disposizione: 2h.

Le risposte vanno adeguatamente giustificate

(1) Si consideri la superficie Σ generata dalla rotazione della curva $\mathcal{C}$: $z = \frac{1}{x}$, $0 < x < 1$ attorno all'asse z. Determinarne le curvature principali e la curvatura gaussiana.

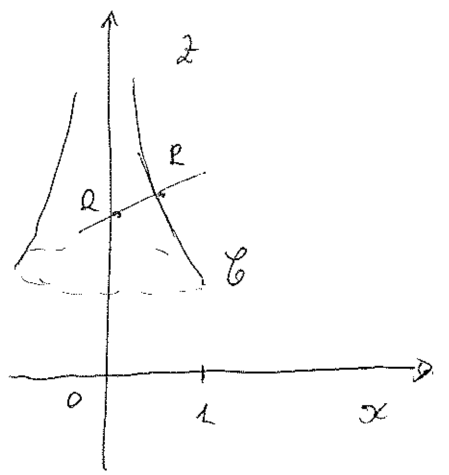
(2) Stesso esercizio, a partire da un'opportuna parametrizzazione (determinando altresì la prima e la seconda forma fondamentale)

(3) Determinare, in due modi, l'evoluta della curva $\mathcal{C}$ dell'es. 1. $\begin{cases} x = t \\ z = \frac{1}{t} \end{cases}$ $0 < t < 1$

Cosa accade per $t \to 0$?

Geometria 1/7/11

(1)



$$z = \frac{1}{x} \qquad 0 < x < 1$$

$$' = \frac{d}{dx}$$

curvatura meridiano k_1: $\quad k_1 = \dfrac{z''}{(1+z'^2)^{3/2}}$

$$= \frac{2x^{-3}}{(1+x^{-4})^{3/2}} = \frac{2}{x^3}\,\frac{1}{\left(1+\frac{1}{x^4}\right)^{3/2}} =$$

$$z = \frac{1}{x}$$
$$z' = -\frac{1}{x^2} = -x^{-2}$$
$$z'' = -(-2)x^{-3} = 2x^{-3}$$

$$= \frac{2}{x^3}\,\frac{x^6}{(1+x^4)^{3/2}} = \frac{2x^3}{(1+x^4)^{3/2}}$$

$$(x^4)^{3/2} = x^6$$

gran normale. $\quad \mathcal{C} \begin{cases} x = t \\ z = \dfrac{1}{t} \end{cases}$

tangente a $\mathcal{C}$ in $P: (t, \frac{1}{t})$ $\qquad z - \dfrac{1}{t} = \underbrace{-\dfrac{1}{t^2}}_{m}(x - t)$

normale: $\qquad z - \dfrac{1}{t} = t^2(x-t)$

$$m^\perp = -\frac{1}{m}$$

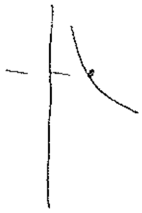
$$Q = \begin{cases} z - \dfrac{1}{t} = t^2(x-t) \\ x = 0 \end{cases} \qquad z - \frac{1}{t} = -t^3$$

$$z = \frac{1}{t} - t^3 = \frac{1-t^4}{t}$$

$$Q = \left(0, \frac{1-t^4}{t}\right)$$

2

Geometria 1/7/11



$$\begin{cases} x = u\cos v \\ y = u \sin v \\ z = \dfrac{1}{u} \end{cases} \qquad 0<u<1 \qquad v \in [0,2\pi)$$

$$\underline{r} = \left(u\cos v,\ u\sin v,\ \frac{1}{u}\right)$$

$$\underline{r}_u = \left(\cos v,\ \sin v,\ -\frac{1}{u^2}\right)$$

$$\underline{r}_v = \left(-u\sin v,\ u\cos v,\ 0\right)$$

$$\underline{r}_{uu} = \left(0,\ 0,\ +2u^{-3}\right)$$

$$\underline{r}_{uv} = \left(-\sin v,\ \cos v,\ 0\right)$$

$$\underline{r}_{vv} = \left(-u\cos v,\ -u\sin v,\ 0\right)$$

$$\underline{N} = \frac{\underline{r}_u \times \underline{r}_v}{\| \quad \|} = \frac{1}{\sqrt{\ }} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos v & \sin v & -\frac{1}{u^2} \\ -u\sin v & u\cos v & 0 \end{vmatrix} =$$

$$= \frac{1}{\sqrt{\ }}\left\{ \underline{i}\left(\frac{1}{u}\cos v\right) - \underline{j}\left(-\frac{1}{u}\sin v\right) + \underline{k}\,u \right\} =$$

$$= \frac{1}{\sqrt{\ }}\left\{ \frac{1}{u}\cos v\,\underline{i} + \frac{1}{u}\sin v\,\underline{j} + u\,\underline{k} \right\}$$

$$\sqrt{\ } = \sqrt{\frac{1}{u^2} + u^2} = \frac{\sqrt{u^4+1}}{u}$$

$$\underline{N} = \frac{u}{\sqrt{1+u^4}}\left(\frac{1}{u}\cos v,\ \frac{1}{u}\sin v,\ u\right) = \left(\frac{\cos v}{\sqrt{\ }},\ \frac{\sin v}{\sqrt{\ }},\ \frac{u^2}{\sqrt{\ }}\right)$$

$$\mathrm{I}^{o}: \begin{cases} E = 1 + \dfrac{1}{u^4} \\ F = 0 \\ G = u^2 \end{cases}$$

$$\mathrm{II}^{a}: \begin{cases} e = \langle \underline{r}_{uu}, \underline{N} \rangle = \dfrac{u^2}{\sqrt{1+u^4}} \cdot \dfrac{2}{u^3} = \\ \quad = \dfrac{2}{u\sqrt{1+u^4}} \\ f = \langle \underline{r}_{uv}, \underline{N} \rangle = \dots = 0 \\ g = \langle \underline{r}_{vv}, \underline{N} \rangle = -\dfrac{u}{\sqrt{1+u^4}} \end{cases}$$

Chiaro anche dalla teoria generale.

$$\mathrm{II}^{a} \begin{cases} e = \dfrac{2}{u\sqrt{1+u^4}} \\ f = 0 \\ g = -\dfrac{u}{\sqrt{1+u^4}} \end{cases}$$

$$K = \frac{eg - f^2}{EG - F^2} = \frac{eg}{EG} = -\frac{2}{u(1+u^4)^{\frac{1}{2}}} \cdot \frac{1}{(1+u^4)^{\frac{1}{2}}} \cdot \frac{u^4}{1+u^4} \cdot \frac{u}{u^2}$$

$$= -\frac{2u^2}{(1+u^4)^2} \qquad \checkmark$$

3

Evoluta di $\mathscr{C}$: $\begin{cases} x = t \\ y = \frac{1}{t} \end{cases}$ $\quad 0 < t < 1$

in più modi

1. inviluppo delle normali

normali $\quad y - \frac{1}{t} = t^2(x-t) \qquad$ (v. ex. 1)

$$\frac{\partial}{\partial t}(\ ) = 0 \longrightarrow \begin{cases} y - \frac{1}{t} = t^2(x-t) \\ +\frac{1}{t^2} - 2tx + 3t^2 = 0 \end{cases}$$

$$2t\,x = 3t^2 + \frac{1}{t^2} = \frac{3t^4+1}{t^2}$$

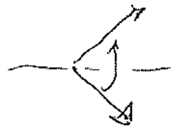
$$x = \frac{3t^4+1}{2t^3}$$

$$y = \frac{1}{t} + t^2\left(\frac{3t^4+1}{2t^3} - t\right) = \frac{1}{t} + t^2\,\frac{3t^4+1-2t^4}{2t^3}$$

$$= \frac{1}{t} + \cancel{t^2}\,\frac{t^4+1}{2t^{\cancel{3}}} = \frac{2+t^4+1}{2t} = \frac{3+t^4}{2t}$$

$$\begin{cases} x = \frac{3t^4+1}{2t^3} \\ y = \frac{3+t^4}{2t} \end{cases}$$

2. <u>luogo dei centri di curvatura</u>



$$\underline{P} = \left(t, \frac{1}{t}\right)$$

$$\dot{\underline{P}} = \left(1, -\frac{1}{t^2}\right) \qquad i\dot{\underline{P}} = \left(\frac{1}{t^2}, 1\right)$$

$$\ddot{\underline{P}} = \left(0, \frac{2}{t^3}\right)$$

$$Q = \underline{P} + \frac{\|\dot{\underline{P}}\|^2}{\langle i\dot{\underline{P}}, \ddot{\underline{P}}\rangle} i\cdot\dot{\underline{P}}$$

$$\|\dot{\underline{P}}\|^2 = 1 + \frac{1}{t^4} = \frac{1+t^4}{t^4}$$

$$\langle i\dot{\underline{P}}, \ddot{\underline{P}}\rangle = \frac{2}{t^3}$$, pertanto

$$Q = \begin{pmatrix} t \\ \frac{1}{t} \end{pmatrix} + \frac{\cancel{t^3}}{2}\,\frac{1+t^4}{\cancel{t^4}} \begin{pmatrix} \frac{1}{t^2} \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} t + \frac{1+t^4}{2t^3} \\ \frac{1}{t} + \frac{1+t^4}{2t} \end{pmatrix} = \begin{pmatrix} \frac{3t^4+1}{2t^3} \\ \frac{3+t^4}{2t} \end{pmatrix} \qquad \checkmark$$

se $t \to 0$ $\quad Q(t) \to \infty$, intuitivamente plausibile (la curvatura $\to 0$, la curva diviene viepiù assimilabile ad una retta)