Introduction to the representation theory of quivers
Second Part

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Warning: In this notes we collect the topics that are discussed during the second part of the course. However, most proofs are omitted or just sketched. The complete arguments will be explained during the lecture!

Contents

1 CONSTRUCTING NEW MODULES

1.1 Our setup

Throughout this chapter, we fix a field k and a finite dimensional algebra Λ over k. We start out by collecting some well-known properties of Λ.

(1) All finitely generated Λ-modules have finite length.

(2) Every finitely generated Λ-module M has an *indecomposable decomposition* $M = \bigoplus_{i=1}^{n} M_i$ with $\operatorname{End}_\Lambda M_i$ local for all $1 \leq i \leq n$ (Theorem of Krull-Remak-Schmidt).

(3) If M, N are finitely generated Λ-modules, then $\operatorname{Hom}_\Lambda(M, N)$ is a finitely generated k-module via the multiplication

$$\alpha \cdot f \colon m \mapsto \alpha f(m) \quad \text{for} \quad \alpha \in k,\ f \in \operatorname{Hom}_\Lambda(M, N)$$

In particular, $\operatorname{End}_\Lambda N$ and $(\operatorname{End}_\Lambda N)^{\mathrm{op}}$ are again finite dimensional k-algebras, and N is a Λ-$(\operatorname{End}_\Lambda N)^{\mathrm{op}}$-bimodule via the multiplication

$$n \cdot s := s(n) \quad \text{for} \quad n \in N,\ s \in \operatorname{End}_\Lambda N$$

Moreover, $\operatorname{Hom}_\Lambda(M, N)$ is an $\operatorname{End} N$-$\operatorname{End} M$-bimodule which has finite length on both sides.

(4) There is a duality

$$D : \Lambda \operatorname{mod} \longrightarrow \ \operatorname{mod} \Lambda,\ M \mapsto \operatorname{Hom}_k(M, k),$$

and ${}_\Lambda D(\Lambda_\Lambda)$ is an injective cogenerator of $\Lambda \operatorname{Mod}$.

(5) The Jacobson radical $J = J(\Lambda)$ is nilpotent, i.e. $J^n = 0$ for some $n \in \mathbb{N}$, and Λ/J is semisimple. Further, $\operatorname{Rad} M = JM$ for every $M \in \Lambda \operatorname{mod}$.

(6) Λ is semiperfect, i.e. there are orthogonal idempotents

$$e_1, \ldots, e_n \in \Lambda \text{ such that } 1 = \sum_{i=1}^{n} e_i,$$

and $e_i \Lambda e_i$ is a local ring for every $1 \leq i \leq n$. This yields the indecomposable decompositions

$${}_\Lambda\Lambda = \bigoplus_{i=1}^{n} \Lambda e_i \ \text{ and } \ {}_\Lambda\Lambda/J \cong \bigoplus_{i=1}^{n} \Lambda e_i/J e_i$$

(7) Λ is Morita equivalent to a *basic* finite dimensional algebra, that is, the category $\Lambda \operatorname{Mod}$ is equivalent to $S \operatorname{Mod}$ where S is a finite dimensional algebra with the property that ${}_S S$ is a direct sum of *pairwise nonisomorphic* projectives, or equivalently, $S/J(S)$ is a product of division rings, see [1, p. 309] or [12, II.2].

From now on, we will assume that Λ is basic. Then

$$\Lambda e_1, \ldots, \Lambda e_n$$

are representatives of the isomorphism classes of the indecomposable projectives in $\Lambda\,\mathrm{Mod}$,

$$\Lambda e_1/Je_1, \ldots, \Lambda e_n/Je_n$$

are representatives of the isomorphism classes of the simples in $\Lambda\,\mathrm{Mod}$, and

$$D(e_1\Lambda), \ldots, D(e_n\Lambda)$$

are representatives of the isomorphism classes of the indecomposable injectives in $\Lambda\,\mathrm{Mod}$. If $1 \le i \le n$, then $D(\Lambda e_i/Je_i) \cong e_i\Lambda/e_iJ$, and ${}_\Lambda D(e_i\Lambda)$ is an injective envelope of $\Lambda e_i/Je_i$.

Starting from these known modules, we want to construct new indecomposable Λ-modules. We first need some preliminaries.

1.2 Reminder on projectives and minimal projective resolutions.

Recall that every Λ-module M has a *projective cover* $p : P \to M$, that is, p is an epimorphism with P being projective and $\operatorname{Ker} p$ being superfluous. Then $\operatorname{Ker} p \subset J\,P$, and no non-zero summand of P is contained in $\operatorname{Ker} p$.
We infer that every Λ-module M has a *minimal projective presentation*

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$$

and a *minimal projective resolution*

$$\cdots P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$$

that is, a long exact sequence where p_0 is a projective cover of M, p_1 is a projective cover of $\operatorname{Ker} p_0$, and so on. In other words, for all $i \ge 0$

$$\operatorname{Im} p_{i+1} = \operatorname{Ker} p_i \subset \operatorname{Rad} P_i = J\,P_i.$$

We will often just consider the *complex* of projectives

$$P^{\cdot} : \qquad \cdots P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \to 0 \to \cdots$$

and will also call it a projective resolution of M (see Section 2.3).

Proposition 1.2.1. *Let M, N be two modules with projective resolutions $P^{\cdot}$ and $Q^{\cdot}$, respectively, and let $f : M \to N$ be a homomorphism.*

1. *There are homomorphisms $f_0, f_1, \ldots$ making the following diagram commutative*

$$\begin{array}{cccccccc} \cdots & P_1 & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & M & \longrightarrow 0 \\ & \downarrow f_1 & & \downarrow f_0 & & \downarrow f & \\ \cdots & Q_1 & \xrightarrow{q_1} & Q_0 & \xrightarrow{q_0} & N & \longrightarrow 0 \end{array}$$

Then $f^{\cdot} = (f_n)_{n\geq 0} : P^{\cdot} \to Q^{\cdot}$ *is called a* chain map.

2. *If* $g^{\cdot} = (g_n)_{n\geq 0} : P^{\cdot} \to Q^{\cdot}$ *is another chain map as above, then there are homomorphisms* $s_n : P_n \to Q_{n+1}, n \geq 0$ *such that, setting* $h_n = f_n - g_n$*, we have*

$$h_0 = q_1 s_0,$$

$$h_n = s_{n-1}p_n + q_{n+1}s_n \text{ for } n \geq 1.$$

Then $s = (s_n)_{n\geq 0}$ *is called a homotopy between* $P^{\cdot}$ *and* $Q^{\cdot}$*, and we say that the chain maps* $f^{\cdot}$ *and* $g^{\cdot}$ *are* homotopic *(or that* $h^{\cdot} = (h_n)_{n\geq 0}$ *is* homotopic to zero*).*

1.3 The Auslander-Bridger transpose

Recall the following property of projective modules.

Lemma 1.3.1. *A module* ${}_RP$ *is projective if and only if it has a* dual basis*, that is, a pair* $(\,(x_i)_{i\in I},\,(\varphi_i)_{i\in I}\,)$ *consisting of elements* $(x_i)_{i\in I}$ *in* P *and homomorphisms* $(\varphi_i)_{i\in I}$ *in* $P^* = \mathrm{Hom}_R(P,R)$ *such that every element* $x \in P$ *can be written as* $x = \sum_{i\in I}\varphi_i(x)\,x_i$ *with* $\varphi_i(x) = 0$ *for almost all* $i \in I$*.*

As a consequence, we obtain the following properties of the contravariant functor ${}^* = \mathrm{Hom}(-,\Lambda) : \Lambda\mathrm{Mod} \longrightarrow \mathrm{Mod}\,\Lambda$.

Proposition 1.3.2. *Let* P *be a finitely generated projective left* Λ*-module. Then* P^* *is a finitely generated projective right* Λ*-module, and* $P^{**} \cong P$*. Moreover, if* I *is an ideal of* Λ*, then* $\mathrm{Hom}_\Lambda(P,I) = P^* \cdot I$*.*

Proof: We only sketch the arguments. First of all, note that the evaluation map $c : P \to P^{**}$ defined by $c(x)(\varphi) = \varphi(x)$ on $x \in P$ and $\varphi \in P^*$ is a monomorphism. Further, if $(\,(x_i)_{1\leq i\leq n},\,(\varphi_i)_{1\leq i\leq n}\,)$ is a dual basis of P, then it is easy to see that $(\,(\varphi_i)_{1\leq i\leq n},\,(c(x_i))_{1\leq i\leq n}\,)$ is a dual basis of P^*. This shows that P^* is finitely generated projective. The isomorphism $P^{**} \cong P$ is proved by showing that the assignment $P^{**} \ni f \mapsto \sum_{i=1}^n f(\varphi_i)\,x_i \in P$ is inverse to c.
For the second statement, the inclusion $\subset$ follows immediately from the fact that $\varphi \in \mathrm{Hom}_\Lambda(P,I)$ satisfies $\varphi(x_i) \in I$ for all $1 \leq i \leq n$, and $\supset$ follows from the fact that for $\varphi \in P^*$ and $a \in I$ we have $(\varphi \cdot a)(x) = \varphi(x) \cdot a \in I$.$\square$

So, the functor ${}^* = \mathrm{Hom}(-,\Lambda) : \Lambda\mathrm{Mod} \longrightarrow \mathrm{Mod}\,\Lambda$ induces a duality between the full subcategories of finitely generated projective modules in $\Lambda\,\mathrm{Mod}$ and $\mathrm{Mod}\Lambda$. The following construction from [9] can be viewed as a way to extend this duality to all finitely presented modules.
We denote by $\Lambda\mathrm{mod}_{\mathcal{P}}$ the full subcategory of $\Lambda\,\mathrm{mod}$ consisting of the modules without non-zero projective summands.
Let $M \in \Lambda\,\mathrm{mod}_{\mathcal{P}}$ and let $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$ be a minimal projective presentation of M. Applying the functor ${}^* = \mathrm{Hom}_\Lambda(-,\Lambda)$ on it, we obtain a minimal projective presentation

$$P_0^* \xrightarrow{p_1^*} P_1^* \to \mathrm{Coker}\,p_1^* \to 0\,.$$

Set $\operatorname{Tr} M = \operatorname{Coker} p_1^*$. Then $\operatorname{Tr} M \in \Lambda\operatorname{mod}_{\mathcal{P}}$. Moreover, the following hold true.

(1) The isomorphism class of $\operatorname{Tr} M$ does not depend on the choice of $P_1 \to P_0 \to M \to 0$.

(2) There is a natural isomorphism $\operatorname{Tr}^2(M) \cong M$.

Let us now consider a homomorphism $f \in \operatorname{Hom}_\Lambda(M, N)$ with $M, N \in \Lambda\operatorname{mod}$. It induces a commutative diagram

$$\begin{tikzcd}
P_1 \arrow[r, "p_1"] \arrow[d, "f_1"] & P_0 \arrow[r, "p_0"] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
Q_1 \arrow[r, "q_1"] & Q_0 \arrow[r, "q_0"] & N \arrow[r] & 0
\end{tikzcd}$$

Applying $^* = \operatorname{Hom}(-, \Lambda)$, we can construct $\widetilde{f} \in \operatorname{Hom}(\operatorname{Tr} N, \operatorname{Tr} M)$ as follows:

$$\begin{tikzcd}
P_0^* \arrow[r, "p_1^*"] & P_1^* \arrow[r] & \operatorname{Tr} M \arrow[r] & 0 \\
Q_0^* \arrow[r, "q_1^*"] \arrow[u, "f_0^*"] & Q_1^* \arrow[r] \arrow[u, "f_1^*"] & \operatorname{Tr} N \arrow[r] \arrow[u, "\widetilde{f}"] & 0
\end{tikzcd}$$

Note that this construction is not unique since $\widetilde{f}$ depends on the choice of f_0, f_1. However, if we choose another factorization of f, say by maps g_0 and g_1, and conctruct $\widetilde{g}$ correspondingly, then the difference $f_0 - g_0 \in \operatorname{Ker} q_0 = \operatorname{Im} q_1$ factors through Q_1, and so $\widetilde{f} - \widetilde{g}$ factors through P_1^*, as illustrated below:

$$\begin{tikzcd}
P_1 \arrow[r, "p_1"] \arrow[d, "f_1"] \arrow[d, dotted, bend right, "g_1"'] & P_0 \arrow[r, "p_0"] \arrow[d, "f_0"] \arrow[d, dotted, bend right, "g_0"'] & M \arrow[r] \arrow[d, "f"] & 0 \\
Q_1 \arrow[r, "q_1"'] & Q_0 \arrow[r, "q_0"'] & N \arrow[r] & 0
\end{tikzcd}
\quad \Rightarrow \quad
\begin{tikzcd}
P_0^* \arrow[r, "p_1^*"] & P_1^* \arrow[r] & \operatorname{Tr} M \arrow[r] & 0 \\
Q_0^* \arrow[r, "q_1^*"'] \arrow[u, "f_0^*"] \arrow[u, dotted, bend right, "g_0^*"'] & Q_1^* \arrow[r] \arrow[u, "f_1^*"] \arrow[u, dotted, bend right, "g_1^*"'] & \operatorname{Tr} N \arrow[r] \arrow[u, "\widetilde{f}"] \arrow[u, dotted, bend right, "\widetilde{g}"'] \arrow[ul] & 0
\end{tikzcd}$$

In other words, if we consider the subgroups
$P(M, N) = \{f \in \operatorname{Hom}(M, N) \mid f \text{ factors through a projective module}\} \leq \operatorname{Hom}_\Lambda(M, N)$, then $\widetilde{f}$ is uniquely determined modulo $P(\operatorname{Tr} N, \operatorname{Tr} M)$.

We set $\underline{\operatorname{Hom}}_\Lambda(M, N) = \operatorname{Hom}_\Lambda(M, N)/P(M, N)$, and let $\Lambda\,\underline{\operatorname{mod}}$ be the category with the same objects as $\Lambda\operatorname{mod}$ and morphisms $\underline{\operatorname{Hom}}_\Lambda(M, N)$. It is called the *stable category* of $\Lambda\operatorname{mod}$ modulo projectives. We obtain the following.

Proposition 1.3.3.

(1) There is a group isomorphism $\underline{\operatorname{Hom}}(M, N) \to \underline{\operatorname{Hom}}(\operatorname{Tr} N, \operatorname{Tr} M)$, $\underline{f} \mapsto \underline{\widetilde{f}}$.

(2) $\operatorname{End}_\Lambda M$ *is local if and only if* $\operatorname{End} \operatorname{Tr} M_\Lambda$ *is local.*

(3) Tr *induces a duality* $\Lambda\,\underline{\operatorname{mod}} \to \underline{\operatorname{mod}}\,\Lambda$.

1.4 The Nakayama functor

We now combine the transpose with the duality D. Denote by

$$\nu : \Lambda\,\mathrm{Mod} \to \Lambda\,\mathrm{Mod},\ X \mapsto D(X^*)$$

the *Nakayama functor.*

Lemma 1.4.1. *The functor ν has the following properties.*

1. *ν is covariant and right exact.*
2. *$\nu(\Lambda e_i) = D(e_i\Lambda)$ is the injective envelope of $\Lambda e_i/Je_i$ for $1 \le i \le n$.*
3. *$\nu({}_\Lambda\Lambda) = D(\Lambda_\Lambda)$ is an injective cogenerator of Λ Mod.*
4. *For $M \in \Lambda$ mod with minimal projective presentation $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \to 0$ there is an exact sequence*

$$0 \to D\,\mathrm{Tr}\,M \to \nu(P_1) \xrightarrow{\nu(p_1)} \nu(P_0) \to \nu(M) \to 0$$

1.5 The Auslander-Reiten translation

We denote

$$\tau(M) = D\,\mathrm{Tr}\,M = \mathrm{Ker}\,\nu(p_1).$$

The functor τ is called *Auslander-Reiten translation*

Denote by $\Lambda\mathrm{mod}_{\mathcal{I}}$ the full subcategory of Λmod consisting of the modules without non-zero injective summands. For $M, N \in \Lambda\,\mathrm{mod}$ consider further the subgroup
$I(M, N) = \{f \in \mathrm{Hom}_\Lambda(M, N) \mid f \text{ factors through an injective module}\} \le \mathrm{Hom}_\Lambda(M, N)$, set $\overline{\mathrm{Hom}}_\Lambda(M, N) = \mathrm{Hom}_\Lambda(M, N)/I(M, N)$, and let $\Lambda\,\overline{\mathrm{mod}}$ be the category with the same objects as $\Lambda\,\mathrm{mod}$ and morphisms $\overline{\mathrm{Hom}}_\Lambda(M, N)$.

Proposition 1.5.1. *(1) The duality D induces a duality $\Lambda\,\underline{\mathrm{mod}} \to \overline{\mathrm{mod}}\,\Lambda$.*

(2) The composition $\tau = D\,\mathrm{Tr}\colon \Lambda\,\underline{\mathrm{mod}} \to \Lambda\,\overline{\mathrm{mod}}$ is an equivalence with inverse $\tau^- = \mathrm{Tr}\,D\colon \Lambda\,\overline{\mathrm{mod}} \to \Lambda\,\underline{\mathrm{mod}}$.

Example 1.5.2. *Let $\Lambda = kA_3$ be the path algebra of the quiver $\underset{1}{\bullet} \to \underset{2}{\bullet} \to \underset{3}{\bullet}$.*
The indecomposable projectives are $P_1,\ P_2 = JP_1,\ P_3 = S_3 = JP_2$, and the indecomposable injectives are $I_1 = S_1 = I_2/S_2,\ I_2 = I_3/S_3,\ I_3 = P_1$.
We compute τS_2. Taking the minimal projective resolution $0 \to P_3 \to P_2 \to S_2 \to 0$, and using that $S_2^* = 0$ and thus $\nu(S_2) = 0$, we obtain an exact sequence

$$0 \to \tau S_2 \to I_3 \to I_2 \to 0$$

showing that $\tau S_2 = S_3$.

2 SOME HOMOLOGICAL ALGEBRA

Throughout this chapter, let R be a ring, and denote by $R\,\mathrm{Mod}$ the category of all left R-modules.

2.1 Push-out and Pull-back

Proposition 2.1.1. [26, pp. 41] *Consider a pair of homomorphisms in $R\mathrm{Mod}$*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \\ C & & \end{array}$$

There is a module $_RL$ together with homomorphisms $\sigma : C \to L$ and $\tau : B \to L$ such that (i) the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow \tau \\ C & \xrightarrow{\sigma} & L \end{array}$$

commutes; and
(ii) given any other module $_RL'$ together with homomorphisms $\sigma' : C \to L'$ and $\tau' : B \to L'$ making the diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow \tau' \\ C & \xrightarrow{\sigma'} & L' \end{array}$$

commute, there exists a unique homomorphism $\gamma : L \to L'$ such that $\gamma\sigma = \sigma'$ and $\gamma\tau = \tau'$.

The module L together with σ, τ is unique up to isomorphism and is called push-out *of f and g.*

Proof: We just sketch the construction. The module L is defined as the quotient $L = B \oplus C \,/\, \{\, (f(a), -g(a)) \mid a \in A\}$, and the homomorphisms are given as $\sigma : C \to L,\ c \mapsto \overline{(0,c)}$, and $\tau : B \to L,\ b \mapsto \overline{(b,0)}$. □

Remark 2.1.2. *If f is a monomorphism, also σ is a monomorphism, and* $\mathrm{Coker}\,\sigma \cong \mathrm{Coker}\,f$.

Dually, one defines the *pull-back* of a pair of homomorphisms

$$\begin{array}{ccc} & & B \\ & & \downarrow f \\ C & \xrightarrow{g} & A \end{array}$$

2.2 A short survey on Ext^1

Aim of this section is to give a brief introduction to the functor Ext^1, as needed in the sequel. For a comprehensive treatment we refer to textbooks in homological algebra, e.g. [26].

Definition. Let A, B be two R-modules. We define a relation on short exact sequences of the form $\mathfrak{E} : 0 \to B \to M \to A \to 0$ by setting

$$\mathfrak{E}_1 : 0 \to B \to E_1 \to A \to 0 \;\sim\; \mathfrak{E}_2 : 0 \to B \to E_2 \to A \to 0$$

if there is $f \in \mathrm{Hom}_R(E_1, E_2)$ making the following diagram commute

$$\begin{array}{cccccccccc}
\mathfrak{E}_1 : & 0 & \to & B & \longrightarrow & E_1 & \longrightarrow & A & \longrightarrow & 0 \\
 & & & \| & & \downarrow f & & \| & & \\
\mathfrak{E}_2 : & 0 & \to & B & \longrightarrow & E_2 & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

It is easy to see that $\sim$ is an equivalence relation, and we denote by $\mathrm{Ext}^1_R(A, B)$ the set of all equivalence classes.

Next, we want to define a group structure on $\mathrm{Ext}^1_R(A, B)$. Let $[\mathfrak{E}]$ be the equivalence class of the short exact sequence $\mathfrak{E} : 0 \to B \to E \to A \to 0$. First of all, for $\beta \in \mathrm{Hom}_R(B, B')$ we can consider the short exact sequence $\beta\,\mathfrak{E}$ given by the push-out diagram

$$\begin{array}{cccccccccc}
\mathfrak{E} : & 0 & \to & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0 \\
 & & & \downarrow \beta & & \downarrow & & \| & & \\
\beta\,\mathfrak{E} : & 0 & \to & B' & \longrightarrow & E' & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

In this way, we can define a map

$$\mathrm{Ext}^1_R(A, \beta) : \mathrm{Ext}^1_R(A, B) \to \mathrm{Ext}^1_R(A, B'),\ [\mathfrak{E}] \mapsto [\beta\,\mathfrak{E}]$$

For $\beta_1 \in \mathrm{Hom}_R(B', B'')$ and $\beta_2 \in \mathrm{Hom}_R(B, B')$ one verifies

$$\mathrm{Ext}^1_R(A, \beta_1)\,\mathrm{Ext}^1_R(A, \beta_2) = \mathrm{Ext}^1_R(A, \beta_1\beta_2)$$

Dually, for $\alpha \in \mathrm{Hom}_R(A', A)$, we use the pull-back diagram

$$\begin{array}{cccccccccc}
\mathfrak{E}\,\alpha : & 0 & \to & B & \longrightarrow & E' & \longrightarrow & A' & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow \alpha & & \\
\mathfrak{E} : & 0 & \to & B & \longrightarrow & E & \longrightarrow & A & \longrightarrow & 0
\end{array}$$

to define a map

$$\mathrm{Ext}^1_R(\alpha, B) : \mathrm{Ext}^1_R(A, B) \to \mathrm{Ext}^1_R(A', B),\ [\mathfrak{E}] \mapsto [\mathfrak{E}\,\alpha]$$

Since

$$\mathrm{Ext}^1_R(\alpha, B')\,\mathrm{Ext}^1_R(A, \beta)[\mathfrak{E}] = \mathrm{Ext}^1_R(A', \beta)\,\mathrm{Ext}^1_R(\alpha, B)[\mathfrak{E}]$$

the composition of the maps above yields a map

$$\mathrm{Ext}^1_R(\alpha,\beta):\mathrm{Ext}^1_R(A,B)\to\mathrm{Ext}^1_R(A',B')$$

Now we are ready to define an addition on $\mathrm{Ext}^1_R(A,B)$, called *Baer sum*. Given two sequences $\mathfrak{E}_1 : 0\to B\to E_1\to A\to 0$ and $\mathfrak{E}_2 : 0\to B\to E_2\to A\to 0$, we consider the direct sum $\mathfrak{E}_1\oplus\mathfrak{E}_2 : 0\to B\oplus B\to E_1\oplus E_2\to A\oplus A\to 0$ together with the diagonal map $\Delta_A : A\to A\oplus A$, $a\mapsto(a,a)$, and the summation map $\nabla_B : B\oplus B\to B$, $(b_1,b_2)\mapsto b_1+b_2$. We then set

$$[\mathfrak{E}_1]+[\mathfrak{E}_2]=\mathrm{Ext}^1_R(\Delta_A,\nabla_B)([\mathfrak{E}_1\oplus\mathfrak{E}_2])\ \in\mathrm{Ext}^1_R(A,B)$$

In this way, $\mathrm{Ext}^1_R(A,B)$ becomes an abelian group. Its zero element is the equivalence class of all split exact sequences. The inverse element of the class $[\mathfrak{E}]$ given by the sequence $\mathfrak{E} : 0\to B\xrightarrow{f}E\xrightarrow{g}A\to 0$ is the equivalence class of the sequence $0\to B\xrightarrow{f}E\xrightarrow{-g}A\to 0$.

Moreover, the maps $\mathrm{Ext}^1_R(A,\beta),\mathrm{Ext}^1_R(\alpha,B)$ are group homomorphisms, and we have a covariant functor $\mathrm{Ext}^1_R(A,-) : R\text{ Mod}\to\mathbb{Z}\,\text{Mod}$ and a contravariant functor $\mathrm{Ext}^1_R(-,B) : R\text{ Mod}\to\mathbb{Z}\,\text{Mod}$.

2.3 The category of complexes

Let R be a ring.
Definitions. (1) A *(co)chain complex* of R-modules $A^\cdot=(A^n,d^n)$ is given by a chain

$$A^\cdot:\qquad \cdots\to A^{-2}\xrightarrow{d^{-2}}A^{-1}\xrightarrow{d^{-1}}A^0\xrightarrow{d^0}A^1\xrightarrow{d^1}A^2\xrightarrow{d^2}\cdots$$

of R-modules A^n with R-homomorphisms $d^n : A^n\to A^{n+1}$, called *differentials*, satisfying

$$d^{n+1}\circ d^n=0$$

for all $n\in\mathbb{Z}$. Given two complexes $A^\cdot,A'^\cdot$, a *(co)chain map* $f^\cdot : A^\cdot\to A'^\cdot$ is given by a family of R-homomorphisms $f^n : A^n\to A'^n$ such that the following diagram is commutative

$$\begin{array}{ccccccc}
\cdots & A^{-1} & \xrightarrow{d^{-1}} & A^0 & \xrightarrow{d^0} & A^1 & \longrightarrow\cdots\\
 & \downarrow f^{-1} & & \downarrow f^0 & & \downarrow f^1 & \\
\cdots & A'^{-1} & \xrightarrow{d'^{-1}} & A'^0 & \xrightarrow{d'^0} & A'^1 & \longrightarrow\cdots
\end{array}$$

Complexes and chain maps form the category of complexes $\mathcal{C}(R\text{ Mod})$.

(2) Given a *complex* of R-modules $A^\cdot=(A^n,d^n)$, the abelian group

$$H^n(A^\cdot)=\mathrm{Ker}\,d^n/\,\mathrm{Im}\,d^{n-1}$$

is called n-th *(co)homology group*. Note that $H^n(A)$ is an R-module, and every chain map $f^{\cdot} : A^{\cdot} \to A'^{\cdot}$ induces R-homomorphisms $H^n(f^{\cdot}) : H^n(A^{\cdot}) \to H^n(A'^{\cdot})$. So, for every $n \in \mathbb{Z}$ there is a functor

$$H^n : \mathcal{C}(R\ \mathrm{Mod}) \to R\ \mathrm{Mod}\,.$$

(3) A chain map $h^{\cdot} : M^{\cdot} \to M'^{\cdot}$ is *homotopic to zero* if there is a *homotopy* $s = (s^n)$ with homomorphisms $s^n : M^n \to M^{n-1}$, $n \in \mathbb{Z}$, such that

$$h^n = s^{n+1}d^n + d'^{n-1}s^n \ \text{ for } n \in \mathbb{Z}.$$

Two chain maps $f^{\cdot}, g^{\cdot} : M^{\cdot} \to M'^{\cdot}$ are *homotopic* if the chain map $h^{\cdot} = f^{\cdot} - g^{\cdot}$ given by $h^n = f^n - g^n$ is homotopic to zero.

Lemma 2.3.1. *Let $f^{\cdot}, g^{\cdot} : M^{\cdot} \to M'^{\cdot}$ be two chain maps.*
(1) If $f^{\cdot}$ and $g^{\cdot}$ are homotopic, then $H^n(f^{\cdot}) = H^n(g^{\cdot})$ for all $n \in \mathbb{Z}$.
(2) If $g^{\cdot}f^{\cdot}$ is homotopic to $\mathrm{id}_{A^{\cdot}}$ and $f^{\cdot}g^{\cdot}$ is homotopic to $\mathrm{id}_{A'^{\cdot}}$, then $H^n(f^{\cdot})$ is an isomorphism for all $n \in \mathbb{Z}$.

Lemma 2.3.2. *Let $\to A^{\cdot} \xrightarrow{f^{\cdot}} B^{\cdot} \xrightarrow{g^{\cdot}} C^{\cdot} \to 0$ be a short exact sequence in $\mathcal{C}(R\ \mathrm{Mod})$, that is, $f^{\cdot}, g^{\cdot}$ are chain maps inducing short exact sequences in each degree. Then there is a long exact sequence of (co-)homology groups*

$$\cdots \to H^{n-1}(C^{\cdot}) \xrightarrow{\delta_{n-1}} H^n(A^{\cdot}) \xrightarrow{H^n(f^{\cdot})} H^n(B^{\cdot}) \xrightarrow{H^n(g^{\cdot})} H^n(C^{\cdot}) \xrightarrow{\delta_n} H^{n+1}(A^{\cdot}) \xrightarrow{H^{n+1}(f^{\cdot})} \cdots$$

given by natural connecting homomorphisms

$$\delta_n : H^n(C^{\cdot}) \to H^{n+1}(A^{\cdot}).$$

2.4 The functors Ext^n

Theorem 2.4.1. *Let A, B be two R-modules, and let the complex*

$$P^{\cdot} : \qquad \cdots P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \to 0 \cdots$$

be a projective resolution of A. Consider the abelian group complex

$$\mathrm{Hom}_R(P^{\cdot}, B) : 0 \to \mathrm{Hom}_R(P_0, B) \xrightarrow{\mathrm{Hom}_R(p_1,B)} \mathrm{Hom}_R(P_1, B) \xrightarrow{\mathrm{Hom}_R(p_2,B)} \mathrm{Hom}_R(P_2, B) \to \cdots$$

Then the homology groups $H^n(\mathrm{Hom}_R(P^{\cdot}, B))$ do not depend from the choice of $P^{\cdot}$, and

$$\mathrm{Hom}_R(A, B) \cong H^0(\mathrm{Hom}_R(P^{\cdot}, B))$$

$$\mathrm{Ext}^1_R(A, B) \cong H^1(\mathrm{Hom}_R(P^{\cdot}, B))$$

Definition. For $n \in \mathbb{N}$ we set

$$\mathrm{Ext}^n_R(A,B) = H^n(\mathrm{Hom}_R(P^\cdot, B))$$

called the n-th *extension group*. We thus obtain additive covariant (respectively, contravariant) functors

$$\mathrm{Ext}^n_R(A,-) : R\ \mathrm{Mod} \to \mathrm{Ab},$$

$$\mathrm{Ext}^n_R(-,B) : R\ \mathrm{Mod} \to \mathrm{Ab}.$$

The Ext-functors "repair" the non-exactness of the Hom-functors as follows.

Lemma 2.4.2. *Let* $\mathfrak{E} :\ 0 \to B \xrightarrow{\beta} B' \xrightarrow{\beta'} B'' \to 0$ *be a short exact sequence in* R Mod, *and* A *an* R*-module. Then there is a long exact sequence*

$$0 \to \mathrm{Hom}_R(A,B) \xrightarrow{\mathrm{Hom}_R(A,\beta)} \mathrm{Hom}_R(A,B') \xrightarrow{\mathrm{Hom}_R(A,\beta')} \mathrm{Hom}_R(A,B'') \xrightarrow{\delta}$$

$$\mathrm{Ext}^1_R(A,B) \xrightarrow{\mathrm{Ext}^1_R(A,\beta)} \mathrm{Ext}^1_R(A,B') \xrightarrow{\mathrm{Ext}^1_R(A,\beta')} \mathrm{Ext}^1_R(A,B'') \longrightarrow \cdots$$

Here $\delta = \delta(A,\mathfrak{E})$ *is given by* $\delta(f) = [\mathfrak{E}\, f]$.
The dual statement for the contravariant functors $\mathrm{Hom}(-,B)$, $\mathrm{Ext}^1_R(-,B)$ *also holds true.*

Note that, since every short exact sequence starting at an injective module is split exact, we have that a module I is injective if and only if $\mathrm{Ext}^1_R(A,I) = 0$ for all modules A. Similarly, a module P is projective if and only if $\mathrm{Ext}^1_R(P,B) = 0$ for all module B. As a consequence, we obtain the following description of Ext^1.

Proposition 2.4.3. *Let* A, B *be left* R*-modules.*
If $0 \to B \to I \xrightarrow{\pi} C \to 0$ *is a short exact sequence where* I *is injective, then*

$$\mathrm{Ext}^1_R(A,B) \cong \mathrm{Coker}\,\mathrm{Hom}_R(A,\pi)$$

Similarly, if $0 \to K \xrightarrow{\iota} P \to A \to 0$ *is a short exact sequence where* P *is projective, then*

$$\mathrm{Ext}^1_R(A,B) \cong \mathrm{Coker}\,\mathrm{Hom}_R(\iota,B)$$

2.5 Homological dimensions

Proposition 2.5.1. *The following statements are equivalent for a module* A.

1. *A has a projective resolution* $0 \to P_n \to \ldots \to P_1 \to P_0 \to A \to 0$
2. $\mathrm{Ext}^{n+1}_R(A,B) = 0$ *for all modules* B
3. $\mathrm{Ext}^m_R(A,B) = 0$ *for all module* B *and all* $m > n$.

If n is the minimum integer for which the conditions above are satisfied, then A is said to have *projective dimension* n, and we set $\operatorname{pdim} A = n$. If there is no such n, then $\operatorname{pdim} A = \infty$. Dually, one defines the *injective dimension* $\operatorname{idim} A$ of a module A.

The supremum of the projective dimensions attained on R Mod coincides with the supremum of the injective dimensions attained on R Mod and is called the *(left) global dimension* of R. It is denoted by $\operatorname{gldim} R$. If R is a right and left noetherian ring, e.g. a finite dimensional algebra, then this number coincides with the right global dimension, that is, with the supremum of the projective (or injective) dimensions attained on right modules.

Theorem 2.5.2. (Auslander) *For any ring R the global dimension is attained on finitely generated modules:*

$$\operatorname{gldim} R = \sup\{ \operatorname{pdim}(R/I) \mid I \text{ left ideal of } R\}.$$

In particular, if R is a finite dimensional algebra, then

$$\operatorname{gldim} R = \max\{ \operatorname{pdim}(S) \mid S \text{ simple left module over } R\}.$$

Proof. Let $n = \sup\{ \operatorname{pdim}(R/I) \mid I \text{ left ideal of } R\}$. In order to verify that $\operatorname{gldim} R = n$, we prove that every module has injective dimension bounded by n. So, let A be an arbitrary left R-module with injective coresolution

$$0 \to A \to E_0 \to E_1 \to \ldots \to E_{n-1} \to C_n \to 0.$$

We have to show that C_n is injective. We use Baer's Lemma stating that C_n is injective if and only if for every left ideal I of R with embedding $I \overset{i}{\hookrightarrow} R$ and for every homomorphism $f \in \operatorname{Hom}_R(I, C_n)$ there is $f' \in \operatorname{Hom}_R(R, C_n)$ making the following diagram commutative:

$$\begin{tikzcd} I \arrow[rr, "i"] \arrow[dr, "f"'] & & R \arrow[dl, "f'"] \\ & C_n & \end{tikzcd}$$

Observe that this means that the map $\operatorname{Hom}_R(i, C_n) : \operatorname{Hom}_R(R, C_n) \to \operatorname{Hom}_R(I, C_n)$ is surjective. Now consider the short exact sequence

$$0 \to I \overset{i}{\hookrightarrow} R \to R/I \to 0$$

and recall from Proposition 2.4.3 that $\operatorname{Coker} \operatorname{Hom}_R(i, C_n) \cong \operatorname{Ext}^1_R(R/I, C_n)$. By dimension shifting $\operatorname{Ext}^1_R(R/I, C_n) \cong \operatorname{Ext}^{n+1}_R(R/I, A)$ which is zero since $\operatorname{pdim} R/I \leq n$ by assumption. This completes the proof.
For the additional statement, recall that over a finite dimensional algebra every finitely generated module M has finite length and is therefore a finite extension of the simple modules $S_1, \ldots, S_n$. Moreover, it follows easily from Lemma 2.4.2 that in a short exact sequence $0 \to A \to B \to C \to 0$ the projective dimension of B is bounded by the maximum of the projective dimensions of A and C. Hence the projective dimension of M is bounded by $\max\{\operatorname{pdim} S_i \mid 1 \leq i \leq n\}$. $\square$

A ring R has global dimension zero if and only if all R-modules are projective, or equivalently, all R-modules are semisimple. This condition is symmetric, that is, all left R-modules are semisimple if and only if so are all right R-modules. Rings with this property are called *semisimple* and are described by the following result. For details we refer to [19, Chapter 1] [26, p. 115], [14, Chapter 2], [16, 2.2], or [22, Chapter 3].

Theorem 2.5.3. (Wedderburn-Artin) *A ring R is semisimple if and only if it is isomorphic to a product of finitely many matrix rings over division rings*

$$R \cong M_{n_1}(D_1) \times \ldots M_{n_r}(D_r).$$

The rings of global dimension one are precisely the hereditary non-semisimple rings.

Theorem 2.5.4. *The following statements are equivalent for a ring R.*

(1) Every left ideal of R is projective.

(2) Every submodule of a projective left R-module is projective.

(3) Every factor module of an injective left R-module is injective.

(4) $\operatorname{gldim} R \leq 1$.

If R is a finite dimensional algebra, then (1) - (4) are also equivalent to

(5) The Jacobson radical J is a projective left R-module.

A ring R satisfying the equivalent conditions above is said to be left hereditary.

Proof: For the implication (1)⇒(2) one needs the following result:

Theorem 2.5.5. (Kaplansky) *Let R be a ring such that every left ideal of R is projective. Then every submodule of a free module is isomorphic to a sum of left ideals.*

For finitely generated modules over a finite dimensional algebra Λ, we can also proceed as follows. Take a finitely generated submodule $M \subset P$ of a projective module P. In order to show that M is projective, we can assume w.l.o.g. that M is indecomposable. P is a direct summand of a free module $\Lambda^{(I)} = \bigoplus\limits_{i=1}^{n} \Lambda e_i^{(I)}$. Choose i such that the composition $M \subset P \subset \bigoplus\limits_{i=1}^{n} \Lambda e_i^{(I)} \xrightarrow{\text{pr}} \Lambda e_i$ is non-zero. The image of this map is contained in $\Lambda e_i \subset \Lambda$ and is therefore a left ideal of Λ, which by assumption must be projective. So the indecomposable module M has a non-zero projective factor module and is thus projective.
(2)⇒ (4) follows immediately from the definition of global dimension.
(4)⇒ (2): Take a submodule $M \subset P$ of a projective module P, and consider the short exact sequence $0 \to M \to P \to P/M \to 0$. For any $N \in R\,\mathrm{Mod}$ we have a long exact sequence

$$\ldots \to \operatorname{Ext}^1_R(P,N) \to \operatorname{Ext}^1_R(M,N) \to \operatorname{Ext}^2(P/M,N) \to \ldots$$

where $\mathrm{Ext}^1_R(P, N) = 0$ as P is projective, and $\mathrm{Ext}^2(P/M, N) = 0$ as all modules have projective dimension bounded by one. Thus $\mathrm{Ext}^1_R(M, N) = 0$ for all $N \in R\,\mathrm{Mod}$, proving that M is projective.
(3)$\Leftrightarrow$ (4) is proven dually, and (2)$\Rightarrow$ (1),(5) is trivial.
It remains to show (5)$\Rightarrow$ (4): The hypothesis (5) states the left module R/J has projective dimension one. Now recall (e.g. from 1.1) that every simple module is a direct summand of R/J and use Theorem 2.5.2. □

Here are some examples of rings of global dimension one.

Example 2.5.6. (1) Principal ideal domains and, more generally, Dedekind domains are (left and right) hereditary.
(2) The upper triangular matrix ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix} = \{\begin{pmatrix} z & q \\ 0 & q' \end{pmatrix} \mid z \in \mathbb{Z}, q, q' \in \mathbb{Q}\}$ (viewed as a subring of $M_2(\mathbb{Q})$) is right hereditary but not left hereditary.

Example 2.5.7. Let Q be a finite quiver without oriented cycles, and let Q_0 be the set of vertices of Q. Let $\Lambda = kQ$ be the path algebra of Q over a field k. Recall that the Jacobson radical $J = J(\Lambda)$ is the ideal of Λ generated by all arrows, and for each vertex $i \in Q_0$, the paths starting in i form a k-basis of Λe_i. Denoting by $\alpha_1, \ldots, \alpha_t$ the arrows $i \bullet \xrightarrow{\alpha_k} \bullet\, j_k$ of Q which start in i, we see that $Je_i = \bigoplus_{k=1}^{t} \Lambda e_{j_k} \alpha_k$. Hence $Je_i \cong \bigoplus_{k=1}^{t} \Lambda e_{j_k}$ is projective for each $i \in Q_0$. In particular, Λ is hereditary.
Moreover, a finite dimensional algebra Λ over an algebraically closed field k is hereditary if and only if $\Lambda \cong kQ$ for some finite acyclic quiver Q (no relations!)

From Theorem 2.5.4 we deduce some important properties of hereditary rings.

Corollary 2.5.8. *Let R be left hereditary, $M \in R\,\mathrm{Mod}$. Then there is a non-zero R-homomorphism $f : M \to P$ with P projective if and only if M has a non-zero projective direct summand. Moreover, if M is indecomposable, then every non-zero R-homomorphism $f : M \to P$ with P projective is a monomorphism.*

Let now Λ be a hereditary finite dimensional algebra. Then the following hold true.

(1) If P is an indecomposable projective Λ-module, then $\mathrm{End}_\Lambda P$ is a division ring.

(2) If $M \in \Lambda\,\mathrm{mod}_{\mathcal{P}}$, then $\mathrm{Hom}_\Lambda(M, P) = 0$ for all projective modules ${}_\Lambda P$.

(3) Tr induces a duality $\Lambda\,\mathrm{mod}_{\mathcal{P}} \to \mathrm{mod}\,\Lambda_{\mathcal{P}}$ which is isomorphic to the functor $\mathrm{Ext}^1_\Lambda(-, \Lambda)$, and τ induces an equivalence $\tau : \Lambda\,\mathrm{mod}_{\mathcal{P}} \longrightarrow \Lambda\,\mathrm{mod}_{\mathcal{I}}$ with inverse τ^-.

Proof: We sketch the argument for (3). By (2) we have $P(M, N) = 0$ for all $M, N \in \Lambda\,\mathrm{mod}_{\mathcal{P}}$, and similarly, $I(M, N) = 0$ for all $M, N \in \Lambda\,\mathrm{mod}_{\mathcal{I}}$. Moreover, if $M \in \Lambda\,\mathrm{mod}_{\mathcal{P}}$, then a minimal projective presentation $0 \to P_1 \to P_0 \to M \to 0$ yields a long exact sequence $0 \to M^* \to P_0^* \to P_1^* \to \mathrm{Ext}^1_\Lambda(M,\Lambda) \to 0$ where $M^* = 0$, so $\mathrm{Ext}^1_\Lambda(M,\Lambda) \cong \mathrm{Tr}\, M$. □

For a more detailed treatment on hereditary rings we refer e.g. to [21, 1.2], [26, p. 120], [14, 3.7], or [16, 5.5].

2.6 The tensor product

Definition. Given a right R-module A and a left R-module B, their *tensor product* $A \otimes_R B$ is an abelian group equipped with a map $\tau : A \times B \to A \otimes_R B$ satisfying the conditions

(i) $\tau(a + a', b) = \tau(a, b) + \tau(a', b)$

(ii) $\tau(a, b + b') = \tau(a, b) + \tau(a, b')$

(iii) $\tau(ar, b) = \tau(a, rb)$

for all $a, a' \in A, b, b' \in B, r \in R$, and having the following universal property:
for any map $\tilde{\tau} : A \times B \to C$ into an abelian group C satisfying the conditions (i)-(iii) there is a unique group homomorphism $f : A \otimes_R B \to C$ making the following diagram commutative

$$\begin{tikzcd} A \times B \arrow[rr, "\tau"] \arrow[dr, "\tilde{\tau}"'] & & A \otimes_R B \arrow[dl, "f"] \\ & C & \end{tikzcd}$$

Construction. By the universal property, the tensor product of two modules A and B is uniquely determined up to isomorphism. Its existence is proven by giving the following explicit construction (which obviously verifies the universal property above):

$$A \otimes_R B = F/K$$

where

F is the free abelian group with basis $A \times B$, that is, every element of F can be written in a unique way as a finite linear combination of elements of the form $(a, b) \in A \times B$ with coefficients in $\mathbb{Z}$, and

K is the subgroup of F generated by all elements of the form

$$(a + a', b) - (a, b) - (a', b)$$

$$(a, b + b') - (a, b) - (a, b')$$

$$(ar, b) - (a, rb)$$

for some $a, a' \in A, b, b' \in B, r \in R$.

The elements of $A \otimes_R B$ are then the images of elements of F via the canonical epimorphism $F \to F/K$ and are thus of the form

$$\sum_{i=1}^{n} a_i \otimes b_i$$

for some $n \in \mathbb{N}$ and $a_i \in A$, $b_i \in B$
(but this representation is not unique! For example $0 \otimes b = a \otimes 0 = 0$ for all $a \in A, b \in B$).

Of course, the following rules hold true for all $a, a' \in A, b, b' \in B, r \in R$:

$$(a + a') \otimes b = a \otimes b + a' \otimes b$$

$$a \otimes (b + b') = a \otimes b + a \otimes b'$$

$$ar \otimes b = a \otimes rb$$

Observe that the tensor product of non-zero modules need not be non-zero.

Example 2.6.1. $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$. Indeed, if $a \otimes b \in \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$, then

$$a \otimes b = a \cdot (3 - 2) \otimes b = a \cdot 3 \otimes b - a \cdot 2 \otimes b = a \otimes 3 \cdot b - a \cdot 2 \otimes b = a \otimes 0 - 0 \otimes b = 0.$$

Homomorphisms. Given a right R-module homomorphism $f : A \to A'$ and a left R-module homomorphism $g : B \to B'$, there is a unique abelian group homomorphism

$$f \otimes g : A \otimes_R B \to A' \otimes_R B'$$

such that $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$ for all $a \in A$ and $b \in B$ (use the universal property!).

In general the tensor product of modules is just an abelian group. When starting with bimodules, however, it becomes a module.
Module structure. If S is a ring and ${}_SA_R$ is an S-R-bimodule, then $A \otimes_R B$ is a left S-module via

$$s \cdot a \otimes b = sa \otimes b$$

Moreover, given $f \in \mathrm{Hom}_R(B, B')$, the map

$$A \otimes_R f = \mathrm{id}_A \otimes f : A \otimes_R B \to A \otimes_R B', \sum_{i=1}^{n} a_i \otimes b_i \mapsto \sum_{i=1}^{n} a_i \otimes f(b_i)$$

is an S-module homomorphism.
The analogous statements hold true if ${}_RB_S$ is a bimodule.

Theorem 2.6.2. (Adjointness of Hom **and** $\otimes$**)** *Let R, S be rings, ${}_SA_R$ be an S-R-bimodule, B a left R-module and C a left S-module. Then there is a natural group homomorphism*

$$\mathrm{Hom}_S(A \otimes_R B, C) \cong \mathrm{Hom}_R(B, \mathrm{Hom}_S(A, C)).$$

Proof. (Sketch) The isomorphism

$$\varphi : \mathrm{Hom}_S(A \otimes_R B, C) \to \mathrm{Hom}_R(B, \mathrm{Hom}_S(A, C))$$

is given by mapping $f \in \mathrm{Hom}_S(A \otimes_R B, C)$ to the R-homomorphism $\varphi(f) : B \to \mathrm{Hom}_S(A, C)$ where $\varphi(f)(b) : A \to C$, $a \mapsto f(a \otimes b)$.
The inverse map

$$\psi : \mathrm{Hom}_S(A \otimes_R B, C) \to \mathrm{Hom}_R(B, \mathrm{Hom}_S(A, C))$$

is given by mapping $g \in \mathrm{Hom}_R(B, \mathrm{Hom}_S(A, C))$ to the S-homomorphism $\psi(g) : A \otimes_R B \to C$ where $\psi(g)(a \otimes b) = g(b)(a)$. □

Corollary 2.6.3. *Let Λ be a finite dimensional algebra over a field k with standard duality $D = \mathrm{Hom}(-, k)$. Then*

$$D(A \otimes B) \cong \mathrm{Hom}_\Lambda(B, D(A))$$

for all right Λ-modules A and left Λ-modules B.

Corollary 2.6.4. *Let R, S be rings, ${}_SA_R$ be an S-R-bimodule. Then*

$$A \otimes_R - : R\ \mathrm{Mod} \to S\,\mathrm{Mod}$$

is an additive, covariant, right exact functor.

The following result will be very useful.

Lemma 2.6.5. *Let $M, P \in R$ Mod, and let P be finitely generated projective. Then there is a natural group homomorphism*

$$\mathrm{Hom}_R(P, M) \cong P^* \otimes_R M.$$

Remark 2.6.6. (1) If V, W are finite dimensional vector spaces over a field k, then $V \otimes_k W$ is isomorphic to the vector space $\mathrm{Bil}(V^* \times W^*, K)$ of all bilinear maps $V^* \times W^* \to K$. Under this bijection an element $v \otimes w$ corresponds to the bilinear map $(\varphi, \psi) \mapsto \varphi(v)\psi(w)$. Indeed, $V^{**} \cong V$, so by Lemma 2.6.5 we have $V \otimes_k W \cong \mathrm{Hom}_k(V^*, W) \cong \mathrm{Hom}_k(V^*, W^{**})$. Further, $\mathrm{Hom}_k(V^*, W^{**} \cong \mathrm{Bil}(V^* \times W^*, K)$ via $g \mapsto \sigma_g$, where σ_g is the bilinear map given by $\sigma_g(\varphi, \psi) = g(\varphi)(\psi)$.
(2) Let B be a left R-module with projective resolution $P^{\cdot}$, and A a right R-module. The homology groups of the complex $A \otimes_R P^{\cdot} : \ldots A \otimes_R P_1 \to A \otimes_R P_0 \to 0$ define the Tor-functors:

$$A \otimes_R B = H^0(A \otimes_R P^{\cdot}),$$
$$\mathrm{Tor}_n^R(A, B) = H^n(A \otimes_R P^{\cdot}) \quad \text{for } n \geq 1.$$

3 AUSLANDER-REITEN THEORY

Let now Λ be again a finite dimensional algebra as in Section 1.1. As we have seen above, over hereditary algebras the functor $\mathrm{Ext}^1_\Lambda(-,\Lambda)$ is isomorphic to the transpose. In general, we have the following result.

Lemma 3.0.7. *Let* $\mathfrak{E}: 0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ *be a short exact sequence, and let* $A \in \Lambda mod_{\mathcal{P}}$. *Then there is a natural homomorphism* $\delta = \delta(A, \mathfrak{E})$ *such that the sequence* $0 \to \mathrm{Hom}_\Lambda(A,X) \to \mathrm{Hom}_\Lambda(A,Y) \to \mathrm{Hom}_\Lambda(A,Z) \xrightarrow{\delta} \mathrm{Tr}\, A \otimes_\Lambda X \to \mathrm{Tr}\, A \otimes_\Lambda Y \to \mathrm{Tr}\, A \otimes_\Lambda Z \to 0$ *is exact.*

Proof: Let $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} A \to 0$ be a minimal projective presentation of A. Since the P_i, $i = 0,1$, are finitely generated projective, we know from 1.3.2 that $\mathrm{Hom}_\Lambda(P_i, M) \cong P_i^* \otimes_\Lambda M$ for any $M \in \Lambda\mathrm{Mod}$. So the cokernel of $\mathrm{Hom}(p_1, M): \mathrm{Hom}_\Lambda(P_0, M) \longrightarrow \mathrm{Hom}_\Lambda(P_1, M)$ is isomorphic to $\mathrm{Tr}\, A \otimes_\Lambda M$. Hence we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & \mathrm{Hom}_\Lambda(A,X) & \longrightarrow & \mathrm{Hom}_\Lambda(A,Y) & \longrightarrow & \mathrm{Hom}_\Lambda(A,Z) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & \mathrm{Hom}_\Lambda(P_0,X) & \longrightarrow & \mathrm{Hom}_\Lambda(P_0,Y) & \longrightarrow & \mathrm{Hom}_\Lambda(P_0,Z) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & \mathrm{Hom}_\Lambda(P_1,X) & \longrightarrow & \mathrm{Hom}_\Lambda(P_1,Y) & \longrightarrow & \mathrm{Hom}_\Lambda(P_1,Z) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{Tr}\, A \otimes X & \longrightarrow & \mathrm{Tr}\, A \otimes Y & \longrightarrow & \mathrm{Tr}\, A \otimes Z & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

and by the snake-lemma [26, 6.5] we obtain the claim. □

3.1 The Auslander-Reiten formula

Theorem 3.1.1 (Auslander-Reiten 1975)**.** *Let* A, C *be* Λ*-modules with* $A \in \Lambda\, mod_{\mathcal{P}}$. *Then there are natural* k*-isomorphisms*

$$\begin{array}{ll}
\text{(I)} & \overline{\mathrm{Hom}}_\Lambda(C, \tau A) \cong D\,\mathrm{Ext}^1_\Lambda(A, C) \\
\text{(II)} & D\,\underline{\mathrm{Hom}}_\Lambda(A, C) \cong \mathrm{Ext}^1_\Lambda(C, \tau A)
\end{array}$$

These formulae were first proven in [10], see also [21]. A more general version of (II), valid for arbitrary rings, is proven in [6, I, 3.4], cf.[17].

The proof uses

Lemma 3.1.2. *Let* $0 \to X \xrightarrow{i} Y \xrightarrow{\pi} Z \to 0$ *be a short exact sequence,* $A \in \Lambda\, mod_{\mathcal{P}}$. *Then there is a* k*-isomorphism* $\mathrm{Coker}\,\mathrm{Hom}_\Lambda(i, \tau A) \cong D\,\mathrm{Coker}\,\mathrm{Hom}_\Lambda(A, \pi)$.

If Λ is hereditary, the Auslander-Reiten-formulae simplify as follows.

Corollary 3.1.3. *Let A, C be Λ-modules with $A \in \Lambda\,\mathrm{mod}_{\mathcal{P}}$.*

1. *If* $\mathrm{pdim} A \leq 1$, *then* $\mathrm{Hom}_\Lambda\,(C, \tau\, A) \cong D\,\mathrm{Ext}^1_\Lambda\,(A, C)$.
2. *If* $\mathrm{idim} \tau A \leq 1$, *then* $D\,\mathrm{Hom}_\Lambda\,(A, C) \cong \mathrm{Ext}^1_\Lambda\,(C, \tau\, A)$.

Here is a first application.

Example 3.1.4. *If $\Lambda = kA_3$ is the path algebra of the quiver $\underset{1}{\bullet} \to \underset{2}{\bullet} \to \underset{3}{\bullet}$, then every short exact sequence $0 \to P_2 \to E \to S_2 \to 0$ splits. Indeed, we know from 3.3 that $\tau S_2 \cong S_3$, so $\mathrm{Ext}^1_\Lambda(S_2, P_2) \cong \mathrm{Hom}_\Lambda(P_2, S_3) = 0$.*

3.2 Almost split maps

Let $\mathcal{M}$ be the category $\Lambda\,\mathrm{Mod}$ or $\Lambda\,\mathrm{mod}$.

Definition.

(1) A homomorphism $g \colon B \to C$ in $\mathcal{M}$ is called *right almost split in $\mathcal{M}$* if

 (a) g is not a split epimorphism, and

 (b) if $h \colon X \to C$ is a morphism in $\mathcal{M}$ that is not a split epimorphism, then h factors through g.

$$\begin{tikzcd} B \arrow[rr, "g"] & & C \\ & X \arrow[ul, dotted] \arrow[ur, "h"'] & \end{tikzcd}$$

(2) $g \colon B \to C$ is called *minimal right almost split in $\mathcal{M}$* if it is right minimal and right almost split in $\mathcal{M}$.

The definition of a *(minimal) left almost split* map is dual.

Remark 3.2.1. (1) Let $g \colon B \to C$ be right almost split in $\mathcal{M}$. Then $\mathrm{End}\, C$ is a local ring and $J(\mathrm{End}\, C) = g \circ \mathrm{Hom}_\Lambda(C, B)$. If C is not projective, then g is an epimorphism. (2) Let $C \in \Lambda\,\mathrm{mod}$ be an indecomposable non-projective module. Then $\mathrm{Tr}\, C$ and τC are indecomposable, and $P(C, C) \subset J(\mathrm{End}\, C)$.

We now use the Auslander-Reiten formulae to prove

Theorem 3.2.2 (Auslander-Reiten 1975)**.** *Let $C \in \Lambda\,\mathrm{mod}$ be an indecomposable non-projective module. Then there is an exact sequence $0 \to A \to B \xrightarrow{g} C \to 0$ such that g is right almost split in $\Lambda\,\mathrm{Mod}$, and $A \cong \tau C$.*

Proposition 3.2.3. *The following statements are equivalent for an exact sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *in* $\mathcal{M}$.

(1) f *is left almost split and* g *is right almost split in* $\mathcal{M}$.

(2) $\operatorname{End}_\Lambda C$ *is local and* f *is left almost split in* $\mathcal{M}$.

(3) $\operatorname{End}_\Lambda A$ *is local and* g *is right almost split in* $\mathcal{M}$.

(4) f *is minimal left almost split in* $\mathcal{M}$.

(5) g *is minimal right almost split in* $\mathcal{M}$.

Definition. An exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{M}$ is called *almost split* (*Auslander-Reiten sequence*) in $\mathcal{M}$ if it satisfies one of the equivalent conditions above.

Remark 3.2.4. [12, V.2, 1.16] Almost split sequences starting (or ending) at a given module are uniquely determined up to isomorphism. More precisely, if $0 \to A \to B \to C \to 0$ and $0 \to A' \to B' \to C' \to 0$ are almost split sequences, then $A \cong A'$ if and only if $C \cong C'$ if and only if there are isomorphisms a, b, c making the following diagram commute

$$\begin{array}{ccccccc} 0 & \to & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c & \\ 0 & \to & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow 0 \end{array}$$

Theorem 3.2.5 (Auslander-Reiten 1975)**.** *(1) For every finitely generated indecomposable non-projective module* M *there is an almost split sequence* $0 \to \tau M \to B \to M \to 0$ *in* Λ *Mod with finitely generated modules.*

(2) For every finitely generated indecomposable non-injective module M *there is an almost split sequence* $0 \to M \to E \to \tau^- M \to 0$ *in* Λ *Mod with finitely generated modules.*

The Theorem above was originally proved in [10]. Another proof, using functorial arguments, is given in [7]. For generalizations of this result to arbitrary rings see [6, 5, 28, 29].

3.3 The Auslander-Reiten quiver

We now use almost split maps to study the category $\Lambda \operatorname{mod}$. First of all, we have to take care of the indecomposable projective and the indecomposable injective modules.

Proposition 3.3.1.

(1) If P *indecomposable projective, then the embedding* $g : \operatorname{Rad} P \hookrightarrow P$ *is minimal right almost split in* Λ*Mod.*

(2) If I *indecomposable injective, then the natural surjection* $f : I \to I/\operatorname{Soc} I$ *is minimal left almost split in* Λ*Mod.*

Proof: (1) Note that $\operatorname{Rad} P = JP$ and P/JP is simple [12, I,3.5 and 4.4], so $\operatorname{Rad} P$ is the unique maximal submodule of P. Thus, if $h\colon X \to P$ is not a split epimorphism, then it is not an epimorphism and therefore $\operatorname{Im} h$ is contained in $\operatorname{Rad} P$. Hence g is right almost split. Moreover, g is right minimal since every $t \in \operatorname{End} \operatorname{Rad} P$ with $gt = g$ has to be a monomorphism, hence an isomorphism.
(2) is proven with dual arguments. □

Let now $M \in \Lambda \operatorname{mod}$ be indecomposable. From 3.3.1 and 3.2.5 we know that there is a map $g : B \to M$ with $B \in \Lambda \operatorname{mod}$ which is minimal right almost split, and there is a map $f : M \to N$ with $N \in \Lambda \operatorname{mod}$ which is minimal left almost split. Consider decompositions

$$B = \bigoplus_{i=1}^{n} B_i \quad \text{and} \quad N = \bigoplus_{k=1}^{m} N_k$$

into indecomposable modules B_i and N_k. The maps

$$g|B_i \quad \text{and} \quad \operatorname{pr}_{N_k} f$$

are characterized by the property of being irreducible in the following sense, see [12, V.5.3].

Definition. A homomorphism $h\colon M \to N$ between indecomposable modules M, N is said to be *irreducible* if h is not an isomorphism, and in any commutative diagram

$$\begin{tikzcd} M \arrow[rr, "h"] \arrow[dr, "\alpha"'] & & N \\ & Z \arrow[ur, "\beta"'] & \end{tikzcd}$$

either α is a split monomorphism or β is a split epimorphism.

In particular, if h is irreducible, then $h \neq 0$ is either a monomorphism or an epimorphism.

Irreducible morphisms can also be described in terms of the following notion, which is treated in detail in [12, V.7].

Definition. For two modules $M, N \in \Lambda \operatorname{mod}$, we define the *radical* of $\operatorname{Hom}_\Lambda(M, N)$ by

$$r(M,N) = \{\, f \in \operatorname{Hom}_\Lambda(M,N) \mid \text{for each indecomposable module } Z \in \Lambda \operatorname{mod}, \text{ every composition of the form } Z \to M \xrightarrow{f} N \to Z \text{ is a non-isomorphism}\}$$

For $n \in \mathbb{N}$ set

$$r^n(M,N) = \{f \in \operatorname{Hom}_\Lambda(M,N) \mid f = gh \ \text{ with } h \in r(M,X), g \in r^{n-1}(X,N), X \in \Lambda \operatorname{mod}\}$$

Proposition 3.3.2. *If $M, N \in \Lambda \operatorname{mod}$ are indecomposable modules, then*

(1) $r(M,N)$ consists of the non-isomorphisms in $\operatorname{Hom}_\Lambda(M,N)$, so $r(M,M) = J(\operatorname{End}_\Lambda M)$.

(2) $f \in \mathrm{Hom}_\Lambda(M,N)$ *is irreducible if and only if* $f \in r(M,N) \setminus r^2(M,N)$.

Since the irreducible morphisms arise as components of minimal right almost split maps and minimal left almost split maps, we obtain the following result.

Proposition 3.3.3. *Let* M, N *be indecomposable modules with an irreducible map* $M \to N$*. Let* $g\colon B \to N$ *be a minimal right almost split map, and* $f\colon M \to B'$ *a minimal left almost split map. Then there are integers* $a, b > 0$ *and modules* $X, Y \in \Lambda\,\mathrm{mod}$ *such that*

(1) $B \cong M^a \oplus X$ *and* M *is not isomorphic to a direct summand of* X*,*

(2) $B' \cong N^b \oplus Y$ *and* N *is not isomorphic to a direct summand of* Y*.*

Moreover,

$$a = \dim r(M,N)/r^2(M,N)_{\mathrm{End}\,M/J(\mathrm{End}\,M)}$$
$$b = \dim_{\mathrm{End}\,N/J(\mathrm{End}\,N)} r(M,N)/r^2(M,N)$$

Thus $a = b$ *provided that* k *is an algebraically closed field.*

Proof: The $\mathrm{End}\,N$-$\mathrm{End}\,M$-bimodule structure on $\mathrm{Hom}_\Lambda(M,N)$ induces an $\mathrm{End}\,N/J(\mathrm{End}\,N)$-$\mathrm{End}\,M/J(\mathrm{End}\,M)$-bimodule structure on $r(M,N)/r^2(M,N)$. Now $\mathrm{End}\,N/J(\mathrm{End}\,N)$ and $\mathrm{End}\,M/J(\mathrm{End}\,M)$ are skew fields. Consider the minimal right almost split map $g : B \longrightarrow N$. If $g_1, \ldots, g_a : M \to N$ are the different components of $g\,|_{M^a}$, then $\overline{g_1}, \ldots, \overline{g_a}$ is the desired $\mathrm{End}\,M/J(\mathrm{End}\,M)$-basis. Dual considerations yield an $\mathrm{End}\,N/J(\mathrm{End}\,N)$-basis of $r(M,N)/r^2(M,N)$. For details, we refer to [12, VII.1].
Finally, since $\mathrm{End}\,N/J(\mathrm{End}\,N)$ and $\mathrm{End}\,M/J(\mathrm{End}\,M)$ are finite dimensional skew field extensions of k, we conclude that $a = b$ provided that k is an algebraically closed field.□

Definition. The *Auslander-Reiten quiver (AR-quiver)* $\Gamma = \Gamma(\Lambda)$ of Λ is constructed as follows. The set of vertices Γ_0 consists of the isomorphism classes $[M]$ of finitely generated indecomposable Λ-modules. The set of arrows Γ_1 is given by the following rule: set an arrow

$$[M] \xrightarrow{(a,b)} [N]$$

if there is an irreducible map $M \to N$ with (a, b) as above in Proposition 3.3.3.

Observe that Γ is a locally finite quiver (i.e. there exist only finitely many arrows starting or ending at each vertex) with the simple projectives as sources and the simple injectives as sinks. Moreover, if k is an algebraically closed field, we can drop the valuation by drawing multiple arrows.

Proposition 3.3.4. *Consider an arrow from* Γ

$$[M] \xrightarrow{(a,b)} [N]$$

(1) Translation of arrows*:*
If M, N are indecomposable non-projective modules, then in Γ there is also an arrow

$$[\tau M] \xrightarrow{(a,b)} [\tau N]$$

(2) Meshes*:*
If N is an indecomposable non-projective module, then in Γ there is also an arrow

$$[\tau N] \xrightarrow{(b,a)} [M]$$

Proof: (1) can be proven by exploiting the properties of the equivalence $\tau = D\,\mathrm{Tr}\colon \Lambda\,\underline{\mathrm{mod}} \to \Lambda\,\overline{\mathrm{mod}}$ from 1.5.1. In fact, the following is shown in [11, 2.2]: If N is an indecomposable non-projective module with a minimal right almost split map $g : B \longrightarrow N$, and $B = P \oplus B'$ where P is projective and $B' \in \Lambda\,\mathrm{mod}_{\mathcal{P}}$ has non non-zero projective summand, then there are an injective module $I \in \Lambda\,\mathrm{mod}$ and a minimal right almost split map $g' : I \oplus \tau B' \longrightarrow \tau N$ such that $\tau(\underline{g}) = \overline{g'}$. Now the claim follows easily.
(2) From the almost split sequence $0 \longrightarrow \tau N \longrightarrow M^a \oplus X \longrightarrow N \longrightarrow 0$ we immediately infer that there is an arrow $[\tau N] \xrightarrow{(b',a)} [M]$ in Γ. So we only have to check $b' = b$. We know from 3.3.3 that $b' = \dim r(\tau N, M)/r^2(\tau N, M)_{\mathrm{End}\,\tau N/J(\mathrm{End}\,\tau N)}$. Now, the equivalence $\tau = D\,\mathrm{Tr}\colon \Lambda\,\underline{\mathrm{mod}} \to \Lambda\,\overline{\mathrm{mod}}$ from 1.5.1 defines an isomorphism $\underline{\mathrm{End}}_\Lambda N \cong \overline{\mathrm{End}}_\Lambda \tau N$, which in turn induces an isomorphism $\mathrm{End}\,N/J(\mathrm{End}\,N) \cong \mathrm{End}\,\tau N/J(\mathrm{End}\,\tau N)$. Moreover, using 3.3.3 and denoting by ℓ the length of a module over the ring k, it is not difficult to verify that $b' \cdot \ell(\mathrm{End}\,\tau N/J(\mathrm{End}\,\tau N)) = a \cdot \ell(\mathrm{End}\,M/J(\mathrm{End}\,M)) = \ell(r(M,N)/r^2(M,N)) = b \cdot \ell(\mathrm{End}\,N/J(\mathrm{End}\,N))$, which implies $b' = b$. $\square$

Remark 3.3.5. *If Q is a finite connected acyclic quiver and $\Lambda = kQ$, then the number of arrows $[\Lambda e_j] \to [\Lambda e_i]$ in Γ coincides with the number of arrows $i \to j$ in Q, and with the number of arrows $[I_j] \to [I_i]$ in Γ ("Knitting procedure").*

Example: Let $\Lambda = K\mathbb{A}_3$ be the path algebra of the quiver $\underset{1}{\bullet} \to \underset{2}{\bullet} \to \underset{3}{\bullet}$.
Λ is a serial algebra. The module $I_3 \cong P_1$ has the composition series $P_1 \supset P_2 \supset P_3 \supset 0$. Furthermore, $I_3/\operatorname{Soc} I_3 \cong I_2$, and $I_2/\operatorname{Soc} I_2 \cong I_1$. So, there are only three almost split sequences, namely $0 \to P_3 \to P_2 \to S_2 \to 0$, and $0 \to P_2 \to S_2 \oplus P_1 \to I_2 \to 0$, and $0 \to S_2 \to I_2 \to I_1 \to 0$. Hence $\Gamma(\Lambda)$ has the form

$$\begin{tikzcd}
 & & P_1 \cong I_3 \arrow[dr] & & \\
 & P_2 \arrow[ur] \arrow[dr] & \cdots\cdots\cdots & I_2 \arrow[dr] & \\
P_3 \arrow[ur] & \cdots\cdots\cdots & S_2 \arrow[ur] & \cdots\cdots\cdots & I_1
\end{tikzcd}$$

4 ALGEBRAS OF FINITE REPRESENTATION TYPE

Definition. A finite dimensional algebra Λ is said to be *of finite representation type* if there are only finitely many isomorphism classes of finitely generated indecomposable left Λ-modules. This is equivalent to the fact that there are only finitely many isomorphism classes of finitely generated indecomposable right Λ-modules.

finite dimensional algebras of finite representation type are completely described by their AR-quiver.

Theorem 4.0.6 (Auslander 1974, Ringel-Tachikawa 1973)**.** *Let Λ be an finite dimensional algebra of finite representation type. Then every module is a direct sum of finitely generated indecomposable modules. Moreover, every non-zero non-isomorphism $f\colon X \to Y$ between indecomposable modules X, Y is a sum of compositions of irreducible maps between indecomposable modules.*

Proof: We sketch the proof of the second statement, and refer to [12, V.7] for details. Take a non-zero non-isomorphism $f\colon X \to Y$ between indecomposable modules X, Y. If $g : B \to Y$ is minimal right almost split, and $B = \bigoplus_{i=1}^{n} B_i$ with indecomposable modules B_i, then we can factor f as follows:

$$\begin{tikzcd} B_i \arrow[r, hook] & B \arrow[r, "g"'] & Y \\ & X \arrow[ul, dotted, "h_i"] \arrow[u, dotted, "h"'] \arrow[ur, "f"'] & \end{tikzcd} \qquad f = gh = \sum_{i=1}^{n} g|_{B_i} \circ h_i \quad \text{with irreducible maps } g|_{B_i}.$$

Moreover, if h_i is not an isomorphism, we can repeat the argument. But this procedure will stop eventually, because we know from the assumption that there is a bound on the length of nonzero compositions of non-isomorphisms between indecomposable modules (e. g. by the *Lemma of Harada and Sai*, see [12, VI.1.3]). So after a finite number of steps we see that f has the desired shape. □

Remark 4.0.7. In [4], Auslander also proved the converse of the first statement in Theorem 4.0.6. Combining this with a result of Zimmermann-Huisgen we obtain that an finite dimensional algebra is of finite representation type if and only if every left module is a direct sum of indecomposable left modules. The question whether the same holds true for any left artinian ring is known as the *Pure-Semisimple Conjecture.*

Observe that for the proof of the second statement in 4.0.6, actually, we only need a bound on the length of the modules involved. In fact, the following was proven in [3].

Theorem 4.0.8 (Auslander 1974)**.** *Let Λ be an indecomposable finite dimensional algebra with AR-quiver Γ. Assume that Γ has a connected component $\mathcal{C}$ such that the lengths of the modules in $\mathcal{C}$ are bounded. Then Λ is of finite representation type, and $\Gamma = \mathcal{C}$.*

In particular, of course, this applies to the case where Γ has a finite component.

We sketch Yamagata's proof of Theorem 4.0.8, see also [12, VI.1.4].

Remark 4.0.9. *For $A, B \in \Lambda\,\mathrm{mod}$ the descending chain $\mathrm{Hom}_\Lambda(A,B) \supset r(A,B) \supset r^2(A,B) \supset \cdots$ of k-subspaces of $\mathrm{Hom}_\Lambda(A,B)$ is stationary.*

Proof of Theorem 4.0.8:
Step 1: The preceding remark, together with the Lemma of Harada and Sai, yields an integer n such that every $A \in \mathcal{C}$ satisfies

$$r^n(A,B) = 0 = r^n(B,A) \quad \text{for every} \quad B \in \Lambda\,\mathrm{mod}.$$

Step 2: If $A \in \mathcal{C}$, and $B \in \Lambda\,\mathrm{mod}$ is an indecomposable module with $\mathrm{Hom}_\Lambda(A,B) \neq 0$ or $\mathrm{Hom}_\Lambda(B,A) \neq 0$, then $B \in \mathcal{C}$. In fact, by similar arguments as in the proof of Theorem 4.0.6, every non-zero map $f \in \mathrm{Hom}_\Lambda(A,B)$ can be written as

$$0 \neq f = \sum g_1 \dots g_{m-1} h$$

where $g_1, \dots, g_{m-1}$ are irreducible maps between indecomposable modules, and by the above considerations, eventually in one of the summands the map h has to be an isomorphism. So, we find a path $A \xrightarrow{g_r} \dots \xrightarrow{g_1} B$ in $\mathcal{C}$ such that, moreover, the composition $g_1 \dots g_r \neq 0$.
Step 3: In particular, if $A \in \mathcal{C}$, we infer that any indecomposable projective module P with $\mathrm{Hom}_\Lambda(P,A) \neq 0$ belongs to $\mathcal{C}$. Since Λ is indecomposable, this shows that all indecomposable projectives are in $\mathcal{C}$. Furthermore, every indecomposable module $X \in \Lambda\,\mathrm{mod}$ satisfies $\mathrm{Hom}(P,X) \neq 0$ for some indecomposable projective P and hence belongs to $\mathcal{C}$ as well. But this means $\Gamma = \mathcal{C}$. Moreover, since there are only finitely many indecomposable projectives and there is a bound on the length of non-zero paths in $\mathcal{C}$, we conclude that $\Gamma = \mathcal{C}$ is finite. $\square$

Theorem 4.0.8 confirms the

First Brauer-Thrall-Conjecture: A finite dimensional algebra is of finite representation type if and only if the lengths of the indecomposable finitely generated modules are bounded.

The following conjecture is verified for finite dimensional algebras over perfect fields but is open in general.

Second Brauer-Thrall-Conjecture: If Λ is a finite dimensional k-algebra where k is an infinite field, and Λ is not of finite representation type, then there are infinitely many $n_1, n_2, n_3, \dots \in \mathbb{N}$ and for each n_k there are infinitely many isomorphism classes of indecomposable Λ-modules of length n_k.

5 TAME AND WILD ALGEBRAS

5.1 The Cartan matrix and the Coxeter transformation

Let us introduce some further techniques that will be useful in the sequel.

Definition. For each module $A \in \Lambda\,\mathrm{mod}$ denote by $\underline{\dim} A = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ the *dimension vector* of A given by the Jordan-Hölder multiplicities, that is, m_i is the number of composition factors of A that are isomorphic to the simple module S_i for aech $1 \leq i \leq n$. We set

$$\underline{e_i} = (0, \ldots, 1, 0, \ldots, 0) = \underline{\dim} S_i$$
$$\underline{p_i} = \underline{\dim}\, \Lambda e_i = \underline{\dim} P_i$$
$$\underline{q_i} = \underline{\dim}\, D(e_i \Lambda) = \underline{\dim} I_i$$

Remark 5.1.1. *(1) For every exact sequence* $0 \to A' \to A \to A'' \to 0$ *we have*

$$\underline{\dim} A = \underline{\dim} A' + \underline{\dim} A''$$

(2) If $\underline{\dim} A = (m_1, \ldots, m_n)$*, then* $l(A) = \sum_{i=1}^{n} m_i$.

(3) Consider the Grothendieck group $K_0(\Lambda)$ *defined as the group generated by the isomorphism classes* $[A]$ *of* $\Lambda\,\mathrm{mod}$ *with the relations* $[A'] + [A''] = [A]$ *whenever* $0 \to A' \to A \to A'' \to 0$ *is exact in* $\Lambda\,\mathrm{mod}$*. Note that* $K_0(\Lambda)$ *is a free abelian group with basis* $[S_1], \ldots, [S_n]$*, see* [12, I,1.7]*. The assignment* $[A] \mapsto \underline{\dim} A$ *defines an isomorphism between* $K_0(\Lambda)$ *and* $\mathbb{Z}^n$.

Lemma 5.1.2. *Let* Λ *be a finite dimensional hereditary algebra. Then the matrix*

$$C = \begin{pmatrix} \underline{p_1} \\ \vdots \\ \underline{p_n} \end{pmatrix} \in \mathbb{Z}^{n \times n}$$

is invertible in $\mathbb{Z}^{n \times n}$.

Proof: We sketch an argument from [24, p. 70]. Take $1 \leq i \leq n$ and a projective resolution $0 \to Je_i \to \Lambda e_i \to S_i \to 0$ of S_i. Then $Je_i = \oplus_{k=1}^{n} {\Lambda e_k}^{r_k}$ with multiplicities $r_k \in \mathbb{Z}$, and by condition (1) in Remark 5.1.1, we see that $\underline{e_i} = \underline{p_i} - \sum r_k \underline{p_k}$ can be written as a linear combination of $\underline{p_1}, \ldots, \underline{p_n}$ with coefficients in $\mathbb{Z}$. This shows that there is a matrix $A \in \mathbb{Z}^{n \times n}$ such that $A \cdot C = E_n$. $\square$

Definition. Let Λ be a finite dimensional hereditary algebra. The matrix $C = \begin{pmatrix} \underline{p_1} \\ \vdots \\ \underline{p_n} \end{pmatrix}$ is called the *Cartan matrix* of Λ. It defines the *Coxeter transformation*

$$c\colon \mathbb{Z}^n \to \mathbb{Z}^n\,, \qquad \underline{z} \mapsto -\,\underline{z}\, C^{-1}\, C^t$$

We are now going to see how the Coxeter transformation can be used to compute τ.

Proposition 5.1.3. *Let Λ be a finite dimensional hereditary algebra over a field k.*

(1) For each $1 \le i \le n$ we have $c(\underline{p_i}) = -\underline{q_i}$.

(2) If $A \in \Lambda\,\mathrm{mod}$ is indecomposable non-projective, then $c(\underline{\dim}A) = \underline{\dim}\,\tau A$.

(3) An indecomposable module $A \in \Lambda\,\mathrm{mod}$ is projective if and only if $c(\underline{\dim}A)$ is negative.

Proof: (1) First of all, note that $\underline{\dim}A = (\dim_k e_1 A, \ldots, \dim_k e_n A)$. In particular

$$\underline{p_i} = (\dim_k e_1\Lambda e_i, \ldots, \dim_k e_n \Lambda e_i)$$

$$\underline{q_i} = (\dim_k e_i\Lambda e_1, \ldots, \dim_k e_i \Lambda e_n) = \underline{\dim} e_i\Lambda.$$

This shows that $C^t = \begin{pmatrix} \underline{q_1} \\ \vdots \\ \underline{q_n} \end{pmatrix}$ and therefore $c(\underline{p_i}) = c(\underline{e_i}\, C) = -\underline{e_i}\, C^t = -\underline{q_i}$.

(2) Consider a minimal projective resolution $0 \longrightarrow Q \longrightarrow P \longrightarrow A \longrightarrow 0$. Then $c(\underline{\dim}A) = c(\underline{\dim}P) - c(\underline{\dim}Q)$. Applying the functor $^* = \mathrm{Hom}_\Lambda(-,\Lambda)$ and using that $\mathrm{Hom}_\Lambda(A,\Lambda) = 0$ by Remark 2.5.8(2), we obtain a minimal projective resolution $0 \longrightarrow P^* \longrightarrow Q^* \longrightarrow \mathrm{Tr}\,A \longrightarrow 0$ and therefore a short exact sequence $0 \longrightarrow \tau A \longrightarrow DQ^* \longrightarrow DP^* \longrightarrow 0$. Thus $\underline{\dim}\tau A = \underline{\dim}DQ^* - \underline{\dim}DP^*$, and the claim follows from (1).
(3) follows immediately from (1) and (2). $\square$

5.2 Gabriel's classification of hereditary algebras

The Cartan matrix is also used to define the Tits form, which plays an essential role in Gabriel's classification of tame hereditary algebras.

Definition. Let Λ be a finite dimensional hereditary algebra.

(1) Consider the (usually non-symmetric) bilinear form

$$B\colon \mathbb{Q}^n \times \mathbb{Q}^n \to \mathbb{Q},\ (\underline{x},\underline{y}) \mapsto \underline{x}\, C^{-1}\underline{y}^t$$

and the corresponding quadratic form

$$\chi\colon \mathbb{Q}^n \to \mathbb{Q}\,, \qquad \underline{x} \mapsto B(\underline{x},\underline{x})$$

χ is called the *Tits form* of Λ.

(2) A vector $\underline{x} \in \mathbb{Z}^n$ is called a *root* of χ provided $\chi(\underline{x}) = 1$.

(3) A vector $\underline{x} \in \mathbb{Q}^n$ is called a *radical vector* provided $\chi(\underline{x}) = 0$. The radical vectors form a subspace of $\mathbb{Q}^n$ which we denote by

$$N = \{\underline{x} \in \mathbb{Q}^n \mid \chi(\underline{x}) = 0\}$$

(4) Finally, we say that a vector $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{Q}^n$ is *positive* if all components $x_i \geq 0$.

The Tits form can be interpreted as follows, see [24, p. 71].

Proposition 5.2.1. *Let Λ be a finite dimensional hereditary algebra over an algebraically closed field k, and let Q be the Gabriel-quiver of Λ. For two vertices $i, j \in Q_0$ denote by d_{ji} the number of arrows $i \to j \in Q_1$. Then*

(1) Homological interpretation of χ *(Euler form): For $X, Y \in \Lambda\,mod$*

$$B(\underline{\dim} X, \underline{\dim} Y) = \dim_k \operatorname{Hom}_\Lambda(X,Y) - \dim_k \operatorname{Ext}^1_\Lambda(X,Y)$$

(2) Combinatorial interpretation of χ *(Ringel form): For $\underline{x} = (x_1, \ldots, x_n) \in \mathbb{Q}^n$*

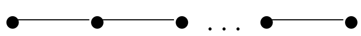
$$\chi(\underline{x}) = \sum_{i \in Q_0} x_i^2 - \sum_{i \to j \in Q_1} d_{ji} x_i x_j$$

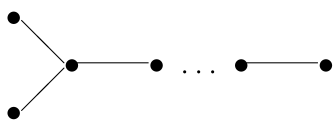
Theorem 5.2.2 (Gabriel 1972)**.** *Let Λ be a finite dimensional hereditary algebra over an algebraically closed field k, and let Q be the Gabriel-quiver of Λ. The following statements are equivalent.*

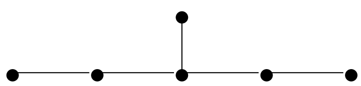
(a) Λ is of finite representations type.

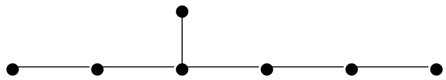
(b) χ is positive definite, i.e. $\chi(x) > 0$ for all $\underline{x} \in \mathbb{Q}^n \setminus \{0\}$.

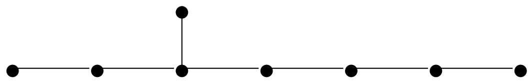
(c) Q is of Dynkin *type, that is, its underlying graph belongs to the following list.*

$\mathbb{A}_n$: 

$\mathbb{D}_n$: 

$\mathbb{E}_6$: 

$\mathbb{E}_7$: 

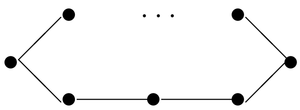
$\mathbb{E}_8$: 

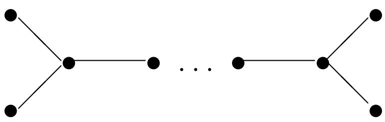
If (a) - (c) are satisfied, the assignment $A \mapsto \underline{\dim} A$ *defines a bijection between the isomorphism classes of indecomposable finite dimensional* Λ*-modules and the positive roots of* χ*. In particular, the finite dimensional indecomposable modules are uniquely determined by their dimension vector.*

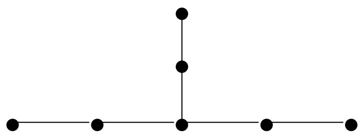
Theorem 5.2.3 (Gabriel 1972)**.** *Let* Λ *be a finite dimensional hereditary algebra over an algebraically closed field* k*, and let* Q *be the Gabriel-quiver of* Λ*. The following statements are equivalent.*

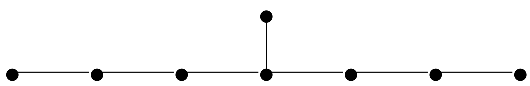
(a) χ *is positive semidefinite, i.e.* $\chi(x) \geq 0$ *for all* $\underline{x} \in \mathbb{Q}^n \setminus \{0\}$*, and there are non-trivial radical vectors.*

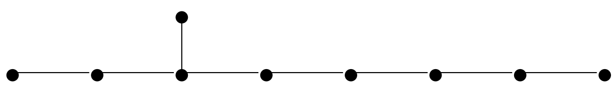
(b) Q *is of* Euclidean *type, that is, its underlying graph belongs to the following list.*

$\widetilde{\mathbb{A}}_n$: 

$\widetilde{\mathbb{D}}_n$: 

$\widetilde{\mathbb{E}}_6$: 

$\widetilde{\mathbb{E}}_7$: 

$\widetilde{\mathbb{E}}_8$: 

If (a) - (b) are satisfied, then Λ *is said to be* tame of infinite representation type.

We will see in the next section that also in the latter case the isomorphism classes of indecomposable finite dimensional modules, though infinite in number, can be classified.

Remark 5.2.4. There is a general definiton of tameness for arbitrary finite dimensional algebras. A finite-dimensional k-algebra Λ over an algebraically closed field k is called *tame* if, for each dimension d, there are finitely many Λ-$k[x]$-bimodules $M_1, \cdots, M_n$ which are free of rank d as right $k[x]$-modules, such that every indecomposable Λ-module of dimension d is isomorphic to $M_i \otimes_{k[x]} k[x]/(x-\lambda)$ for some $1 \leq i \leq n$ and $\lambda \in k$. In other words, Λ is tame iff for each dimension d there is a finite number of one-parameter families of indecomposable d-dimensional modules such that all indecomposable modules of dimension d belong (up to isomorphism) to one of these families.

Moreover, Λ is said to be *of wild representation type* if there is a representation embedding from $k < x, y >$mod into $\Lambda\,$mod, where $k < x, y >$ denotes the free associative algebra in two non-commuting variables. Observe that in this case there is a representation embedding AMod $\to \Lambda\,$Mod for any finite dimensional k-algebra A, and furthermore, any finite dimensional k-algebra A occurs as the endomorphism ring of some Λ-module.

A celebrated theorem of Drozd [13] states that every finite dimensional algebra Λ over an algebraically closed field k is either tame or wild.

5.3 The AR-quiver of a hereditary algebra

Definition. The component $\mathbf{p}$ of Γ containing all projective indecomposable modules is called the *preprojective component*. The component $\mathbf{q}$ of Γ containing all injective indecomposable modules is called the *preinjective component*. The remaining components of Γ are called *regular*.

We denote by $\mathbb{N}Q^{\text{op}}$ and $-\mathbb{N}Q^{\text{op}}$ the quivers obtained from Q by drawing the opposite quiver Q^{op}, and applying the "Knitting Procedure" described in 3.3.4 and 3.3.5.

Theorem 5.3.1 (Gabriel-Riedtmann 1979)**.** *[15] Let Λ and Q be as above.*

(1) If Q is a Dynkin quiver, then $\Gamma = \mathbf{p} = \mathbf{q}$ is a full finite subquiver of $\mathbb{N}Q^{\text{op}}$.

(2) If Q is not a Dynkin quiver, then $\mathbf{p} = \mathbb{N}Q^{\text{op}}$, and $\mathbf{q} = -\mathbb{N}Q^{\text{op}}$, and the modules in $\mathbf{p}$ and $\mathbf{q}$ are uniquely determined by their dimension vectors. Moreover $\mathbf{p} \cap \mathbf{q} = \emptyset$, and $\mathbf{p} \cup \mathbf{q} \subsetneqq \Gamma$.

So, regular components only occur when Λ is of infinite representation type. They have a rather simple shape, as shown independently in [8] and [23].

Theorem 5.3.2 (Auslander-Bautista-Platzeck-Reiten-Smalø; Ringel 1979)**.** *Let Λ be of infinite representation type. Let $\mathcal{C}$ be a regular component of Γ. For each $[M]$ in $\mathcal{C}$ there are at most two arrows ending in $[M]$.*

For a proof, we refer to [12, VIII.4]. Here we only explain the
Construction of the regular component $\mathcal{C}$: For each $M \in \mathcal{C}$ we consider a minimal right almost split map $g : B \longrightarrow M$, and we denote by $\alpha(M)$ the number of summands in an indecomposable decomposition $B = B_1 \oplus \ldots \oplus B_{\alpha(M)}$ of B. We have stated the Theorem in a weak form; actually, it is even known that $\alpha(M) \leq 2$.
In order to construct $\mathcal{C}$, let us start with a module $C_0 \in \mathcal{C}$ of minimal length. Such a module is called *quasi-simple* (or *simple regular*).
Note that $\alpha(C_0) = 1$. Otherwise there is an almost split sequence of the form $0 \to \tau C_0 \to X_1 \oplus X_2 \xrightarrow{(g_1,g_2)} C_0 \to 0$ with non-zero modules X_1, X_2, and one can check that g_i cannot be both epimorphisms. But then $l(X_i) < l(C_0)$ for some i, a contradiction.
Now $\alpha(C_0) = 1$ implies that in Γ there is a unique arrow $[X] \xrightarrow{(1,1)} [C_0]$ ending in C_0, and therefore by 3.3.4(3), also a unique arrow starting in $[C_0]$ with valuation $(1, 1)$. So we have an almost split sequence $0 \longrightarrow C_0 \xrightarrow{f_0} C_1 \xrightarrow{g_0} \tau^- C_0 \longrightarrow 0$ with C_1 being indecomposable. Moreover, we have an almost split sequence $0 \longrightarrow \tau C_1 \longrightarrow C_0 \oplus Y \xrightarrow{(f_0,\, h)} C_1 \longrightarrow 0$ where $Y \neq 0$ because f_0 is an irreducible monomorphism. Hence $\alpha(C_1) = 2$ and Y is indecomposable. Furthermore, one checks that h must be an irreducible epimorphism.
Setting $C_2 = \tau^- Y$ and $g_1 = \tau^- h$, we obtain an almost split sequence $0 \longrightarrow C_1 \xrightarrow{(f_1,\, g_0)^t} C_2 \oplus \tau^- C_0 \xrightarrow{(g_1,\, \tau^- f_0)} \tau^- C_1 \longrightarrow 0$ where g_0, g_1 are irreducible epimorphisms and $f_1, \tau^- f_0$ are irreducible monomorphisms.
Proceeding in this manner, we obtain a chain of irreducible monomorphisms $C_0 \hookrightarrow C_1 \hookrightarrow C_2 \ldots$ with almost split sequences $0 \longrightarrow C_i \longrightarrow C_{i+1} \oplus \tau^- C_{i-1} \longrightarrow \tau^- C_i \longrightarrow 0$ for all i.
The component $\mathcal{C}$ thus has the shape

$$\begin{tikzcd}[column sep=small]
 & & & \vdots & & \vdots & & \vdots \\
 & & & \vdots & & \vdots & & \vdots \\
 & & & Y = \tau C_2 \arrow[dr, "h"] & & C_2 \arrow[dr, "g_1"] & & \tau^- C_2 \\
 & \cdots & \tau C_1 \arrow[ur] \arrow[dr] & & C_1 \arrow[ur, "f_1"] \arrow[dr, "g_0"'] & & \tau^- C_1 \arrow[ur] & \cdots \\
\cdots & \tau C_0 \arrow[ur] & & C_0 \arrow[ur, "f_0"'] & & \tau^- C_0 \arrow[ur, "\tau^- f_0"'] & \cdots &
\end{tikzcd}$$

and every module in $\mathcal{C}$ has the form $\tau^r C_i$ for some i and some $r \in \mathbb{Z}$.
Observe that if $\tau^r C_i \cong C_i$ for some i and r, then $\tau^r C \cong C$ for all C in $\mathcal{C}$.

Corollary 5.3.3. *Let* $\mathbb{A}_\infty$ *be the infinite quiver* $\bullet \to \bullet \to \bullet \to \bullet \cdots$
Then $\mathcal{C}$ *has either the form* $\mathbb{Z}A_\infty$ *or it has the form* $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ *where* $n = \min\{r \in \mathbb{N} \mid \tau^r C \cong C$ *for some* $C \in \mathcal{C}\}$.

Definition. We call $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ a *(stable) tube*, and we call it *homogeneous* if $n = 1$.

Stable tubes do not occur in the wild case. In the tame case, the regular components form a family of tubes $\mathbf{t} = \bigcup \mathbf{t}_\lambda$ indexed over the projective line $\mathbb{P}_1 k$, and all but at most three $\mathbf{t}_\lambda$ are homogeneous.

5.4 The Tame Hereditary Case

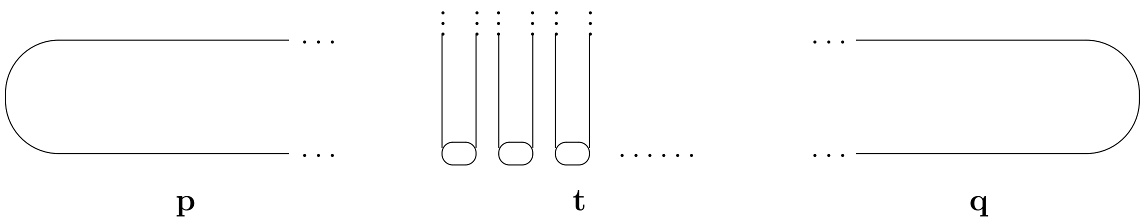
Let k be an algebraically closed field, and let Λ be a finite dimensional hereditary k-algebra with Gabriel-quiver Q of Euclidean type.
The following properties are shown e.g. in [24].

(1) The $\mathbb{Q}$-subspace $N = \{\underline{x} \in \mathbb{Q}^n \mid \chi(\underline{x}) = 0\}$ formed by the radical vectors is one-dimensional and can be generated by a vector $\underline{v} = (v_1, \ldots, v_n) \in \mathbb{N}^n$ with at least one component $v_i = 1$.

(2) There is a $\mathbb{Q}$-linear map $\delta\colon \mathbb{Q}^n \to \mathbb{Q}$ which is invariant under c, that is, $\delta(c\underline{x}) = \delta(\underline{x})$ for all $\underline{x} \in \mathbb{Q}^n$, and moreover satisfies $\delta(\underline{p_i}) \in \mathbb{Z}$ for each $1 \le i \le n$ and $\delta(\underline{p_i}) = -1$ for at least one i.

    The map δ is called the *defect*, and an indecomposable projective module $P = \Lambda e_i$ with defect -1 is called *peg*.

(3) As we have seen in the last section, the AR-quiver Γ has the shape



    where $\mathbf{t} = \bigcup \mathbf{t}_\lambda$ and $\mathbf{t}_\lambda$ are tubes of rank n_λ with almost all $n_\lambda = 1$.

(4) The categories $\mathbf{p}, \mathbf{q}, \mathbf{t}$ are *numerically determined*:

    If X is an indecomposable Λ-module, then

$$X \text{ belongs to } \mathbf{p} \text{ if and only if } \delta(\underline{\dim} X) < 0$$
$$X \text{ belongs to } \mathbf{q} \text{ if and only if } \delta(\underline{\dim} X) > 0$$
$$X \text{ belongs to } \mathbf{t} \text{ if and only if } \delta(\underline{\dim} X) = 0$$

(5) The dimension vectors $\underline{\dim} X$ of the indecomposable Λ-modules X are either positive roots of χ or positive radical vectors of χ. The assignment $X \mapsto \underline{\dim} X$ defines bijections

$$\{\text{isomorphism classes of } \mathbf{p}\} \longrightarrow \{\text{positive roots of } \chi \text{ with negative defect}\}$$
$$\{\text{isomorphism classes of } \mathbf{q}\} \longrightarrow \{\text{positive roots of } \chi \text{ with positive defect}\}$$

For any positive radical vector $\underline{x} \in \mathbb{Z}^n$ of χ there is a whole $\mathbb{P}_1 k$-family of isomorphism classes of $\mathbf{t}$ having dimension vector $\underline{x}$.

(6) $\mathbf{p}$ is *closed under predecessors*: If $X \in \Lambda\,\mathrm{Mod}$ is an indecomposable module with $\mathrm{Hom}(X, P) \neq 0$ for some $P \in \mathbf{p}$, then $X \in \mathbf{p}$.

In fact, $\mathbf{p}$ inherits "closure properties" from the projective modules. This can be proven employing the notion of preprojective partition together with the existence of almost split sequences in $\Lambda\,\mathrm{Mod}$. For finitely generated X there is also an easier argument: Since by Proposition 1.5.1 the functor $\tau\colon\ \mathrm{mod}\,\Lambda_{\mathcal{P}} \to \Lambda\,\mathrm{mod}_{\mathcal{I}}$ is an equivalence, $\mathrm{Hom}(X, P) \neq 0$ implies that either X is projective or $\mathrm{Hom}(\tau X, \tau P) \neq 0$. Continuing in this way and using that $\tau^n P$ is projective for some n, we infer that there exists an $m \leq n$ such that $\tau^m X$ is projective, which proves $X \in \mathbf{p}$.

(7) $\mathbf{q}$ is *closed under successors*: If $X \in \Lambda\,\mathrm{Mod}$ is an indecomposable module with $\mathrm{Hom}(Q, X) \neq 0$ for some $Q \in \mathbf{q}$, then $X \in \mathbf{q}$.

This is shown with dual arguments.

(8) The additive closure $\mathrm{add}\mathbf{t}$ of $\mathbf{t}$ is an *exact abelian serial subcategory of* Λmod: Each object is a direct sum of indecomposable objects, and each indecomposable object X has a unique chain of submodules in $\mathrm{add}\mathbf{t}$

$$X = X_m \supset X_{m-1} \supset \cdots \supset X_1 \supset X_0 = 0$$

such that the consecutive factors are simple objects of $\mathrm{add}\,\mathbf{t}$. The simple objects of $\mathrm{add}\,\mathbf{t}$ are precisely the quasi-simple modules introduced in 5.3.2. Their endomorphism rings are skew fields.

(9) The tubular family $\mathbf{t}$ is *separating*, that is:

(a) $\mathrm{Hom}(\mathbf{q}, \mathbf{p}) = \mathrm{Hom}(\mathbf{q}, \mathbf{t}) = \mathrm{Hom}(\mathbf{t}, \mathbf{p}) = 0$

(b) Any map from a module in $\mathbf{p}$ to a module in $\mathbf{q}$ factors through any $\mathbf{t}_\lambda$.

So, between the components of the AR-quiver, there are only maps from left to right. Actually, even inside $\mathbf{p}$ and $\mathbf{q}$ there are only maps from left to right.

(10) $\mathbf{t}$ is *stable*, i.e. it does not contain indecomposable modules that are projective or injective, and it is *sincere*, i.e. every simple module occurs as the composition factor of at least one module from $\mathbf{t}$.

Let us illustrate the above properties with an example.

5.5 The Kronecker Algebra

Consider the quiver

$$Q = \widetilde{\mathbb{A}_1}: \qquad \overset{1}{\bullet} \rightrightarrows \overset{2}{\bullet}$$

Then $\Lambda = kQ$ is called the Kronecker algebra, cf. [18].

(1) The Coxeter transformation and the Tits form:

$$\left.\begin{aligned} \underline{p_1} &= \underline{\dim}\Lambda e_1 = (1,2) \\ \underline{p_2} &= \underline{\dim}\Lambda e_2 = (0,1) \end{aligned}\right\} \text{ hence } \quad C = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad C^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

So we have

$$c(\underline{x}) = -\underline{x}\, C^{-1} C^t = \underline{x} \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$$

$$\chi(\underline{x}) = x_1^2 + x_2^2 - 2x_1x_2 = (x_1 - x_2)^2$$

$$N = \left\{\underline{x} \in \mathbb{Q}^2 \mid x_1 = x_2\right\} \quad \text{is generated by } \underline{v} = (1,1)\,.$$

We can then write

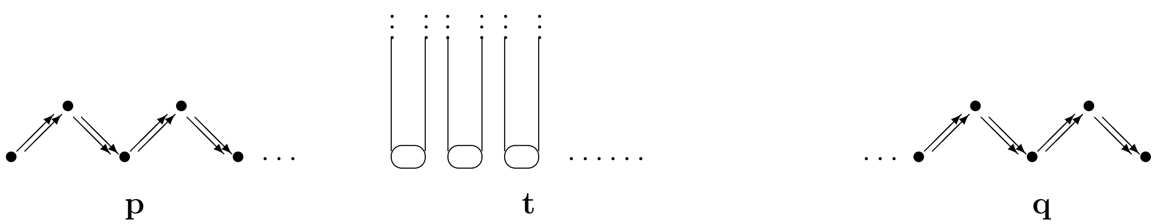
$$c(\underline{x}) = \underline{x}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\right) = \underline{x} + 2(x_1 - x_2)\underline{v}$$

and since $c(\underline{v}) = \underline{v}$, we have

$$c^m \underline{x} = \underline{x} + 2m(x_1 - x_2)\underline{v} \quad \text{for each } m.$$

(2) Take $\delta\colon \mathbb{Q}^2 \to \mathbb{Q}$, $\underline{x} \mapsto B(\underline{v}, \underline{x}) = x_1 - x_2$. The $\mathbb{Q}$-linear map δ is the *defect*.
Then $\delta(\underline{p_1}) = -1 = \delta(\underline{p_2})$, so $P_1 = \Lambda e_1$ and $P_2 = \Lambda e_2$ are *pegs*.

(3) The AR-quiver Γ:



The shape of **t** is explained below. For **p** and **q** we refer to Theorem 5.3.1.

We can now compute the dimension vectors. For example, from the first two arrows on the left we deduce that there is an almost split sequence $0 \longrightarrow P_2 \longrightarrow P_1 \oplus P_1 \longrightarrow C \longrightarrow 0$ and $\underline{\dim}C = (1,2) + (1,2) - (0,1) = (2,3)$. In this way we observe

(4) $\mathbf{p}$ consists of the modules X with $\underline{\dim}X = (m, m+1)$, so $\delta(\underline{\dim}X) = -1$.

$\mathbf{q}$ consists of the modules X with $\underline{\dim}X = (m+1, m)$, so $\delta(\underline{\dim}X) = 1$.

The modules in $\mathbf{t}$ are precisely the modules X with $\underline{\dim}X = (m, m)$, so $\delta(\underline{\dim}X) = 0$.

Let us check the last statement. Let $X \in \mathbf{t}$ and $\underline{\dim}X = (l, m)$. If $l < m$, then

$$c^m(\underline{\dim}X) = (l, m)\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + m\begin{pmatrix}2 & 2\\ -2 & -2\end{pmatrix}\right) = (l, m) + 2m(l-m, l-m)$$

is negative. By 5.1.3 we have $c^m(\underline{\dim}X) = c(\underline{\dim}\tau^{m-1}X)$, thus $\tau^{m-1}X$ is projective, and $X \in \mathbf{p}$. Dually, $l > m$ implies $X \in \mathbf{q}$. Hence we conclude $l = m$.

(5) Let us now compute $\mathbf{t}$. First of all, the quasi-simple modules, that is, the indecomposable regular modules of minimal length, are precisely the modules X with $\underline{\dim}X = \underline{v} = (1, 1)$. A complete irredundant set of quasi-simples is then given by

$$V_\lambda : K \underset{\lambda}{\overset{1}{\rightrightarrows}} K,\ \lambda \in K, \quad \text{and} \quad V_\infty : K \underset{\lambda}{\overset{0}{\rightrightarrows}} K$$

Note that each V_λ is sincere with composition factors S_1, S_2.

Furthermore, applying $\mathrm{Hom}(-, V_\mu)$ on the projective resolution $0 \to \Lambda e_2 \to \Lambda e_1 \to V_\lambda \to 0$ we see that V_λ, V_ν are "perpendicular":

$$\dim_k \mathrm{Hom}_\Lambda(V_\lambda, V_\mu) = \dim_k \mathrm{Ext}\,1_\Lambda(V_\lambda, V_\mu) = \begin{cases}1 & \mu = \lambda\\ 0 & \text{else}\end{cases}$$

Next, we check that each V_λ defines a homogeneous tube $\mathbf{t}_\lambda$.
In fact, $\tau V_\lambda \cong V_\lambda$ for all $\lambda \in K \cup \{\infty\}$:

$$\underline{\dim}\tau V_\lambda = c(\underline{\dim}V_\lambda) = (1, 1)\begin{pmatrix}3 & 2\\ -2 & -1\end{pmatrix} = (1, 1)\,,$$

$$\text{hence} \quad \tau V_\lambda \cong V_\mu \text{ with } \mathrm{Ext}^1(V_\lambda, V_\mu) \neq 0\,, \text{ so } \mu = \lambda\,.$$

So, for each $\lambda \in K \cup \{\infty\}$ there is a chain of irreducible monomorphisms

$$V_\lambda = V_{\lambda,1} \hookrightarrow V_{\lambda,2} \hookrightarrow \ldots$$

that gives rise to a homogeneous tube $\mathbf{t}_\lambda \cong \mathbb{Z}A_\infty \setminus \langle\tau\rangle$ consisting of modules $V_{\lambda,j}$ with $\tau V_{\lambda,j} \cong V_{\lambda,j}$, $\underline{\dim}V_{\lambda,j} = (j, j)$, $\delta(\underline{\dim}V_{\lambda,j}) = 0$, and $V_{\lambda,j+1}/V_{\lambda,j} \cong V_\lambda$.
Moreover, there are neither nonzero maps nor extensions between different tubes $\mathbf{t}_\lambda$.

Finally, let us indicate how to show that every indecomposable regular module X is contained in some tube $\mathbf{t}_\lambda$. We already know that X has the form $X : K^m \underset{\beta}{\overset{\alpha}{\rightrightarrows}} K^m$. Now, suppose that α is a isomorphism. Then, since k is algebraically closed, $\alpha^{-1}\beta$ has an eigenvalue λ, and, as explained in [12, VIII.7.3], it is possible to embed $V_\lambda \subset X$. This proves that X belongs to $\mathbf{t}_\lambda$. Similarly, if $\operatorname{Ker}\alpha \neq 0$, it is possible to embed $V_\infty \subset X$, which proves that X belongs to t_∞.

(6) To show that $\mathbf{t}$ is separating, we check that every $f\colon P \to Q$ with $P \in \mathbf{p}$, and $Q \in \mathbf{q}$, factors through any $\mathbf{t}_\lambda$. The argument is taken from [24, p.126].

Let $\lambda \in K \cup \{\infty\}$ be arbitrary, and let $\underline{\dim} P = (l, l+1)$ and $\underline{\dim} Q = (m+1, m)$. Choose an integer $j \geq l+m+1$. We are going to show that f factors through $V_{\lambda,j}$.

Note that $\operatorname{Ext}^1_\Lambda(P, V_{\lambda,j}) = 0$. So, using the homological interpretation of B in Proposition 5.2.1 we obtain $\dim_k \operatorname{Hom}_\Lambda(P, V_{\lambda,j}) = \dim_k \operatorname{Hom}_\Lambda(P, V_{\lambda,j}) - \dim_k \operatorname{Ext}^1_\Lambda(P, V_{\lambda,j}) = B(\underline{\dim} P, \underline{\dim} V_{\lambda,j}) = (l, l+1) \left(\begin{smallmatrix} 1 & -2 \\ 0 & 1 \end{smallmatrix}\right) \binom{j}{j} = j$.

So, the k-spaces $\operatorname{Hom}_\Lambda(P, V_{\lambda,j})$, $j \geq 0$, form a strictly increasing chain. Hence there exists a map $g\colon P \to V_{\lambda,j}$ such that $\operatorname{Im} g \not\subset V_{\lambda,j-1}$, and by length arguments we infer that $\operatorname{Im} g$ is a proper submodule of $V_{\lambda,j}$. Thus $\operatorname{Im} g$ is not regular. Then it must contain a preprojective summand P', and we conclude that g is a monomorphism. Consider the exact sequence

$$0 \longrightarrow P \xrightarrow{g} V_{\lambda,j} \longrightarrow Q' \longrightarrow 0$$

The module Q' cannot have regular summands, so it is a direct sum of preinjective modules and satisfies

$$\delta(\underline{\dim} Q') = \delta(\underline{\dim} V_{\lambda,j}) - \delta(\underline{\dim} P) = 1$$

This shows $Q' \in q$. Furthermore, $\underline{\dim} Q' = (s+1, s)$ with $s = j - (l+1) \geq m$, which proves $\operatorname{Ext}^1_\Lambda(Q', Q) = 0$. Thus we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[rr, "g"] \arrow[dr, "f"'] & & V_{\lambda,j} \arrow[r] \arrow[dl, dotted] & Q' \arrow[r] & 0 \\ & & Q & & & \end{tikzcd}$$

and the claim is proven.

References

[1] F. W. ANDERSON, K. R. FULLER, Rings and categories of modules, second ed., Springer, New York, 1992.

[2] M. AUSLANDER, Representation theory of artin algebras I, Comm. Algebra **1** (1974), 177-268.

[3] M. AUSLANDER, Representation theory of artin algebras II, Comm. Algebra **1** (1974), 269-310.

[4] M. AUSLANDER, Large modules over artin algebras, in Algebra, Topology, Categories; Academic Press 1976, 1-17.

[5] M. AUSLANDER, A survey of existence theorems for almost split sequences, Representations of algebras (Durham, 1985), 81-89, London Math. Soc. Lecture Note Ser. **116**, Cambridge Univ. Press, Cambridge, 1986.

[6] M. AUSLANDER, Functors and morphisms determined by objects, Lecture Notes in Pure and Appl. Math. **37** (1978), 1-244.

[7] M. AUSLANDER, A functorial approach to representation theory, Lecture Notes in Math. **944** (1982), 105-179.

[8] M. AUSLANDER, R. BAUTISTA, M. I. PLATZECK, I. REITEN, S. O. SMALØ, Almost split sequences whose middle term has at most two indecomposable summands, Canad. J. Math. **31** (1979), 942-960.

[9] M. AUSLANDER, M. BRIDGER, Stable module theory, Mem. Amer. Math. Soc. **94** (1969).

[10] M. AUSLANDER, I. REITEN, Representation theory of artin algebras III, Comm. Algebra **3** (1975), 239-294.

[11] M. AUSLANDER, I. REITEN, Representation theory of artin algebras V. Comm. Algebra **5** (1977), 443-518.

[12] M. AUSLANDER, I. REITEN, S. O. SMALØ, *Representation theory of artin algebras*, Cambridge University Press (1994).

[13] YU. DROZD, Tame and wild matrix problems, *in* Representations and quadratic forms (Institute of Mathematics, Academy of Sciences, Ukranian SSR, Kiev 1979), 39-74. Amer. Math. Soc. Transl. **128** (1986), 31-55.

[14] YU. DROZD, V. V. KIRICHENKO, *Finite dimensional algebras*, Springer Verlag 1994.

[15] P. GABRIEL, C. RIEDTMANN, Group representations without groups, Comm. Math. Helv. **54** (1979) 240-287.

[16] M. HAZEWINKEL, N. GUBARENI, V.V. KIRICHENKO, *Algebras, Rings, and Modules*, Kluwer Academic Publishers 2010.

[17] H. KRAUSE, A short proof for Auslander's defect formula, Linear Algebra and its Appl. **365** (2003), 267-270.

[18] L. KRONECKER, Algebraische Reduction der Scharen Bilinearer Formen, Sitzungsber. Akad. Berlin (1890) 1225-1237.

[19] T.Y. LAM, *A First Course in Noncommutative Rings*, Graduate Texts in Mathematics **131**, Springer New York 2001.

[20] T.Y. LAM, *Lectures on Modules and Rings*, Graduate Texts in Mathematics **189**, Springer New York 1999.

[21] H. LENZING, Auslander's work on Artin algebras, CMS Conf. Proc. **23** (1998), 83-105.

[22] R.S.PIERCE, *Associative Algebras*, Springer Verlag 1982.

[23] C. M. RINGEL, Finite dimensional hereditary algebras of wild representation type, Math. Z. **161** (1978) 235-255.

[24] C. M. RINGEL, Tame algebras and integral quadratic forms. Springer Lect. Notes Math. 1099 (1984).

[25] C. M. RINGEL, H. TACHIKAWA, QF-3 rings, J. Reine Angew. Math. **272** (1975), 49-72.

[26] J. J. ROTMAN, *An introduction to homological algebra*, Academic Press 1979.

[27] B.ZIMMERMANN-HUISGEN, Rings whose right modules are direct sums of indecomposables, Proc. Amer. Math. Soc. **77** (1979), 191-197.

[28] W. ZIMMERMANN, Auslander-Reiten sequences over artinian rings, J.Algebra $\underline{119}$ (1988), 366-392.

[29] W. ZIMMERMANN, Auslander-Reiten sequences over derivation polynomial rings, J. Pure and Appl. Algebra $\underline{74}$ (1991), 317-332.