Appunti del corso di GEOMETRIA (Prof. M. Spera)

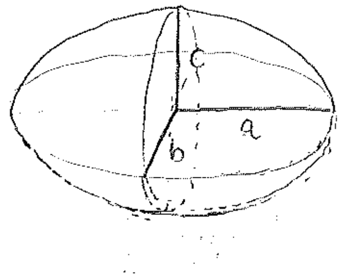
Lezione XII-1

ESEMPI, ESERCIZI e COMPLEMENTI VARI

◇ - Sulle quadriche (sup. algebriche del 2° ordine)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (a \geq b \geq c > 0)$$

☆ ellissoide



coordinate ellissoidali
$$\begin{cases} x = a \cos u \cos v \\ y = b \cos u \sin v \\ z = c \sin u \end{cases} \qquad u \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad v \in [0, 2\pi)$$

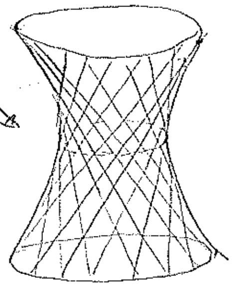
pti ellittici

$K > 0$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

☆ iperboloide ad una falda (iperboloide iperbolico) (o rigato)

ellisse di gola (è la linea di stringimento)



$$\begin{cases} x = a \cosh u \cos v \\ y = b \cosh u \sin v \\ z = c \sinh u \end{cases} \qquad u \in \mathbb{R} \quad v \in [0, 2\pi)$$

pti iperbolici $K < 0$

l'iperboloide ad una falda è <u>doppiamente</u> rigato;

le generatrici appartenenti ad una stessa <u>schiera</u> sono <u>sghembe</u>;

una generatrice qualsiasi incontra invece tutte quelle dell'altra schiera.. Descrizione:

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2}$$

$$\left(\frac{y}{b} + \frac{z}{c}\right)\left(\frac{y}{b} - \frac{z}{c}\right) = \left(1 + \frac{x}{a}\right)\left(1 - \frac{x}{a}\right)$$

che si ottiene da

$$\begin{cases} 1 + \dfrac{x}{a} = t\left(\dfrac{y}{b} + \dfrac{z}{c}\right) \\ 1 - \dfrac{x}{a} = \dfrac{1}{t}\left(\dfrac{y}{b} - \dfrac{z}{c}\right) \end{cases}$$

fasci di piani

1ª schiera

eliminando t $(t \neq 0 ..)$

oppure da

$$\begin{cases} 1 + \dfrac{x}{a} = t'\left(\dfrac{y}{b} - \dfrac{z}{c}\right) \\ 1 - \dfrac{x}{a} = \dfrac{1}{t'}\left(\dfrac{y}{b} + \dfrac{z}{c}\right) \end{cases}$$

2ª schiera

eliminando t'

asse del fascio:

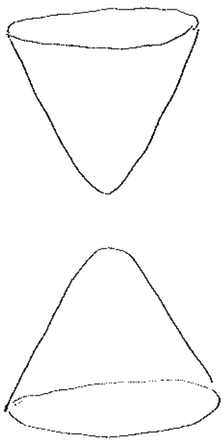
$t = 0$ $\quad 1 + \frac{x}{a} = 0$ $\quad \begin{cases} 1 + \frac{x}{a} = 0 \\ \frac{y}{b} + \frac{z}{c} = 0 \end{cases}$ è una retta della 2ª schiera $(t' = 0)$

$t = \infty$ $\quad \frac{y}{b} + \frac{z}{c} = 0$

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

✱ iperboloide ellittico (o a due falde)

oppure $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$



$$\begin{cases} x = a \sinh u \cos v & u \in \mathbb{R} \\ y = b \sinh u \sin v & v \in [0, 2\pi) \\ z = c \cosh u \end{cases}$$

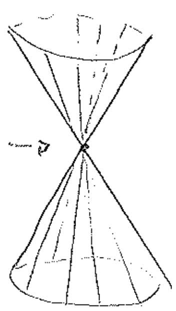
$K > 0$

pti ellittici

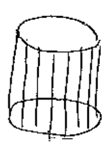
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

✱ cono

linea di stringimento ridotta ad un punto



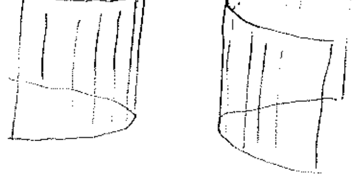
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



cilindro ellittico

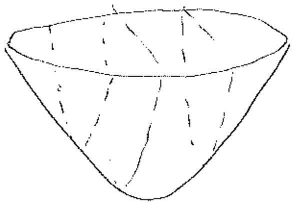
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

cilindro iperbolico



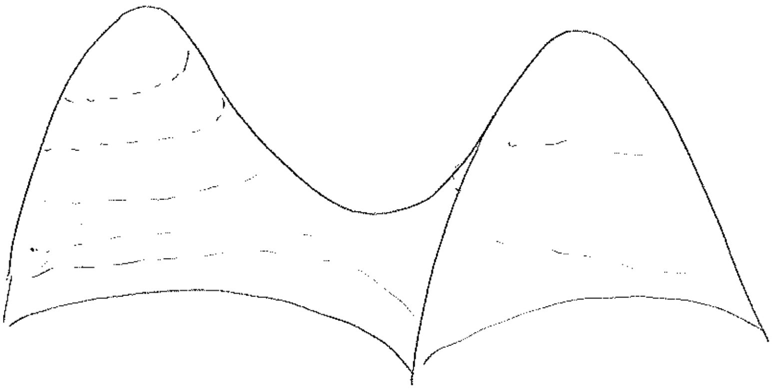
$$2z = \frac{x^2}{p} + \frac{y^2}{q} \qquad p, q > 0$$

paraboloide ellittico



$$2z = \frac{x^2}{p} - \frac{y^2}{q}$$

paraboloide iperbolico (o a sella)



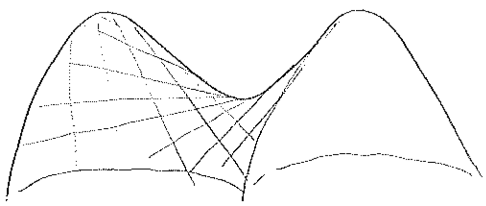
è una superficie rigata (doppiamente)

$$\begin{cases} \dfrac{x}{\sqrt{p}} + \dfrac{y}{\sqrt{q}} = 2tz \\[2ex] \dfrac{x}{\sqrt{p}} - \dfrac{y}{\sqrt{q}} = \dfrac{1}{t} \end{cases}$$

direzioni appartenenti ad una stessa giacitura sono // a $\frac{x}{\sqrt{p}} - \frac{y}{\sqrt{q}} = 0$

$$\begin{cases} \dfrac{x}{\sqrt{p}} + \dfrac{y}{\sqrt{q}} = \dfrac{1}{t'} \\[2ex] \dfrac{x}{\sqrt{p}} - \dfrac{y}{\sqrt{q}} = 2t'z \end{cases}$$

sono // a $\frac{x}{\sqrt{p}} + \frac{y}{\sqrt{q}} = 0$



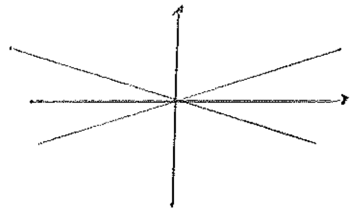
* Curve asintotiche

$$\boxed{e\,u'^2 + 2f\,u'v' + g\,v'^2 = 0}$$

$$(u'\ v')\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix} = 0$$

(nel piano dei parametri..)

pto per punto danno le direzioni asintotiche ($k_n = 0$)
cf. Meusnier



* Linee di curvatura

Rodrigues: $dN = \lambda\, \alpha'(t)$ $\qquad N' = \lambda \alpha'$

$S = -dN$

$$\mathcal{M}_{\mathcal{BB}}(dN) = \begin{pmatrix} \overbrace{\dfrac{fF - eG}{EG - F^2}}^{a_{11}} & \overbrace{\dfrac{gF - fG}{EG - F^2}}^{a_{12}} \\ \underbrace{\dfrac{eF - fE}{EG - F^2}}_{a_{21}} & \underbrace{\dfrac{fF - gE}{EG - F^2}}_{a_{22}} \end{pmatrix}$$

$\mathcal{B} = (r_u, r_v)$

* Weingarten

$$(fE - eF)(u')^2 + (gE - eG)\,u'v' + (gF - fG)(v')^2 = 0$$

in forma compatta

$$\begin{vmatrix} v'^2 & -u'v' & u'^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0$$

in un intorno di un pto non ombelicale, le curve C_u e C_v sono linee di curvatura $\Leftrightarrow$ $F = f = 0$

(risulta $u'v' = 0$ $\Rightarrow$ $\begin{cases} u = u_0 \\ v = v \end{cases}$ $\quad \begin{cases} u = u \\ v = v_0 \end{cases}$

Se $F = 0$, le formule di Weingarten diventano

$$m_{\mathcal{B}\mathcal{B}}(dN) = \begin{pmatrix} -\dfrac{e\cancel{G}}{E\cancel{G}} & -\dfrac{f\cancel{G}}{E\cancel{G}} \\[2ex] -\dfrac{f\cancel{E}}{\cancel{E}G} & -\dfrac{g\cancel{E}}{\cancel{E}G} \end{pmatrix}$$

$$\|$$

$$\begin{pmatrix} -\dfrac{e}{E} & -\dfrac{f}{E} \\[2ex] -\dfrac{f}{G} & -\dfrac{g}{G} \end{pmatrix}$$

★ Dettagli sul calcolo delle direzioni principali

Si ricordi il calcolo di H e K; riprendiamo l'equazione

$$\left[\begin{array}{l} S\underline{v} = k\underline{v} \\ \underline{v} = \alpha \underline{r}_u + \beta \underline{r}_v \end{array}\right]$$

$$\begin{cases} (e - kE)\alpha + (f - kF)\beta = 0 \\ (f - kF) + (g - kG)\beta = 0 \end{cases}$$

cambiamo $\underline{v} \longmapsto \dot{\underline{\alpha}} = \dot{u}\,\underline{r}_u + \dot{v}\,\underline{r}_v$, con $\dot{u} \equiv \alpha$, $\dot{v} \equiv \beta$

raggruppiamo, ponendo $\alpha \equiv \dot{u}$ $\beta \equiv \dot{v}$

$$\begin{cases} (e\dot{u} + f\dot{v}) - (E\dot{u} + F\dot{v})k = 0 \\ (f\dot{u} + g\dot{v}) - (F\dot{u} + G\dot{v})k = 0 \end{cases}$$

$$\begin{pmatrix} e\dot{u} + f\dot{v} & E\dot{u} + F\dot{v} \\ f\dot{u} + g\dot{v} & F\dot{u} + G\dot{v} \end{pmatrix} \begin{pmatrix} 1 \\ -k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

per avere una sol. non banale deve essere $\det(\quad) = 0$

ossia:

$$(e\dot{u} + f\dot{v})(F\dot{u} + G\dot{v}) - (E\dot{u} + F\dot{v})(f\dot{u} + g\dot{v}) = 0$$

$$(eF - Ef)\dot{u}^2 + (eG + \cancel{fF} - \cancel{fF} - Eg)\dot{u}\dot{v} + (fG - Fg)\dot{v}^2 = 0$$

$$\boxed{(eF - Ef)\dot{u}^2 + (eG - Eg)\dot{u}\dot{v} + (fG - Fg)\dot{v}^2 = 0}$$

★ <u>Sull'iperboloide ad una falda</u>

Sia $S^1 : \{ x^2+y^2=1 \ , z=0 \}$ (circ. unitaria sul piano x.y

$$\underline{\alpha}=\underline{\alpha}(s) : \begin{cases} x = \cos s \\ y = \sin s \\ z = 0 \end{cases}$$

$s = \varphi$ = l. d'arco $\qquad \varphi \in [0, 2\pi)$

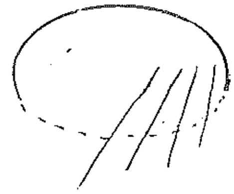
$$\underline{\alpha} = (\cos s \ , \ \sin s, \ 0)$$

$$\underline{\alpha}' = (-\sin s, \ \cos s, \ 0) = -\sin s \ \underline{i} + \cos s \ \underline{j}$$

$$\underline{w}(s) := (\underline{\alpha}' + \underline{k}) = -\sin s \ \underline{i} + \cos s \ \underline{j} + \underline{k}$$

consideriamo la <u>superficie rigata</u> $\qquad \| \underline{w} \| = 2$

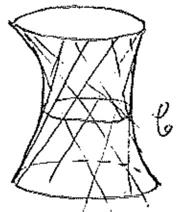
$$\underline{r}(s,t) = \underline{\alpha}(s) + t \ \underline{w}(s)$$



$$= (\underbrace{\cos s - t \sin s}_{x}, \ \underbrace{\sin s + t \cos s}_{y}, \ \underbrace{t}_{z})$$

$$\begin{aligned} x^2+y^2-z^2 &= \cos^2 s \cancel{- 2t \sin s \cos s} + t^2 \sin^2 s \\ &+ \sin^2 s + t^2 \cos^2 s \cancel{+ 2t \sin s \cos s} + t^2 \\ &- t^2 \\ &= 1 \end{aligned}$$

★ iperboloide iperbolico di rivoluzione



$$x^2+y^2-z^2 = 1$$

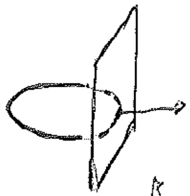
Sia $\underline{P} = (1, 0, 0) \in \mathcal{C}$

Determiniamo, in $T_P \Sigma$, le dir. asintotiche e le direzioni principali, col calcolo implicito. [per esercizio, si calcoli il tutto tramite la parametrizzazione precedente]

$$x^2 + y^2 - z^2 = 1 \qquad \text{si osservi che, in } \underline{P}, \quad \frac{\partial f}{\partial x} = 2x$$

$$f(x,y,z) = 0 \qquad\qquad = 2 \neq 0$$

$\Rightarrow$ (Dini) $\quad x = x(y,z) \equiv \varphi(y,z)$



in $\underline{P}: (1,0,0)$

$x = \varphi(0,0) = +1$

$$\varphi^2 + y^2 - z^2 = 1$$

$\dfrac{\partial}{\partial y}$ $\qquad \cancel{2}\varphi\varphi_y + \cancel{2}y = 0 \qquad \varphi\varphi_y + y = 0 \qquad \varphi_y(\underline{P}) = 0$

$\dfrac{\partial}{\partial z}$ $\qquad +\cancel{2}\varphi\varphi_z - \cancel{2}z = 0 \qquad \varphi\varphi_z - z = 0 \qquad \varphi_z(\underline{P}) = 0$

(Chiaro...)

$\underline{N}(\underline{P}) = (1,0,0)$

$\dfrac{\partial^2}{\partial y^2}$ $\qquad \varphi_y^2 + \varphi\varphi_{yy} + 1 = 0 \qquad \varphi_{yy}(\underline{P}) = -1$

$\dfrac{\partial^2}{\partial y \partial z}$ $\qquad \varphi_z\varphi_y + \varphi\varphi_{yz} = 0 \qquad \varphi_{yz}(\underline{P}) = 0$

$\dfrac{\partial^2}{\partial z^2}$ $\qquad \varphi_z^2 + \varphi\varphi_{zz} - 1 = 0 \qquad \varphi_{zz}(\underline{P}) = +1$

$$I^a \begin{cases} E = 1 + \varphi_y^2 & \text{in } \underline{P}: \ E = 1 \\ F = \varphi_y\varphi_z & \text{in } \underline{P} \ \ F = 0 \\ G = 1 + \varphi_z^2 & \text{in } \underline{P} \ \ G = 1 \end{cases}$$

$$\begin{cases} \underline{r} = (\varphi, y, z) \\ \underline{r}_y = (\varphi_y, 1, 0) \\ \underline{r}_z = (\varphi_z, 0, 1) \end{cases}$$

$$\begin{cases} \underline{r}_{yy} = (\varphi_{yy}, 0, 0) \\ \underline{r}_{yz} = (\varphi_{yz}, 0, 0) \\ \underline{r}_{zz} = (\varphi_{zz}, 0, 0) \end{cases}$$

$e = \varphi_{yy}$ in P: $e = -1$

$f = \varphi_{yz}$ in P $f = 0$

$g = \varphi_{zz}$ in P $g = +1$

I^a $E = G = 1$ $F = 0$ (chiaro a priori...)

II^a $e = -1$ $g = 1$ $f = 0$

$K = -1$ (in $\underline{\underline{P}}$)

$$H = \frac{1}{2}\,\frac{Ge - 2Ff + Eg}{EG - F^2} = \frac{1}{2}\,\frac{-1+1}{1} = 0 \quad (\text{in } \underline{\underline{P}})$$

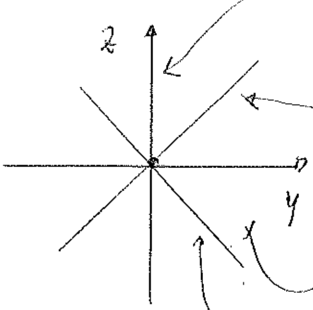
$$S(P) = -dN(P) = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

direzioni principali $u = y$, $v = z$

$k_1 = -1$

$k_2 = +1$

in una sup. di rivoluzione, i meridiani e i paralleli sono linee di curvatura !!



$z = 0$ $y = 0$

direzioni asintotiche $y'^2 - z'^2 = 0$

$(y' + z')(y' - z') = 0$

già trovata

la retta $P + t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -t \\ t \end{pmatrix}$ è contenuta in Σ:

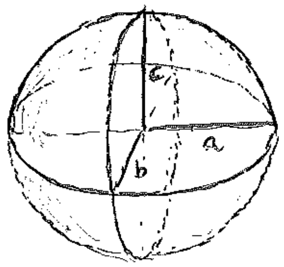
$x^2 + y^2 - z^2 = 1$

(Σ è doppiamente rigato)

4 Curvatura dell'ellissoide

$$\underset{\substack{\parallel \\ 0}}{f(x,y,z)} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad a \geq b \geq c$$

utilizziamo il calcolo diff. implicito
(cf. l'esercizio sulla finestra di Viviani)



Sia $P: (x_0, y_0, z_0)$ tale che $\frac{\partial f}{\partial z}(P) \neq 0$

$\Rightarrow$ (Dini) localmente $z = \varphi(x,y)$ ..

Per una superficie cartesiana: $z = \varphi(x,y)$ $\begin{pmatrix} x = x \\ y = y \\ z = \varphi(x,y) \end{pmatrix}$

$$\underline{r} = (x, \; y, \; \varphi) \qquad \underline{r} = x\underline{i} + y\underline{j} + \varphi\underline{k}$$

$$\underline{r}_x = (1, \; 0, \; \varphi_x)$$

$$\underline{r}_y = (0 \; 1, \; \varphi_y)$$

$$\underline{N} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & \varphi_x \\ 0 & 1 & \varphi_y \end{vmatrix} \frac{1}{\| \quad \|}$$

$$\Downarrow$$

$$\underline{N} = \frac{1}{\sqrt{1+\varphi_x^2+\varphi_y^2}} (-\varphi_x, -\varphi_y, 1)$$

$$\underline{r}_{xx} = (0 \quad 0 \quad \varphi_{xx})$$

$$\underline{r}_{yx} = \underline{r}_{xy} = (0 \quad 0 \quad \varphi_{xy})$$

$$\underline{r}_{yy} = (0 \quad 0 \quad \varphi_{yy})$$

$$\Rightarrow \quad E = 1 + \varphi_x^2 \qquad F = \varphi_x \varphi_y \qquad G = 1 + \varphi_y^2$$

$$EG - F^2 = 1 + \varphi_x^2 + \varphi_y^2 + \varphi_x^2\varphi_y^2 - \varphi_x^2\varphi_y^2 = 1 + \varphi_x^2 + \varphi_y^2$$

$$e = \frac{\varphi_{xx}}{\sqrt{1+\varphi_x^2+\varphi_y^2}} \qquad f = \frac{\varphi_{yy}}{\sqrt{}} \qquad g = \frac{\varphi_{yy}}{\sqrt{}}$$

$$K = \frac{\overbrace{\varphi_{xx}\varphi_{yy} - \varphi_{xy}^2}^{\text{Hessiano}}}{(1+\varphi_x^2+\varphi_y^2)^2}$$

$$H = \dots$$

Procuriamoci ciò che ci serve da

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{\varphi^2}{c^2} = 1 \qquad (\diamondsuit) \qquad \varphi \neq 0 ..$$

$\frac{\partial}{\partial x}$:
$$\frac{\cancel{2}\, x}{a^2} + \frac{\cancel{2}\,\varphi\,\varphi_x}{c^2} = 0 \quad (*) \qquad \boxed{\varphi_x = -\frac{c^2 x}{a^2 \cdot \varphi}}$$

$\frac{\partial}{\partial y}$!
$$\frac{\cancel{2}\, y}{b^2} + \frac{2\varphi\,\varphi_y}{c^2} = 0 \quad (**) \qquad \boxed{\varphi_y = -\frac{c^2 y}{b^2 \varphi}}$$

$\frac{\partial}{\partial x}$ $(*)$
$$\frac{1}{a^2} + \frac{1}{c^2}\left(\varphi_x^2 + \varphi\,\varphi_{xx}\right) = 0$$

$$\varphi_x^2 + \varphi\cdot\varphi_{xx} = -\frac{c^2}{a^2}$$

$$\varphi_{xx} = -\left(\frac{c^2}{a^2} + \varphi_x^2\right)\frac{1}{\varphi} = -\left(\frac{c^2}{a^2} + \frac{c^4}{a^4}\frac{x^2}{\varphi^2}\right)\frac{1}{\varphi}$$

$$= -\frac{c^2}{a^2}\left(1 + \frac{c^2}{a^2}\frac{x^2}{\varphi^2}\right)\frac{1}{\varphi}$$

$$= -\frac{c^2}{a^2}\,\frac{a^2\varphi^2 + c^2x^2}{a^2\varphi^3} = -\frac{c^2}{a^4}\,\frac{a^2\varphi^2 + c^2x^2}{\varphi^3}$$

$$\varphi_{yy} = -\frac{c^2}{b^4}\,\frac{b^2\varphi^2 + c^2y^2}{\varphi^3} \qquad \varphi_{xy} = \frac{c^4}{a^2b^2}\,\frac{xy}{\varphi^3}$$

$$\parallel$$

$\frac{\partial}{\partial y}$ $(*)$
$$\varphi_y\varphi_x + \varphi\,\varphi_{xy} = 0 \qquad \varphi_{xy} = -\frac{\varphi_x \cdot \varphi_y}{\varphi}$$

$$\begin{cases} \varphi_{xx} = -\dfrac{c^2}{a^4}\dfrac{a^2\varphi^2+c^2x^2}{\varphi^3} \\[2ex] \varphi_{xy} = -\dfrac{c^4}{a^2b^2}\dfrac{xy}{\varphi^3} \\[2ex] \varphi_{yy} = -\dfrac{c^2}{b^4}\dfrac{b^2\varphi^2+c^2y^2}{\varphi^3} \end{cases}$$

calcoliamo $\varphi_{xx}\varphi_{yy}-\varphi_{xy}^2$. Si ha

$$\frac{c^4}{a^4b^4}\frac{1}{\varphi^6}(a^2\varphi^2+c^2x^2)(b^2\varphi^2+c^2y^2) - \frac{c^8}{a^4b^4}\frac{x^2y^2}{\varphi^6}$$

$$= \frac{c^4}{a^4b^4\varphi^6}\left[(a^2\varphi^2+c^2x^2)(b^2\varphi^2+c^2y^2) - c^4x^2y^2)\right]$$

$$= \frac{c^4\varphi^2}{a^4b^4\varphi^6}\overbrace{(a^2b^2\underset{z}{\varphi^2} + c^2b^2x^2 + a^2c^2y^2)}^{a^2b^2c^2}$$

$\leftarrow \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$

$a^2b^2z^2+$
$c^2a^2y^2+$
$b^2c^2x^2 = a^2b^2c^2$

$$= \frac{1}{\varphi^4}\frac{c^4}{a^4b^4}a^2b^2c^2 = \frac{c^6}{\varphi^4a^2b^2}$$

$$EF - G^2 = 1 + \varphi_x^2 + \varphi_y^2 = 1 + \frac{c^4}{a^4}\frac{x^2}{\varphi^2} + \frac{c^4}{b^4}\frac{y^2}{\varphi^2} =$$

$$= \frac{a^4b^4\varphi^2 + b^4c^4x^2 + a^4c^4y^2}{a^4b^4\varphi^2}$$

$\Rightarrow$

$$K = \frac{c^6}{\varphi^4 a^2 b^2} \cdot \left[\frac{a^4b^4\varphi^2}{a^4b^4\varphi^2 + b^4c^4x^2 + a^4c^4y^2}\right]^2$$

$$= \frac{a^6 b^6 c^6}{\left(b^4c^4x^2 + c^4a^4y^2 + a^4b^4z^2\right)^2}$$

(Controllo: sfera: $a = b = c = R$ $\quad K = \frac{R^{18}}{(R^2R^4R^4)^2} = \frac{R^{18}}{R^{20}}$

$= \frac{1}{R^2}$

ok!

in definitiva:

$$\boxed{K = \frac{a^6b^6c^6}{\left(b^4c^4x^2 + c^4a^4y^2 + a^4b^4z^2\right)^2}}$$

in un pto di $\mathcal{E}$:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

il denom. non è mai nullo...

* Controesempio all' inverso del "theorem egregium"

$$(K_1 = K_2 \not\Rightarrow \Sigma_1 \text{ e } \Sigma_2 \text{ loc isometriche})$$

$$(\Leftarrow \text{ è l'egregium})$$

1

2

$\underline{x}(u,v) = (u\cos v,\ u\sin v,\ \log u)$ $u>0$, $v\in[0,2\pi)$ sup. di rot di log

$\underline{y}(u,v) = (u\cos v,\ u\sin v,\ v)$ elicoide

$$\underline{x}_u = \left(\cos v,\ \sin v,\ \frac{1}{u}\right) \qquad \underline{y}_u = (\cos v,\ \sin v,\ 0)$$

$$\underline{x}_v = (-u\sin v,\ u\cos v,\ 0) \qquad \underline{y}_v = (-u\sin v,\ u\cos v,\ 1)$$

$$E_1 = 1 + \frac{1}{u^2} \qquad E_2 = 1$$

$$F_1 = 0 \qquad F_2 = 0$$

$$G_1 = u^2 \qquad G_2 = u^2 + 1$$

$$\underline{N}_1 = \frac{\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos v & \sin v & \frac{1}{u} \\ -u\sin v & u\cos v & 0 \end{vmatrix}}{\|\ \cdot\ \|} = \frac{\underline{i}\,(-\cos v) - \underline{j}\,\sin v + \underline{k}\,u}{\|\quad\|} =$$

$$= \frac{(-\cos v)\,\underline{i} + (-\sin v)\,\underline{j} + u\,\underline{k}}{\sqrt{1+u^2}}$$

$$\underline{N}_2 = \frac{\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix}}{\|\ \cdot\ \|} = \frac{\underline{i}\,(\sin v) - \underline{j}\,\cos v + \underline{k}\,u}{\sqrt{1+u^2}}$$

$$\underline{x}_{uu} = \left(0 \quad 0 \quad -\frac{1}{u^2}\right) \qquad e_1 = -\frac{1}{u}\cdot\frac{1}{\sqrt{1+u^2}}$$

$$\underline{x}_{uv} = (-\sin v \quad \cos v, \ 0) \qquad f_1 = \sin v\cos v - \sin v \cos v + 0 = 0$$

$$\underline{x}_{vv} = (-u\cos v \quad -u\sin v, \ 0) \qquad g_1 = +u(\cos^2 v + \sin^2 v) = \frac{u}{\sqrt{1+u^2}}$$

$$\underline{y}_{uu} = (0 \quad 0 \quad 0) \qquad e_2 = 0$$

$$\underline{y}_{uv} = (-\sin v \quad \cos v \quad 0) \qquad f_2 = (-\sin^2 v - \cos^2 v)\frac{1}{r} = \frac{-1}{\sqrt{1+u^2}}$$

$$\underline{y}_{vv} = (-u\cos v \quad -u\sin v \quad 0) \qquad g_2 = \ldots = 0$$

$$K_1 = \frac{-\frac{1}{1+u^2}}{\left(1+\frac{1}{u^2}\right)u^2} = \frac{-\frac{1}{1+u^2}}{(1+u^2)} = -\frac{1}{(1+u^2)^2}$$

$$=? \ \boxed{K_1 = K_2}$$

$$K_2 = \frac{-\frac{1}{1+u^2}}{u^2+1} = -\frac{1}{(1+u^2)^2}$$

Sia ora $u = \text{cost} = u_0$ $\quad \underline{x}(u_0, v) = (u_0\cos v, u_0 \sin v, \log u_0)$

$\underline{y}(u_0, v) = (u_0 \cos v, u_0 \sin v, v)$

tratti corrispondenti di tali curve non hanno la stessa lunghezza.

$$\boxed{\frac{ds_1^2}{ds_2^2} = \frac{u_0^2}{u_0^2+1}} \neq \left[\begin{array}{l} ds_1^2 = u_0^2\, dv^2 \\ ds_2^2 = (u_0^2+1)\, dv^2 \end{array}\right]$$

XII-15

★ Curvatura dell'elicoide, in altro modo

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right) \qquad (F=0)$$

$$E = 1 \qquad F = 0 \qquad G = u^2+1$$

$$K = -\frac{1}{2\sqrt{1+u^2}}\frac{d}{du}\left(\frac{2u}{\sqrt{1+u^2}}\right) =$$

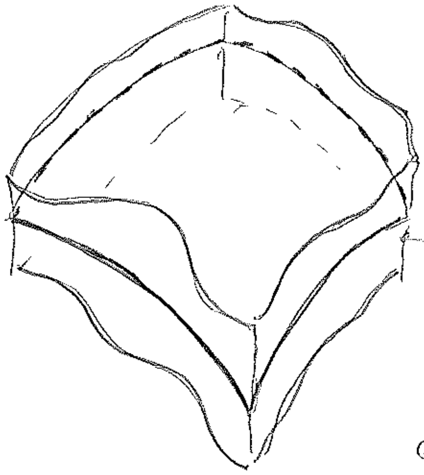
$$= -\frac{1}{2\sqrt{1+u^2}}\frac{2\sqrt{1+u^2} - 2u\,\frac{1}{2}\,\frac{1}{\sqrt{1+u^2}}\cdot 2u}{(1+u^2)} =$$

$$= -\frac{1}{2\sqrt{1+u^2}}\frac{2(1+u^2) - 2u^2}{(1+u^2)\sqrt{1+u^2}}$$

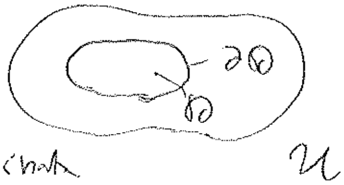
$$= -\frac{1}{(u^2+1)^2}$$

* Superfici minime $(H = 0)$ [Lagrange, 1760]

↑ curvatura media

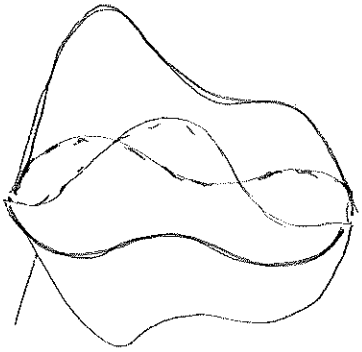


$\underline{r} = \underline{r}(u,v)$ $\qquad (u,v) \in \mathcal{U}$



Variazione normale determinata da $h : \overline{D} \to \mathbb{R}$ (liscia)

$$\underline{\varphi}(\underbrace{u,v}_{\in\, \overline{D}}, \underbrace{t}_{\in\,(-\varepsilon,\varepsilon)}) := \underline{r}(u,v) + t\,h(u,v)\,\underline{N}(u,v)$$



per t fissato ho $\underline{r}^t = \underline{r}^t(u,v)$

si ha:

$$\begin{cases} \underline{r}^t_u = \underline{r}_u + th\underline{N}_u + t\,h_u\underline{N} \\ \underline{r}^t_v = \underline{r}_v + th\underline{N}_v + t\,h_v\underline{N} \end{cases}$$

profilo fissato

ricerca di una superficie di area minima:

* problema di Plateau

[o della bolla di sapone]

e, pertanto, per la relativa metrica, si ha:

$$E^t = E + th\left[\underbrace{\langle \underline{r}_u, \underline{N}_u\rangle}_{-e} + \underbrace{\langle \underline{r}_u, \underline{N}_u\rangle}_{-e}\right] + \dots$$

$$F^t = F + th\left[-2f\right] + \dots$$

$$G^t = G + th\left[-2g\right] + \dots$$

Sicché $\quad E^t G^t - {F^t}^2 = EG - F^2 +$

$$+ (-2th)\left[Eg - 2Ff + Ge \right] + \dots$$

$$= (EG - F^2)\left[1 - 4th\,H \right] + o(t)$$

elemento d'area $\longrightarrow \quad d\sigma^t = d\sigma \cdot (1 - 2tHh)$

$$A(t) = \int_{\mathcal{D}} d\sigma^t \qquad\qquad \frac{d\sigma^t}{dt} = -2tHh\,d\sigma$$

$$\mathring{A}(0) = \int_{\mathcal{D}} \frac{d\sigma^t}{dt} = -\int_{\mathcal{D}} 2hH\,d\sigma$$

si ha $\mathring{A}(0) = 0 \quad \forall\, h$ (stazionarietà)

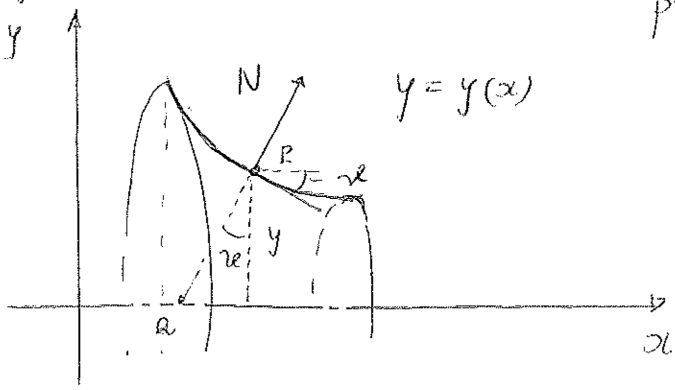
variazione normale

$\Longleftrightarrow \quad H = 0$ (interpretazione variazionale della condizione $H = 0$)

È un'equazione di E-L.

☆ La catenoide è l'unica superficie di rivoluzione minima

Una dimostrazione tramite le eq. di Eulero-Lagrange è stata già fornita prima (v. cap. XI). Qui procediamo per via geometrica.



Dalla teoria generale delle sup. di rivoluzione, le curvature principali sono quella del (generico) meridiano e l'inverso della grannormale

$$\overline{PQ} = \frac{y}{\cos\nu} = y\cdot\sqrt{1+\tan^2\nu} = y\sqrt{1+y'^2}$$

$H = \frac{1}{2}(k_1 + k_2) = 0$ diventa

$$\frac{y''}{(1+y'^2)^{\frac{3}{2}}} = \frac{1}{y}\,\frac{1}{(1+y'^2)^{\frac{1}{2}}}$$

⚠ I segni sono corretti...

$\Rightarrow$ (assumendo $y' \neq 0$...) $\quad \dfrac{2y''y'}{1+y'^2} = \dfrac{2y'}{y} \Rightarrow$

$$d\log(1+y'^2) = d\log y^2 \Rightarrow$$

$$1+y'^2 = k^2y^2 \quad (k \text{ costante})$$

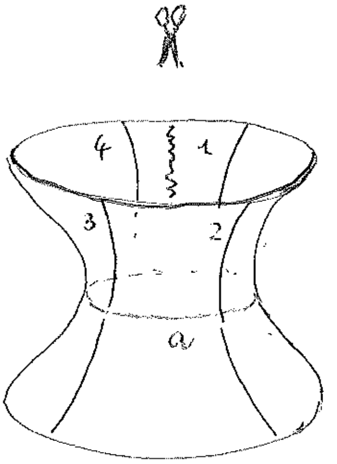
$$y'^2 = k^2y^2 - 1 \,. \quad y' = \pm\sqrt{(ky)^2-1}$$

$$\frac{dy}{\sqrt{(ky)^2-1}} = \pm\, dx$$

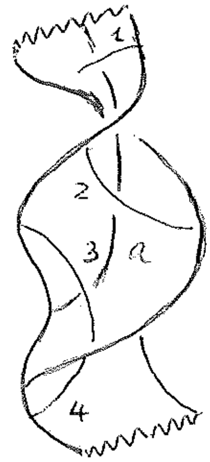
⇝ ricordando $\cosh^2\xi - \sinh^2\xi = 1$ si arriva a

$$y = \frac{1}{k}\cosh(kx + c)$$

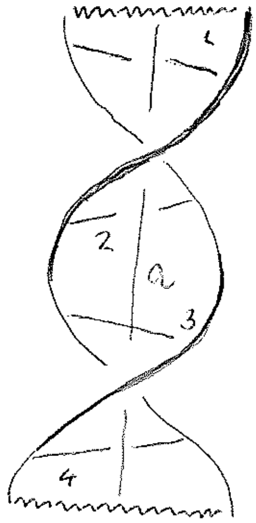
i.e. si ha una catenaria



Catenoide



Superficie di Scherk



Elicoide

★ Sup. di Scherk $\quad \underline{r}(x,y) = \left(x, y, \log\left(\frac{\cos y}{\cos x}\right)\right)$

definita sui "quadrati neri" della "scacchiera" (v. fig.)

$$H = 0$$