GEOMETRIA

Formulario

★ CALCOLO VETTORIALE

- vettori geometrici $\underline{a} = (a_1, a_2\ a_3)$ coordinate cartesiane (ortogonali)

- prodotto scalare $\langle \underline{a}, \underline{b} \rangle = \sum_{i=1}^{3} a_i b_i$

  $\| \underline{a} \| := \langle \underline{a}, \underline{a} \rangle^{\frac{1}{2}}$ lunghezza (norma) di $\underline{a}$

- angolo tra $\underline{a}$ e $\underline{b}$ $\cos\alpha = \dfrac{\langle \underline{a}, \underline{b} \rangle}{\|\underline{a}\| \|\underline{b}\|}$ $\underline{a} \neq \underline{0}$, $\underline{b} \neq \underline{0}$

- ortogonalità: $\langle \underline{a}, \underline{b} \rangle = 0$

- prodotto vettoriale

$$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- $\| \underline{a} \times \underline{b} \| = \| \underline{a} \| \| \underline{b} \| \, |\sin\alpha| = \left(\| \underline{a} \|^2 \| \underline{b} \|^2 - \langle \underline{a}, \underline{b} \rangle^2 \right)^{\frac{1}{2}}$

$\underline{a} \parallel \underline{b}$ ($\underline{a}$, $\underline{b}$ l. d.) $\underline{a} \times \underline{b} = \underline{0}$
(parallelismo)

- prodotto misto $\det(\underline{a}, \underline{b}, \underline{c}) = \langle \underline{a}, \underline{b} \times \underline{c} \rangle$
  $\equiv | \underline{a}, \underline{b}, \underline{c} |$

- complanarità di $\underline{a}, \underline{b}, \underline{c}$ $\langle \underline{a}, \underline{b} \times \underline{c} \rangle = 0$
  ($\underline{a}$, $\underline{b}$, $\underline{c}$ l. d.)

★ CURVE $\underline{r} = \underline{r}(t)$ $t \in I$ (intervallo aperto...) $\dot{\underline{r}} = \dfrac{d\underline{r}}{dt}$ $\dot{\underline{r}} \neq \underline{0}$ curve regolari

- lunghezza d'arco $ds = \| \dot{\underline{r}} \| \, dt$

$' = \dfrac{d}{ds}$ $\| \underline{r}' \| = 1$

$$\underline{t} = \underline{r}' = \frac{\underline{\dot{r}}}{\|\underline{\dot{r}}\|} \qquad \text{versore \underline{tangente}}$$

$$\underline{n} = \frac{\underline{r}''}{\|\underline{r}''\|} \qquad \|\underline{r}''\| = \kappa \quad \text{curvatura}$$

- normale principale

$\kappa > 0$: biregolarità

$$\underline{b} = \underline{t} \times \underline{n}$$

- versore binormale

* Formule di Frénet:

$$\begin{cases} \underline{t}' = \kappa \, \underline{n} \\ \underline{n}' = -\kappa \, \underline{t} - \tau \, \underline{b} \\ \underline{b}' = \tau \, \underline{n} \end{cases} \qquad \tau : \text{torsione}$$

- Formule generali per κ e τ

$$\kappa = \frac{\|\underline{\dot{r}} \times \underline{\ddot{r}}\|}{\|\underline{\dot{r}}\|^3} = \frac{\left[\|\underline{\dot{r}}\|^2 \|\underline{\ddot{r}}\|^2 - \langle \underline{\dot{r}}, \underline{\ddot{r}} \rangle^2\right]^{\frac{1}{2}}}{\|\underline{\dot{r}}\|^3}$$

$$\rho = \frac{1}{\kappa} \qquad \text{• raggio di curvatura}$$

curva piana: $\kappa = \dfrac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$

con segno

$$\tau = \langle \underline{b}', \underline{n} \rangle = - \frac{|\underline{r}' \; \underline{r}'', \underline{r}'''|}{\underbrace{\|\underline{r}''\|^2}_{\kappa^2}} = - \frac{|\underline{\dot{r}} \; \underline{\ddot{r}} \; \underline{\dddot{r}}|}{\left(\|\underline{\dot{r}}\|^2 \|\underline{\ddot{r}}\|^2 - \langle \underline{\dot{r}}, \underline{\ddot{r}} \rangle^2\right)}$$

$$= - \frac{\langle \underline{\dot{r}} \times \underline{\ddot{r}}, \underline{\dddot{r}} \rangle}{\|\underline{\dot{r}} \times \underline{\ddot{r}}\|^2}$$

- Sfera osculatrice

$$\underset{\text{centro}}{C} = P + \rho \, \underline{n} + \frac{\rho'}{\tau} \, \underline{b} \qquad \tau \neq 0$$

$$\underset{\text{raggio}}{R} = \sqrt{\rho^2 + \left(\frac{\rho'}{\tau}\right)^2}$$

★ SUPERFICIE

$$\underline{r} = \underline{r}(u,v) \quad (u,v) \in \mathcal{U} \text{ regione}$$

$$\underline{N} = \frac{\underline{r}_u \times \underline{r}_v}{\|\underline{r}_u \times \underline{r}_v\|}$$

$\underline{r}_u \times \underline{r}_v \neq \underline{0}$ (regolarità)

• forme fondamentali

$$\mathrm{I} \begin{cases} E = \langle \underline{r}_u, \underline{r}_u \rangle \\ F = \langle \underline{r}_u, \underline{r}_v \rangle \\ G = \langle \underline{r}_v, \underline{r}_v \rangle \end{cases} \qquad \mathrm{II} \begin{cases} e = \langle \underline{r}_{uu}, \underline{N} \rangle \\ f = \langle \underline{r}_{uv}, \underline{N} \rangle \\ g = \langle \underline{r}_{vv}, \underline{N} \rangle \end{cases}$$

$$e = \frac{\langle \underline{r}_{uu}, \underline{r}_u \times \underline{r}_v \rangle}{\|\underline{r}_u \times \underline{r}_v\|} = \frac{\langle \underline{r}_{uu}, \underline{r}_u \times \underline{r}_v \rangle}{\sqrt{EG - F^2}} \quad \text{ecc.}$$

$$\frac{\langle \underline{w}\, S\, \underline{w} \rangle}{\|\underline{w}\|^2} = \frac{\mathrm{II}(\underline{w})}{\|\underline{w}\|^2} = k_n = \frac{e\dot{u}^2 + 2f\dot{u}\dot{v} + g\dot{v}^2}{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}$$

$\underline{w} \in T_p\Sigma$, $\neq \underline{0}$

curvatura normale nella direzione determinata da $\underline{w}$

$$\underline{w} = \underline{r}_u \dot{u} + \underline{r}_v \dot{v}$$

$$K = \frac{eg - f^2}{EG - F^2} \quad \text{• curvatura gaussiana} \quad = k_1 k_2$$

$$H = \frac{1}{2}\,\frac{eG - 2Ff + Eg}{EG - F^2} = \frac{1}{2}(k_1 + k_2) \quad \text{• curv. principali}$$

• curvatura media

$$k_n = k_1 \cos^2\vartheta + k_2 \sin^2\vartheta$$

Eulero

• curvature principali : $\quad k_i^2 - 2H k_i + K = 0$

• linee di curvatura (direzioni principali)

$$\begin{vmatrix} \dot{v}^2 & -\dot{u}\dot{v} & \dot{u}^2 \\ E & F & G \\ e & f & g \end{vmatrix} = 0$$

• linee asintotiche (direzioni asintotiche)

$$e\dot{u}^2 + 2f\dot{u}\dot{v} + g\dot{v}^2 = 0$$

★ Formule di Weingarten

$$\mathcal{M}_{\mathcal{B}\mathcal{B}}(S) = \begin{pmatrix} \frac{eG - fF}{EG - F^2} & \frac{fG - gF}{EG - F^2} \\ \frac{-eF + fE}{EG - F^2} & \frac{gE - fF}{EG - F^2} \end{pmatrix}$$

$\mathcal{B} = \{\underline{r}_u, \underline{r}_v\}$, $S = -dN$

★ operatore di forma

$$F = 0 \;\leadsto\; \begin{pmatrix} \frac{e}{E} & +\frac{f}{E} \\ +\frac{f}{G} & \frac{g}{G} \end{pmatrix}$$

- Simboli di Christoffel $\qquad ds^2 = g_{ij}\, dx^i dx^j$

$i, j, k = 1, 2$

$$\Gamma^i_{jk} = \frac{1}{2} g^{ih} \left(\frac{\partial g_{kh}}{\partial x^j} + \frac{\partial g_{jh}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^h} \right)$$

elementi della matrice inversa di (g_{ij})

+ convenzione di Einstein
[qui si somma su $h = 1, 2$]

- equazione del trasporto parallelo

$$\dot{w}^i + \Gamma^i_{kh}\, \dot{x}^k w^h = 0 \qquad i = 1, 2$$

- equazione delle geodetiche

$$\ddot{x}^i + \Gamma^i_{kh}\, \dot{x}^k \dot{x}^h = 0 \qquad i = 1, 2$$

- Equazioni di Lagrange

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

nel nostro caso

$$\mathcal{L} = \frac{1}{2} g_{ij}\, \dot{x}^i \dot{x}^j$$

$\Longrightarrow$ eq. geodetica

se $\quad \dot{} = {}' = \dfrac{d}{ds}$ $\qquad g_{ij}\, x'^i x'^j = 1$ $\qquad$ "conservazione dell'energia"

se $\quad \dfrac{\partial \mathcal{L}}{\partial q} = 0$, è $\quad \dfrac{\partial \mathcal{L}}{\partial \dot{q}} = \text{cost}$ $\qquad$ (integrale primo)

- Formula intrinseca per la curvatura

se $F = 0 \qquad K = -\dfrac{1}{2\sqrt{EG}} \left[\left(\dfrac{E_v}{\sqrt{EG}} \right)_v + \left(\dfrac{G_u}{\sqrt{EG}} \right)_u \right]$