CORTONA 1983

revised 2009

Summer course on Differential Geometry
Prof. F. TRICERRI and J.C. WOOD

# MORSE THEORY

Seminar: MAURO SPERA

main references: J. MILNOR - Morse theory
R. BOTT, L. TU - Differential forms in algebraic topology

## Contents:

to be said immediately { 0 - Main idea: use functions to get knowledge of the topology of the manifold

1 - Attaching cells: CW-complexes (alteration of homology)
( Homology, Betti numbers )

Ex: circle, sphere, torus (I)

2 - Theorem (Milnor) "Every $M$ (compact) is $\sim$ CW" (statement

3 - Critical points, index, morse functions, MORSE LEMMA (critical points are isolated (and finite)

TORUS (II) $\Rightarrow$ functions + attaching $k$-cells at every critical value of index $k$

4 - The theorems

1 - Retraction   $a$ $b$   no crit. points inside

2 - Attaching cells   $a$ $k$ $b$   $M^b \simeq M^a \cup e^k$

(3 - Existence of Morse functions) by Sard (on $M$ embedded

4 - MILNOR'S THEOREM end of proof.

5 - Applications

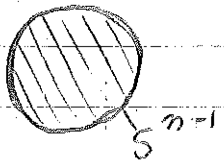
1 - $\mathbb{CP}^n$

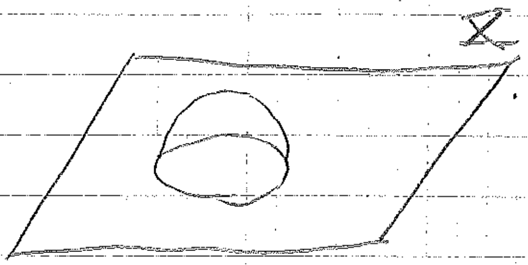
2 - Morse inequalities

3 - Reeb's theorem (Milnor spheres)

TOPOLOGIA E GEOMETRIA DIFFERENZIALE
Prof. M. Spera, a.a. 2009/10   Lezione A6

## 1 - Attaching cells

$e^n =$ 


$f: S^{n-1} \longrightarrow X$

$x \longmapsto f(x)$

$X \cup_f e^n = X \amalg e^n / f(x) \sim x$

$\boxed{\text{CW complex}}$ (C closed iff C ∩ cell closed = weak topology)

$\chi = \sum (-1)^k b_k \overset{\text{Hopf Poincaré}}{=} \sum (-1)^k c_k$

$\boxed{c_k \geq b_k}$

Betti

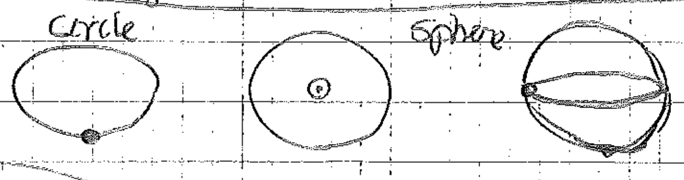
independent cycles (singular or simplicial) $c$, $\partial c = 0$ $\xleftrightarrow{\text{De Rham}}$ independent closed forms $\omega$, $d\omega = 0$

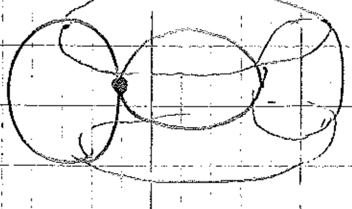
$\langle c, \omega \rangle = \int_c \omega$

Stokes: $\langle \partial c, \omega \rangle = \langle c, d\omega \rangle$

Hodge (harmonic forms)

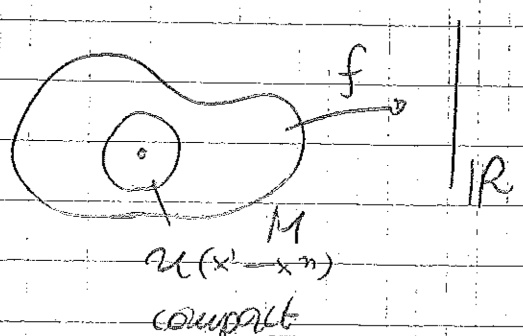
**Examples**




## 2 - Milnor's theorem (statement)

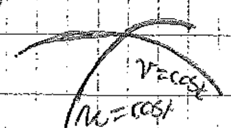
3 – Critical points etc




critical point: $df(p) = 0 \qquad \frac{\partial f}{\partial x^i}(p) = 0 \ \forall i$

non degeneracy $\det H(p) = \det\left(\frac{\partial f}{\partial x_i \partial x_j}\right) \neq 0$

$\Rightarrow \prod \lambda_i \neq 0$ change of coordinates $\rightarrow \begin{pmatrix} \frac{\partial^2 f}{\partial u^2} & 0 \\ 0 & \frac{\partial^2 f}{\partial v^2} \end{pmatrix}$




intrinsic / $x \rightarrow y \qquad \frac{\partial f}{\partial y_l} = \frac{\partial f}{\partial x_i}\frac{\partial x_j}{\partial y_l}$

$$\frac{\partial^2 f}{\partial y_k \partial y_l} = \frac{\partial^2 f}{\partial x_i \partial x_j}\frac{\partial x_i \partial x_j}{\partial y_k \partial y_l} + \underbrace{\frac{\partial f}{\partial x_i}}_{0}\frac{\partial^2 x_j}{\partial y_k \partial y_l}$$

$$\boxed{H(y) = J^t H(x) J}$$

Index = $n^\circ$ of negative eigenvalues

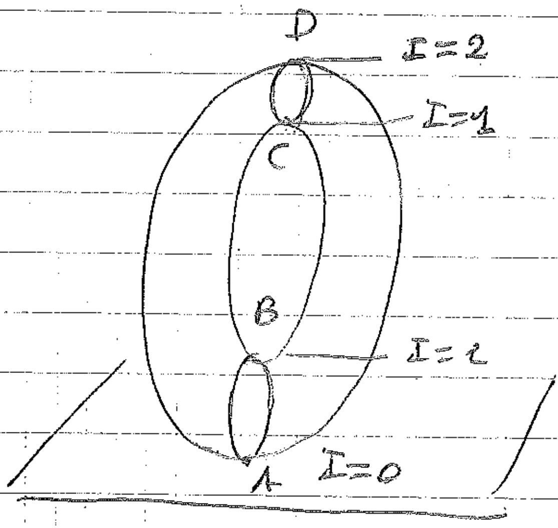
intrinsic! (algebra or intuition)

Morse function ; Morse lemma : you can choose coordinates in p critical non deg. s. that.

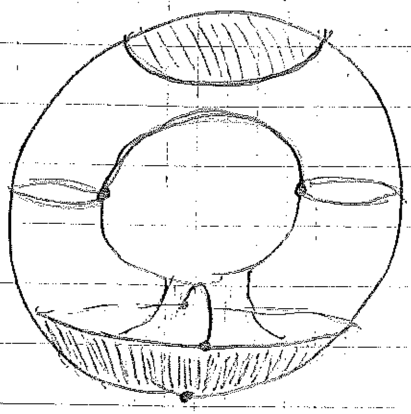
$$f = \underset{c}{f(p)} - x_1^2 - x_2^2 - \dots \underset{\text{index}}{x_k^2} + x_{k+1}^2 + \dots x_n^2$$

$\Rightarrow$ Critical (non deg) are isolated $\overset{\text{M compact}}{\Rightarrow}$ finite

## THE TORUS




$\Leftrightarrow$

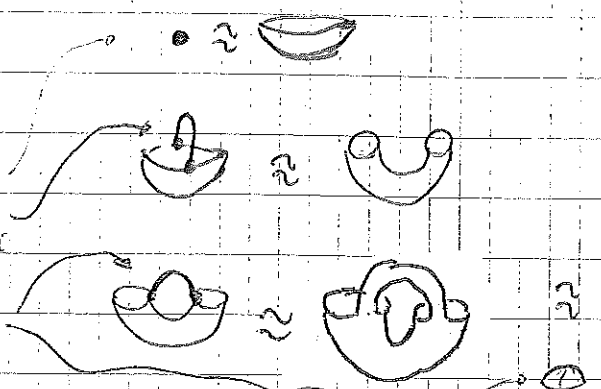


$\{f \leq c\}$

sublevels

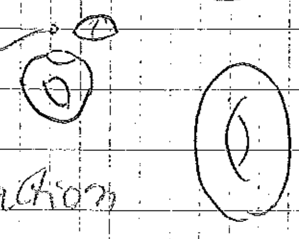
$f =$ height

corrispondono agli autovalori negativi di $H(f)$



## MILNOR'S PROOF

$f$ morse



1 — $M_a = f^{-1}[-\infty, a]$

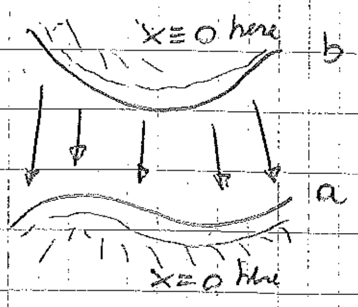
Retraction

If $f^{-1}[a,b]$ compact and does not contain crit. points

then $M_a \simeq M_b$

PROOF choose $\langle\ \rangle$ metric. $\langle \nabla f, Y \rangle = df(Y)$

$X = \dfrac{-\nabla f}{\|\nabla f\|}$ makes sense



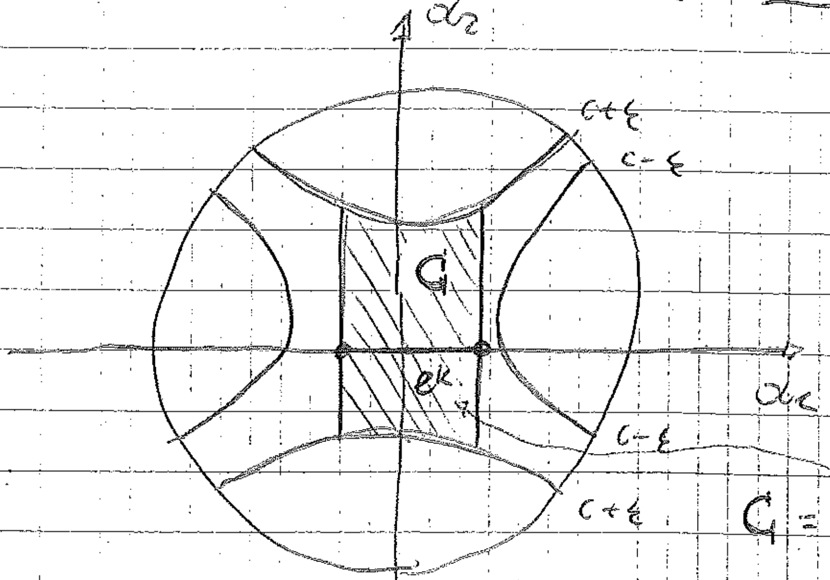
## 2- Attaching cells

$f^{-1}[a,b]$ compact. $\exists$ $p$ crit. index $p = k$

$M_b \simeq M_a \cup e^k$

see Figure: $k = 1$ $n = 2$ Use Morse Lemma

$\varepsilon$ suitable

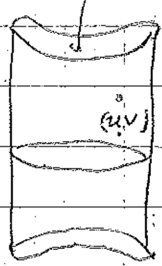



$G = \{ f \le c+\varepsilon , \sum_{i=1}^{k} x_i^2 \le \delta \}$

$\overset{f}{c+\varepsilon} = c - x_1^2 + x_2^2$

$\varepsilon = -x_1^2 + x_2^2$

$\boxed{G \simeq e^k}$



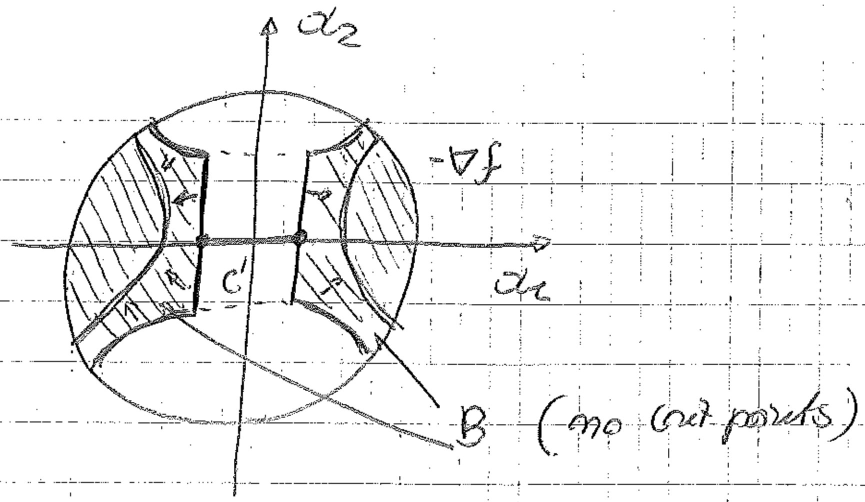

$u' = (1+\rho)u$

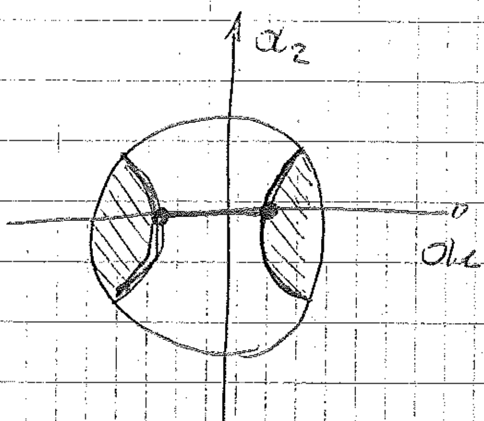
$v' = (1-\frac{\rho}{|v|})v$

$\rho = \min\left(|v| \quad \frac{\sqrt{\delta}-u}{|u|}\right)$

$\rho = \frac{\sqrt{\delta}-|u|}{|u|}$

$\rho = |v|$




□

3 - (There exist morse functions)
(embedding in Euclidean space + Sard)

4) Milnor: idea of proof

$M \to$ Whitney** $\to M \hookrightarrow \mathbb{R}^k \to f$ of 3)

crit are isol, $M_a$ compact are finite

$|p_1 \text{——} p_r|$ $\to$ attaching process $\to$
finite!! index 0
a $\lambda$ cell for every $p(\lambda)$ ($\lambda$: index)
compare TORUS □

5 - Applications

1) $\boxed{\mathbb{CP}^n \text{ as } CW}$ $\quad c_0 \in \mathbb{R}$

$(z_0 : z_1 : \dots z_n)$ $\quad \left( \sum |z_i|^2 = 1 \quad z_i \to e^{i\lambda} z_i \right)$

$f(z_0 : z_1 : \dots z_n) = \sum c_i |z_i|^2$ well defined

$(U_i : z_i \neq 0)$ $\quad |z_0|^2 = 1 - \sum_i x_i^2 + y_i^2$

$|z_0| \frac{z_j}{z_0} = x_i + i y_j$

$f = c_0 |z_0|^2 + \sum c_i |z_i|^2 = c_0 |z_0|^2 + \sum c_i (x_i^2 + y_i^2)$

$= c_0 \left( 1 - \sum x_i^2 + y_i^2 \right) + \sum c_i (x_i^2 + y_i^2)$

$= c_0 + \sum_{i \geq 1} (c_i - c_0)(x_i^2 + y_i^2)$

critical $p_0 = (1, 0 0 0 0)$ non deg. $c_j < c_0$

$\Rightarrow p_0 \; p_1 \; p_n$ $\quad \underset{p_k}{\text{index}} = 2 \; (c_j < c_k)$

$\Downarrow$

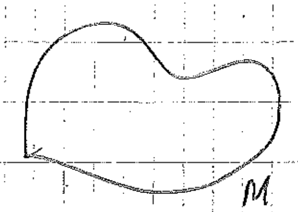
$\mathbb{CP}^n \approx e^0 \cup e^2 \cup \dots e^4 \cup e^{2n}$

$\Rightarrow H_i(\mathbb{CP}_n, \mathbb{Z}) = \begin{cases} \mathbb{Z} \\ 0 \end{cases}$

2 - Morse inequalities

$$\begin{cases} \sum (-1)^\lambda c_\lambda = \chi(M) \\ c_k \geq \beta_k \end{cases}$$

3 - Reeb

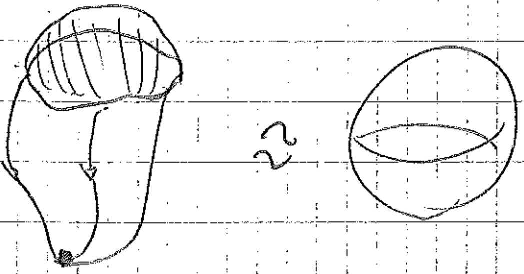


$M$

$f$ only two n.d. crit. points $\Rightarrow M \underset{\text{homeo}}{\cong} S^n$

$p_2 = \max \to 2 = i$

$p_0 = \min \to 0 = i$

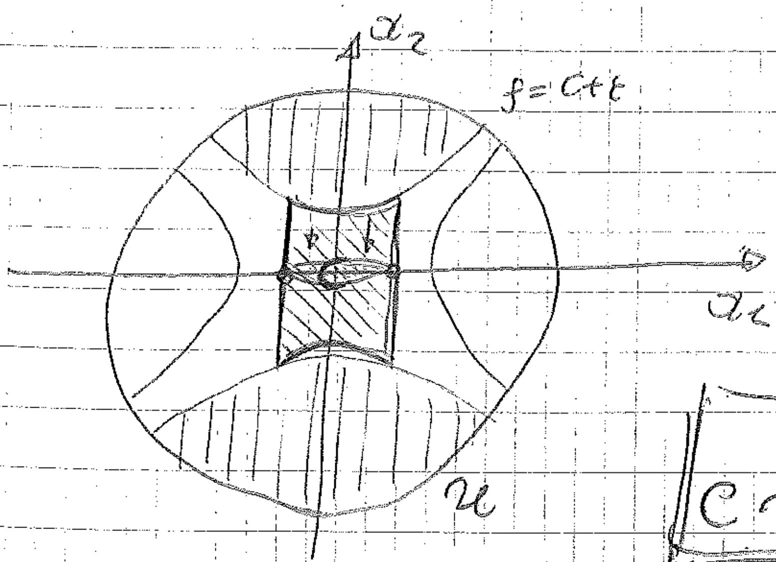


$S^n$

M is obtained by attaching a 2-cell on a 0-cell (or 2 2-cells along their common boundary)

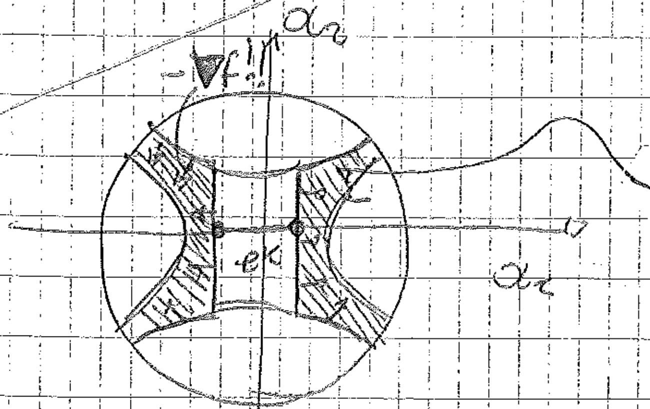
Milnor spheres : The spheres obtained in this way may not be diffeomorphic! (ex: on $S^7$ one gets 28 different differentiable structures)

Picture : $k = 1 \quad n = 2$



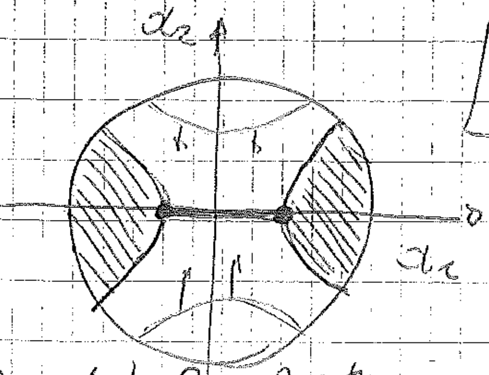

$\boxed{C \simeq e^k}$

$f = c+\varepsilon = c - x_1^2 + x_2^2$

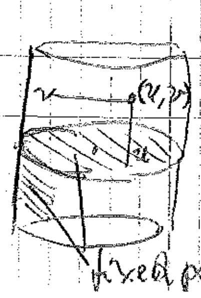



no critical points!

$\boxed{M_{c+\varepsilon} \simeq M_{c-\varepsilon} \cup e^1}$




If one wants to describe the retraction

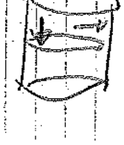



$u' = (1+\rho)u$ $\qquad \rho = \min\left(|v|, \frac{\sqrt{\varepsilon}-|u|}{|u|}\right)$

$v' = \left(1 - \frac{\rho}{|v|}\right)v$

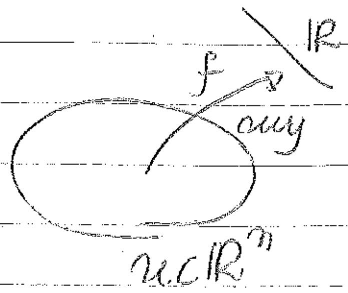
$\rho = \frac{\sqrt{\varepsilon}-|u|}{|u|} \longrightarrow$

$\rho = |v| \Rightarrow$



(10)

There exist Morse functions (sketch)




$a = (a_1 \ldots a_n)$

$f_a(x) = f(x) + a_1 x_1 + \ldots a_n x_n$

$J$ Jacobian

$g_{(a)}(x) = \left( \frac{\partial f_{(x)}}{\partial x_1} \ldots \frac{\partial f_{(x)}}{\partial x_n} \right)$

$H(f) = J(g)$

$x$ non deg for $f \Leftrightarrow g(x) = 0$ and $J(g)$ non sing

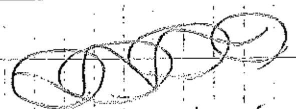
$g_a(x) = g(x) + a \qquad J(g_a) = J(g)$

$x$ critical for $f_a \Leftrightarrow g(x) = -a$

non deg for $f_a \Rightarrow J(g)(x) \neq 0$ i.e. $-a$ is

Regular $\Rightarrow$ SARD They exist.

On a manifold:



$M \hookrightarrow \mathbb{R}^r$ $\quad \mathcal{U}_i$ countable

non reg. value

$x_1 \ldots x_n$ local coord. Fix $(a_{n+1} \ldots a_r)$, define

$f(x) = a_{n+1} x_{n+1} + a_r x_r$

a.e.: $a$: $f(a) + a_1 x_1 \ldots a_n x_n$ morse on $\mathcal{U}_i$

$\Rightarrow$ a.e. $\quad f_a(x) = a x_1 + \ldots + a_r x_r$ is Morse

$A_i = \{ a : f_a(x)$ non Morse on $\mathcal{U}_i \}$

$\cup A_i$ has measure 0 (Sard)

$\Rightarrow \square$