GEOMETRIA (Prof. M. Spera) a.a. 2010/11

Prova scritta del 19/9/2011

(1) Sia Σ la sup. di rotazione di $x = \sin z$ attorno all'asse z $\quad (z \in (0, \pi))$

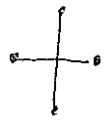
La si parametrizzi usando (z, φ) $\quad \varphi \in [0, 2\pi)$

Se ne calcoli la prima e la seconda forma fondamentale, la curvatura gaussiana e le curvature principali.

(2) Con riferimento all'es. 1, data

$$\underline{\alpha} = \underline{\alpha}(t) = (\sin t \cos t, \sin t \cdot \sin t, t) \qquad t > 0,$$

dire se risulta essere una geodetica di Σ utilizzando il teorema di Clairaut.

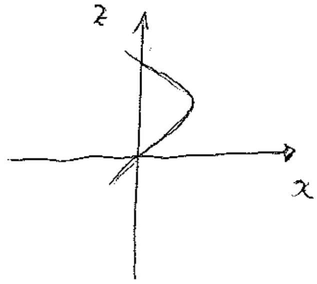
(3) Siano dati, nel piano, i seguenti spazi topologici (top. relativa): $X =$  $\quad Y =$ 

Dire se essi risultano omeomorfi

Tempo a disposizione: 2h. Le risposte vanno adeguatamente giustificate

Geometria 19/9/11

(1) Σ : sup. di rot. di $x = \sin z$ attorno all'asse z



$z \in [0, \pi]$

"lanterna"

usare (z, φ)

$$\begin{cases} x = \sin z \cos\varphi \\ y = \sin z \sin\varphi \\ z = z \end{cases}$$

$$\underline{r} = (\sin z \cos\varphi, \sin z \sin\varphi, z)$$

$$\underline{r}_z = (\cos z \cos\varphi, \cos z \sin\varphi, 1)$$

$$\underline{r}_\varphi = (-\sin z \sin\varphi, \sin z \cos\varphi, 0)$$

$$\underline{N} = \frac{1}{\sqrt{}} \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos z \cos\varphi & \cos z \sin\varphi & 1 \\ -\sin z \sin\varphi & \sin z \cos\varphi & 0 \end{vmatrix} =$$

$$= \frac{1}{\sqrt{}} \left\{ \underline{i} \left(-\sin z \cos\varphi \right) - \underline{j} \left(\sin z \sin\varphi \right) + \underline{k} \left(\sin z \cos z \cos^2\varphi + \sin z \cos z \sin^2\varphi \right) \right\}$$

$$= \frac{1}{\sqrt{}} \left\{ \underline{i} \left(-\sin z \cos\varphi \right) - \underline{j} \sin z \sin\varphi + \underline{k} \sin z \cos z \right\}$$

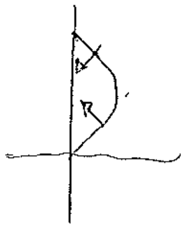
$$\| \ \|^2 = \sin^2 z \cos^2\varphi + \sin^2 z \sin^2\varphi + \sin^2 z \cos^2 z$$

$$= \sin^2 z \left(1 + \cos^2 z\right) = \left(1 - \cos^2 z\right)\left(1 + \cos^2 z\right) =$$

$$= 1 - \cos^4 z$$

$$\underline{N} = \frac{1}{\sqrt{1-\cos^4 z}} \left(-\sin z \cos\varphi, \quad -\sin z \sin\varphi, \quad \sin z \cos z \right)$$

orient.
"interno"



$$E = \|r_z\|^2 = \cos^2 z + 1$$
$$F = \langle \underline{r}_z, \underline{r}_\varphi \rangle = \langle \underline{r}_\varphi, \underline{r}_z \rangle = 0$$
$$G = \sin^2 z$$

$$\underline{r}_{zz} = (-\sin z \cos\varphi, \ -\sin z \sin\varphi, \ 0)$$
$$r_{\varphi z} = \underline{r}_{z\varphi} = (-\cos z \sin\varphi, \ \cos z \cos\varphi, \ 0)$$
$$\underline{r}_{\varphi\varphi} = (-\sin z \cos\varphi, \ -\sin z \sin\varphi, \ 0)$$

$$e = \langle \underline{r}_{zz}, \underline{N} \rangle = \frac{\sin^2 z \cos^2\varphi + \sin^2 z \sin^2\varphi}{\sqrt{}} = \frac{\sin^2 z}{\sqrt{1-\cos^4 z}}$$
$$f = \langle \underline{r}_{z\varphi}, \underline{N} \rangle = \ldots \ 0$$

$$g = \langle \underline{r}_{\varphi\varphi}, \underline{N} \rangle = \frac{1}{\sqrt{1-\cos^4 z}} \left(\sin^2 z \cos^2\varphi + \sin^2 z \sin^2\varphi \right)$$

$$= \frac{\sin^2 z}{\sqrt{1-\cos^4 z}} = \frac{\sin^2 z}{\sin z \cdot \sqrt{1+\cos^2 z}} = \frac{\sin z}{\sqrt{1+\cos^2 z}}$$

$\sin z > 0$
per $z \in (0,\pi)$

$$K = \frac{e\,g}{E\,G} = \left(\frac{e}{E}\right)\left(\frac{g}{G}\right) = \underbrace{\frac{\frac{\sin^2 z}{\sqrt{}}}{\cos^2 z + 1}}_{=k_1} \cdot \underbrace{\frac{\frac{\sin^2 z}{\sqrt{}}}{\sin^2 z}}_{=k_2}$$

$$= \frac{\sin^2 z}{(1-\cos^4 z)(1+\cos^2 z)} = \frac{1-\cos^2 z}{(1-\cos^2 z)(1+\cos^2 z)^2} = \frac{1}{(1+\cos^2 z)^2}$$

k_1 e k_2 più in dettaglio:

$$\boxed{\frac{\sin^2 z}{\sqrt{1-\cos^4 z}} = \frac{\sin z}{\sqrt{1+\cos^2 z}}}$$

$$k_1 = \frac{\dfrac{\sin^2 z}{\sqrt{1-\cos^4 z}}}{\cos^2 z + 1} = \frac{\sin z}{(1+\cos^2 z)^{3/2}}$$

$$k_2 = \frac{\sin z}{\sqrt{1+\cos^2 z}} \cdot \frac{1}{\sin^2 z} = \frac{1}{\sin z\sqrt{1+\cos^2 z}}$$

(2) $\underline{\alpha}(t) = (\sin t \cos t, \sin t \sin t, t)$

$\underline{\alpha} = \underline{\alpha}(t)$ è

una geodetica di Σ ? Utilizzare Clairaut.

$$\dot{\underline{\alpha}} = \left(\overset{2\cos^2 t - 1}{\cos^2 t - \sin^2 t}, \cos t \cdot \sin t + \sin t \cdot \cos t, 1 \right)$$

$\cos t \cos t + \sin t(-\sin t)$ $\quad 2 \cos t \sin t$

$\frac{d}{dt} \sin^2 t = 2 \sin t \cos t$

$$= (\cos 2t, \sin 2t, 1)$$

$\| \dot{\underline{\alpha}} \|^2 = 1 + 1 = 2 \qquad \| \dot{\underline{\alpha}} \| = \sqrt{2} \qquad \underline{v} := \frac{\dot{\underline{\alpha}}}{\sqrt{2}}$

Direzione dell'azimut mol. da $\underline{r}_2(t,t) =$

$= (\cos t \cos t, \cos t \sin t, 1)$

$\| \; \| = \sqrt{\cos^2 t \cos^2 t + \cos^2 t \sin^2 t + 1} = \sqrt{1 + \cos^2 t}$.

$\underline{n} = \frac{r_2(t,t)}{\| \; \|}$

calcoliamo $\langle \underline{n}, \underline{v} \rangle = \underset{\text{azimut}}{\cos \alpha} = \frac{\{\sin t \cos t \cdot \cos t \cos t + \sin t \sin t \cos t \sin t + t\}}{\sqrt{2} \sqrt{1+\cos^2 t}} =$

$$= \frac{\sin t \cos^3 t + \sin^3 t \cos t + t}{\sqrt{2(1+\cos^2 t)}} = \frac{\sin t \cos t (\cos^2 t + \sin^2 t) + t}{\sqrt{2(1+\cos^2 t}}$$

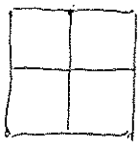
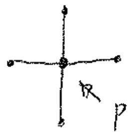
$$= \frac{t + \sin t \cos t}{\sqrt{2(1+\cos^2 t}}$$

$$|\sin\alpha| = \sqrt{1-\cos^2\alpha} = \sqrt{1 - \frac{t^2 + 2t\cos t\sin t + \cos^2 t\sin^2 t}{2(1+\cos^2 t)}}$$

$$= \sqrt{\frac{2 + 2\cos^2 t - t^2 - 2t\cos t\sin t - \cos^2 t\sin^2 t}{2(1+\cos^2 t)}}$$

raggio parallelo: $\sin t$. $\sin t \cdot \sin\alpha$ non è costante al variare di t.

(3)

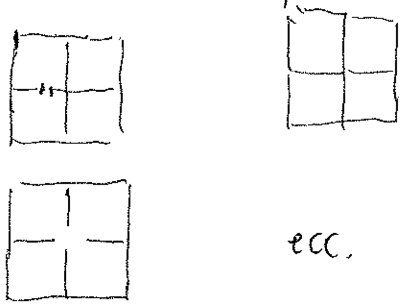
$Y =$  $X =$ 

$X \not\approx Y$ se $\exists\, f : X \to Y$ omeom

$f|_{X \setminus \{p\}} : X \setminus \{p\} \longrightarrow Y \setminus \{f(p)\}$

risulterebbe un omeomorfismo, ma $X \setminus \{p\}$ non è connesso, $Y \setminus \{f(p)\}$ lo è in ogni caso

(e la connessione è una prop. topologica, ovvero è invariante per omeomorfismi)



ecc.