Predicate Logic
(first order logic)

formula	intuitive meanings
$\exists x P(x)$	there is an x with property P
$\forall y P(y)$	?
$\forall x \exists y (x = 2y)$	?
$\forall \varepsilon (\varepsilon > 0 \rightarrow \exists n (1 < \varepsilon))$	?
$x < y \rightarrow \exists z (x < z \wedge z < y)$	?
$\forall x \exists y (x.y = 1)$	?

formula	intuitive meanings
$\exists x P(x)$	there is an x with property P
$\forall y P(y)$	for all y P holds (all y have the property P)
$\forall x \exists y (x = 2y)$	for all x there is a y such that x is two times y
$\forall \varepsilon (\varepsilon > 0 \rightarrow \exists n (1 < \varepsilon))$	for all positive ε there is an n such that 1< ε
$x < y \rightarrow \exists z (x < z \wedge z < y)$	if x < y, then there is a z such that x < z and z < y
$\forall x \exists y (x.y = 1)$	for each x there exists an inverse y

The semantics of predicate logics

Structure

$$\langle A\ ,\ R_1,\ldots,R_n\ ,\ F_1,\ldots,F_m\ ,\ \{c_i | i \in I\}\rangle$$

Structure

$$\mathfrak{A} = \langle A, \boxed{R_1,\ldots,R_n}, \boxed{F_1,\ldots,F_m}, \{c_i | i \in I\} \rangle$$

A non-empty set

relations on A

functions on A

elements of A

notation $|\mathfrak{A}| = A$

$\langle R, +, \cdot, ^{-1}, 0, 1 \rangle$ – the field of real numbers,

$\langle N, < \rangle$ – the ordered set of natural numbers.

Definition 2.2.2 *The similarity type of a structure* $\mathfrak{A} = \langle A, R_1, \ldots, R_n, F_1, \ldots, F_m, \{c_i | i \in I\}\rangle$ *is a sequence,* $\langle r_1, \ldots, r_n; a_1, \ldots, a_m; \kappa\rangle$*, where* $R_i \subseteq A^{r_i}$, $F_j : A^{a_j} \to A$, $\kappa = |\{c_i \ | i \in I\}|$ *(cardinality of* I*).*

what is A^0 ?

what is $f: A^0 \to A$?

what is A^0 ?

what is $f: A^0 \to A$?

what is A^n ?

what is $f: \varnothing \to A$?

Write down the similarity type for the following structures:

(i) $\langle \mathbb{Q}, <, 0 \rangle$

(ii) $\langle \mathbb{N}, +, \cdot, S, 0, 1, 2, 3, 4, \ldots, n, \ldots \rangle$,where$S(x) = x + 1$,

(iii) $\langle \mathcal{P}(\mathbb{N}), \subseteq, \cup, \cap, ^c, \emptyset \rangle$,

(iv) $\langle \mathbb{Z}/(5), +, \cdot, -, ^{-1}, 0, 1, 2, 3, 4 \rangle$,

(v) $\langle \{0, 1\}, \wedge, \vee, \rightarrow, \neg, 0, 1 \rangle$, where $\wedge, \vee, \rightarrow, \neg$ operate according to the ordinary truth tables,

(vi) $\langle \mathbb{R}, 1 \rangle$,

(vii) $\langle \mathbb{R} \rangle$,

Give structures with type $\langle 1, 1; -; 3 \rangle$, $\langle 4; -; 0 \rangle$.

alphabet

$\langle r_1,\ldots,r_n ; a_1,\ldots,a_m;\kappa\rangle$, with $r_i \geq 0, a_j > 0$.

1. Predicate symbols: sequence $P_1, \ldots, P_n$, plus $=$.
2. Function symbols: sequence $f_1,\ldots,f_m$
3. Constant symbols $\mathbf{c}_i$ for $i \in I$ with $|I|=\kappa$
4. Variables: $x_0,x_1,x_2,\ldots$(countably many)
5. Connectives: $\vee,\wedge,\rightarrow,\neg,\leftrightarrow,\bot\ \forall,\exists$
6. auxiliary symbols: (,),

we write also $\langle <P_1, \ldots, P_n; f_1,\ldots,f_m,\{\mathbf{c}_i\}_{i\in I}\rangle$ to relate with

$$\langle r_1,\ldots,r_n ;a_1,\ldots,a_m;\kappa\rangle$$

$\langle r_1,\ldots,r_n ; a_1,\ldots,a_m;\kappa\rangle$, with $r_i \geq 0, a_j > 0$.

Definition *TERM is the smallest set X with the properties*

(i) $\overline{c}_i \in X\,(i \in I)$ *and* $x_i \in X(i \in N)$,

(ii) $t_1,\ldots,t_{a_i} \in X \Rightarrow f_i(t_1,\ldots,t_{a_i}) \in X$, *for* $1 \leq i \leq m$

TERM is our *set of terms.*

Definition *FORM is the smallest set X with the properties:*

(i) $\perp \in X; P_i \in X$ *if* $r_i = 0$; $t_1,\ldots,t_{r_i} \in \mathit{TERM} \Rightarrow$ $P_i(t_1,\ldots,t_{r_i}) \in X; t_1, t_2 \in \mathit{TERM} \Rightarrow t_1 = t_2 \in X$,

(ii) $\varphi, \psi \in X \Rightarrow (\varphi\Box\psi) \in X$, *where* $\Box \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$,

(iii) $\varphi \in X \Rightarrow (\neg\varphi) \in X$,

(iv) $\varphi \in X \Rightarrow ((\forall x_i)\varphi), ((\exists x_i)\varphi) \in X$.

proof by induction

Lemma *Let $A(t)$ be a property of terms. If $A(t)$ holds for t a variable or a constant, and if $A(t_1), A(t_2), \ldots, A(t_n) \Rightarrow A(f(t_1, \ldots, t_n))$, for all function symbols f, then $A(t)$ holds for all $t \in TERM$.*

Lemma *Let $A(\varphi)$ be a property of formulas. If*

(i) $A(\varphi)$ *for atomic* φ,
(ii) $A(\varphi), A(\psi) \Rightarrow A(\varphi \Box \psi)$,
(iii) $A(\varphi) \Rightarrow A(\neg\varphi)$,
(iv) $A(\varphi) \Rightarrow A((\forall x_i)\varphi), A((\exists x_i)\varphi)$ *for all* i, *then* $A(\varphi)$ *holds for all* $\varphi \in FORM$.

Example of a language of type $\langle 2;2,1;1 \rangle$.

predicate symbols: L, =
function symbols: p, i
constant symbol: e

Definition by Recursion on TERM: Let $H_0 : Var \cup Const \to A$ (i.e.H_0 is defined on variables and constants), $H_i : A^{a_i} \to A$, then there is a unique mapping $H : TERM \to A$ such that

$$\begin{cases} H(t) = H_0(t) \text{ for } t \text{ a variable or a constant,} \\ H(f_i(t_1, \ldots, t_{a_i})) = H_i(H(t_1), \ldots, H(t_{a_i})). \end{cases}$$

Definition by Recursion on FORM:

$$\begin{array}{ll} \text{Let } H_{at} : At \to A & (\text{i.e.} H_{at} \text{ is defined on atoms}), \\ \quad H_\Box : A^2 \to A, & (\Box \in \{\vee, \wedge, \to, \leftrightarrow\}) \\ \quad H_\neg : A \to A, & \\ \quad H_\forall : A \times N \to A, & \\ \quad H_\exists : A \times N \to A. & \end{array}$$

then there is a unique mapping $H : FORM \to A$ such that

$$\begin{cases} H(\varphi) & = H_{at}(\varphi) \text{ for atomice } \varphi, \\ H(\varphi \Box \psi) & = H_\Box(H(\varphi), H(\psi)), \\ H(\neg\varphi) & = H_\neg(H(\varphi)), \\ H(\forall x_i \varphi) & = H_\forall(H(\varphi), i), \\ H(\exists x_i (\varphi) & = H_\exists(H(\varphi), i). \end{cases}$$

free variables

Definition *The set $FV(t)$ of free variables of t is defined by*

$$\begin{array}{lll} (i) & FV(x_i) & := \{x_i\}, \\ & FV(\overline{c}_i) & := \emptyset \\ (ii) & FV(f(t_1, \ldots, t_n)) & := FV(t_1) \cup \ldots \cup FV(t_n). \end{array}$$

Definition *The set $FV(\varphi)$ of free variables of φ is defined by*

$$\begin{array}{lll} (i) & FV(P(t_1, \ldots, t_p)) & := FV(t_1) \cup \ldots \cup FV(t_p), \\ & FV(t_1 = t_2) & := FV(t_1) \cup FV(t_2), \\ & FV(\perp) = FV(P) & := \emptyset \textit{ for } P \textit{ a proposition symbol}, \\ (ii) & FV(\varphi \Box \psi) & := FV(\varphi) \cup FV(\psi), \\ & FV(\neg\varphi) & := FV(\varphi), \\ (iii) & FV(\forall x_i \varphi) := FV(\exists x_i \varphi) & := FV(\varphi) - \{x_i\}. \end{array}$$

- t or φ is called closed if $FV(t) = \varnothing$, resp. $FV(\varphi) = \varnothing$.
- a closed formula is also called a sentence.
- a formula without quantifiers is called open.
- $TERM_c$ denotes the set of closed terms;
- SENT denotes the set of sentences.

Exercise:

- define the set $BV(\varphi)$ of bound variables of φ
- $FV(\varphi) \cap BV(\varphi) = \varnothing$?

The notion of SUBFORMULa

- $Sub(\varphi) = \{\varphi\}$ for atomic φ
- $Sub(\varphi_1 \square \varphi_2) = Sub(\varphi_1) \cup Sub(\varphi_2) \cup \{\varphi_1 \square \varphi_2\}$ for $\square \in \{\wedge, \vee, \rightarrow\}$
- $Sub^{+}(\neg\varphi) = Sub(\varphi) \cup \{\neg\varphi\}$
- $Sub^{+}(Qx.\varphi) = Sub(\varphi) \cup \{Qx.\varphi\}$ for $Q \in \{\forall, \exists\}$

Free and Bound occurrences of variables

- an occurrence of a variable x in δ is BOUND, if x occurs in $\varphi \in SUB\{\delta\}$ and $\varphi \equiv \{Qx.\theta\}$ for $Q \in \{\forall, \exists\}$
- an occurrence of a variable x in δ is FREE, if x does not occur in any $\varphi \in SUB\{\delta\}$ with $\varphi \equiv \{Qx.\theta\}$ for $Q \in \{\forall, \exists\}$

SUBSTITUTION

$\varphi[t/x]$?

$\phi = (\ldots\ldots\ x\ \ldots\ldots\ldots) \Rightarrow \phi[t/x] = (\ldots\ldots\ t\ \ldots\ldots\ldots)$

free

$\phi = (\ldots\ldots\ x\ \ldots\ldots\ldots) \Rightarrow \phi[t/x] = (\ldots\ldots\ x\ \ldots\ldots\ldots)$

bound

SUBSTITUTION

Definition *Let s and t be terms, then $s[t/x]$ is defined by:*

$$
\begin{aligned}
(i)\quad & y[t/x] && := \begin{cases} y \text{ if } y \not\equiv x \\ t \text{ if } y \equiv x \end{cases} \\
& c[t/x] && := c \\
(ii)\quad & f(t_1, \ldots, t_p)[t/x] && := f(t_1[t/x], \ldots, t_p[t/x]).
\end{aligned}
$$

Definition *$\varphi[t/x]$ is defined by:*

$$
\begin{aligned}
(i)\quad & \bot\ [t/x] && := \bot, \\
& P[t/x] && := P \text{ for propositions } P, \\
& P(t_1, \ldots, t_p)[t/x] && := P(t_1[t/x], \ldots, t_p[t/x]), \\
& (t_1 = t_2)[t/x] && := t_1[t/x] = t_2[t/x], \\
(ii)\quad & (\varphi \Box \psi)[t/x] && := \varphi[t/x] \Box \psi[t/x], \\
& (\neg\varphi)[t/x] && := \neg\varphi[t/x] \\
(iii)\quad & (\forall y \varphi)[t/x] && := \begin{cases} \forall y \varphi[t/x] \text{ if } x \not\equiv y \\ \forall y \varphi \text{ if } x \equiv y \end{cases} \\
& (\exists y \varphi)[t/x] && := \begin{cases} \exists y \varphi[t/x] \text{ if } x \not\equiv y \\ \exists y \varphi \text{ if } x \equiv y \end{cases}
\end{aligned}
$$

Define symultaneus substitution $\delta[t_1,\ldots,t_n/x_1,\ldots,x_n]$

$$\exists x(y < x)[x/y] = \exists x\,(x < x)$$

?

$$\cancel{\exists x(y < x)[x/y] = \exists x\,(x < x)}$$

We must forbid dangerous substitutions

$t = (\ldots y \ldots)$ $\quad \phi = (\ldots(\exists y \ldots x \ldots)\ldots)$

$$\phi[t/x] = (\ldots(\exists y \ldots(\ldots y \ldots)\ldots)\ldots)$$

y is now bound!

Definition

t is free for x in ϕ if

(i) ϕ is atomic,

(ii) $\phi := \phi_1 \Box \phi_2$ (or $\phi := \neg\phi_1$) and t is free for x in ϕ_1 and ϕ_2 (resp. ϕ_1),

(iii) $\phi := \exists y \psi$ (or $\phi := \forall y \psi$) and if $x \in FV(\phi)$, then $y \notin FV(t)$ and t is free for x in ψ.

$t = (\ldots y \ldots)$ $\quad$ $\phi = (\ldots(\exists y \ldots x \ldots)\ldots)$

$\phi[t/x] = (\ldots(\exists y \ldots(\ldots y \ldots)\ldots)\ldots)$

t is NOT free for x in ϕ

proposition

t is free for x in φ $\Leftrightarrow$ the variables of t in $\varphi[t/x]$ are not bound by a quantifier.

proof by induction (exercise!)

SUBSTITUTION

Definition *Let s and t be terms, then $s[t/x]$ is defined by:*

$$
\begin{array}{lll}
(i) & y[t/x] & := \begin{cases} y \text{ if } y \not\equiv x \\ t \text{ if } y \equiv x \end{cases} \\
 & c[t/x] & := c \\
(ii) & f(t_1,\ldots,t_p)[t/x] & := f(t_1[t/x],\ldots,t_p[t/x]).
\end{array}
$$

let t be free for x in ϕ $\varphi[t/x]$ *is defined by:*

$$
\begin{array}{lll}
(i) & \perp[t/x] & := \perp, \\
 & P[t/x] & := P \text{ for propositions } P, \\
 & P(t_1,\ldots,t_p)[t/x] & := P(t_1[t/x],\ldots,t_p[t/x]), \\
 & (t_1 = t_2)[t/x] & := t_1[t/x] = t_2[t/x], \\
(ii) & (\varphi \Box \psi)[t/x] & := \varphi[t/x] \Box \psi[t/x], \\
 & (\neg\varphi)[t/x] & := \neg\varphi[t/x] \\
(iii) & (\forall y \varphi)[t/x] & := \begin{cases} \forall y \varphi[t/x] \text{ if } x \not\equiv y \\ \forall y \varphi \text{ if } x \equiv y \end{cases} \\
 & (\exists y \varphi)[t/x] & := \begin{cases} \exists y \varphi[t/x] \text{ if } x \not\equiv y \\ \exists y \varphi \text{ if } x \equiv y \end{cases}
\end{array}
$$

Define symultaneus substitution $\delta[t_1,\ldots,t_n/x_1,\ldots,x_n]$

Check which terms are free in the following cases, and carry out the substitution:

(a) x for x in $x = x$,
(b) y for x in $x = x$,
(c) $x + y$ for y in $z = \overline{0}$,
(d) $\overline{0} + y$ for y in $\exists x(y = x)$,
(e) $x + y$ for z in $\exists w(w + x = \overline{0})$,
(f) $x + w$ for z in $\forall w(x + z = \overline{0})$,
(g) $x + y$ for z in $\forall w(x + z = \overline{0}) \wedge \exists y(z = x)$,
(h) $x + y$ for z in $\forall u(u = v) \rightarrow \forall z(z = y)$.

NATURAL DEDUCTION

Notation

in the same context **φ(x)** and **φ(t)** denote respectively **φ** and **φ[t/x]**

hp$\mathcal{D}$

$$\forall I \; \dfrac{\begin{matrix}\mathcal{D}\\ \varphi(x)\end{matrix}}{\forall x \varphi(x)}$$

x∉FV(hp$\mathcal{D}$)

$$\forall E \; \dfrac{\begin{matrix}\mathcal{D}\\ \forall x \varphi(x)\end{matrix}}{\varphi(t)}$$

t free for x in φ

$$\dfrac{\dfrac{\dfrac{\dfrac{[x = 0]}{\forall x(x = 0)}}{x = 0 \to \forall x(x = 0)}}{\forall x(x = 0 \to \forall x(x = 0))}}{0 = 0 \to \forall x(x = 0)}$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{[x = 0]}{\forall x(x = 0)}\ \textbf{NO!}}{x = 0 \to \forall x(x = 0)}}{\forall x(x = 0 \to \forall x(x = 0))}}{0 = 0 \to \forall x(x = 0)}
$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[x = 0]}{\forall x(x = 0)}}{x = 0 \rightarrow \forall x(x = 0)}}{\forall x(x = 0 \rightarrow \forall x(x = 0))}}{0 = 0 \rightarrow \forall x(x = 0)}$$

$$\dfrac{\dfrac{[\forall x \neg \forall y (x = y)]}{\neg \forall y (y = y)}}{\forall x \neg \forall y (x = y) \to \neg \forall y (y = y)}$$

$$\forall I \;\dfrac{\begin{matrix}\mathcal{D}\\ \varphi(x)\end{matrix}}{\forall x \varphi(x)}$$

$x \notin FV(hp\mathcal{D})$

$$\forall E \;\dfrac{\begin{matrix}\mathcal{D}\\ \forall x \varphi(x)\end{matrix}}{\varphi(t)}$$

t free for x in φ

$$\cfrac{\cfrac{[\forall x \neg \forall y (x = y)]}{\neg \forall y (y = y)} \textbf{ NO!}}{\forall x \neg \forall y (x = y) \to \neg \forall y (y = y)}$$

$$\cancel{\dfrac{\dfrac{[\forall x \neg \forall y (x = y)]}{\neg \forall y (y = y)}}{\forall x \neg \forall y (x = y) \rightarrow \neg \forall y (y = y)}}$$

$$\forall x \forall y \varphi(x, y) \rightarrow \forall y \forall x \varphi(x, y)$$

$$\frac{[\forall x \forall y \varphi(x,y)]}{\forall y \varphi(x,y)}\,\forall E$$

$$\dfrac{\dfrac{[\forall x \forall y \varphi(x,y)]}{\forall y \varphi(x,y)}\,\forall E}{\varphi(x,y)}\,\forall E$$

$$
\dfrac{\dfrac{\dfrac{[\forall x \forall y \varphi(x,y)]}{\forall y \varphi(x,y)}\ \forall E}{\varphi(x,y)}\ \forall E}{\forall x \varphi(x,y)}\ \forall I
$$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\forall x \forall y \varphi(x, y)]}{\forall y \varphi(x, y)}\ \forall E}{\varphi(x, y)}\ \forall E}{\forall x \varphi(x, y)}\ \forall I}{\forall y \forall x (\varphi(x, y)}\ \forall I$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x\forall y\varphi(x,y)]}{\forall y\varphi(x,y)}\;\forall E}{\varphi(x,y)}\;\forall E}{\forall x\varphi(x,y)}\;\forall I}{\forall y\forall x(\varphi(x,y)}\;\forall I}{\forall x\forall y\varphi(x,y)\to\forall y\forall x\varphi(x,y)}\;\to I
$$

$$\overline{\forall x(\varphi \wedge \psi) \to \forall x\varphi \wedge \forall x\psi}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x(\varphi(x) \land \psi(x))]}{\varphi(x) \land \psi(x)}}{\varphi(x)}}{\forall x \varphi(x)} \qquad \dfrac{\dfrac{\dfrac{[\forall x(\varphi(x) \land \psi(x))]}{\varphi(x) \land \psi(x)}}{\psi(x)}}{\forall x \psi(x)}}{\forall x \varphi(x) \land \forall x \psi(x)}}{\forall x(\varphi \land \psi) \to \forall x \varphi \land \forall x \psi}$$

Let $x \notin FV(\varphi)$

$$\forall x(\varphi \longrightarrow \psi(x)) \longrightarrow (\varphi \longrightarrow \forall x(\psi(x))$$

Let $x \notin FV(\varphi)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x(\varphi \to \psi(x))]}{\varphi \to \psi(x)}\,\forall E \qquad [\varphi]}{\psi(x)}\,\to E}{\forall x \psi(x)}\,\forall I}{\varphi \to \forall x \psi(x)}\,\to I}{\forall x(\varphi \to \psi(x)) \to (\varphi \to \forall x(\psi(x))}$$

$$\mathit{\Gamma} \vdash \varphi(x) \Rightarrow \mathit{\Gamma} \vdash \forall x \varphi(x) \text{ if } x \notin FV(\psi) \text{ for all } \psi \in \mathit{\Gamma}$$
$$\mathit{\Gamma} \vdash \forall x \varphi(x) \Rightarrow \mathit{\Gamma} \vdash \varphi(t) \text{ if } t \text{ is free for } x \text{ in } \varphi.$$

1. Show: (i)$\vdash \forall x(\varphi(x) \to \psi(x)) \to (\forall x\varphi(x) \to \forall x\psi(x))$,

(ii)$\vdash \forall x\varphi(x) \to \neg\forall x\neg\varphi(x)$,

(iii)$\vdash \forall x\varphi(x) \to \forall z\varphi(z)$ if z does not occur in $\varphi(x)$,

SEMANTICS

$$\langle r_1, \ldots, r_n; a_1, \ldots, a_m; \kappa \rangle$$

$$\langle \mathbf{P}_1, \ldots, \mathbf{P}_n; \mathbf{f}_1, \ldots, \mathbf{f}_m, \{\mathbf{c}_i\}_{i<k} \rangle$$

$$\mathfrak{A} = \langle A, P_1, \ldots, P_n; f_1, \ldots, f_m, \{\mathbf{c}_i\}_{i<k} \rangle$$

Bijective mapping I
between the alphabet and the elements
of a mathematical structure

$$I(\mathbf{P}_i) = P_i$$

$$I(\mathbf{f}_i) = f_i$$

$$I(\mathbf{c}_i) = c_i$$

Given $\mathfrak{U}$. and a corresponding assignment I

an environment, is a function

ρ: VAR $\rightarrow$ A

ENV={ρ | ρ: VAR $\rightarrow$ A}

an interpretation of the terms of L in $\mathfrak{U}$, is a mapping

$[\![-]\!]$: TERM x ENV$\rightarrow$ $|\mathfrak{U}|$ satisfying:

(i)$[\![\mathbf{c}]\!]_\rho = I(c)$

(ii)$[\![\mathbf{x}]\!]_\rho = \rho(x)$,

(iii)$[\![\mathbf{f}(t_1,\ldots,t_k)]\!]_\rho = I(f)([\![t_1]\!]_\rho,\ldots,[\![t_k]\!]_\rho)$

The relation $\models$

1. $\mathfrak{U}, \rho \models P(t_1,\ldots,t_n) \Leftrightarrow (\llbracket t_1 \rrbracket_\rho,\ldots,\llbracket t_n \rrbracket_\rho)$

2. $\mathfrak{U}, \rho \models \varphi \wedge \psi \Leftrightarrow \mathfrak{U}, \rho \models \varphi$ and $\mathfrak{U}, \rho \models \psi$,

3. $\mathfrak{U}, \rho \models \varphi \vee \psi \Leftrightarrow \mathfrak{U}, \rho \models \varphi$ or $\mathfrak{U}, \rho \models \psi$

4. $\mathfrak{U}, \rho \models \neg\varphi \Leftrightarrow \mathfrak{U}, \rho \not\models \varphi$,

5. $\mathfrak{U}, \rho \models \varphi \rightarrow \psi \Leftrightarrow (\mathfrak{U}, \rho \models \varphi \Rightarrow \mathfrak{U}, \rho \models \psi)$,

6. $\mathfrak{U}, \rho \models \forall x \varphi \Leftrightarrow \mathfrak{U}, \rho[\mathbf{a} \mapsto x] \models \varphi$, for each $a \in |\mathfrak{U}|$.

7. $\mathfrak{U}, \rho \models \exists x \varphi \Leftrightarrow \mathfrak{U}, \rho[\mathbf{a} \mapsto x] \models \varphi$, for some $a \in |\mathfrak{U}|$.

where

$\rho[\mathbf{a} \mapsto x](y) = (\textbf{if } y = x \textbf{ then } a \textbf{ else } \rho(y))$

- $\mathfrak{A} \models \varphi$ ($\mathfrak{A}$ is a model of φ) $\Leftrightarrow$ **for each** ρ, $\mathfrak{A}, \rho \models \varphi$,
- $\models \varphi$ (φ is valid/true) $\Leftrightarrow$

  $\mathfrak{A} \models \varphi$ for all $\mathfrak{A}$ (of the appropriate type),
- $\mathfrak{A}, \rho \models \Gamma$ ($\mathfrak{A}$ satisfies Γ) $\Leftrightarrow$

  $\mathfrak{A}, \rho \models \psi$ for all $\psi \in \Gamma$,
- $\Gamma \models \varphi$ (φ is consequence of Γ) $\Leftrightarrow$

  for each ρ $\mathfrak{A}$ and **for each** $\rho(\mathfrak{A}, \rho \models \Gamma \Rightarrow \mathfrak{A}, \rho \models \varphi)$,

exercises

$$
\begin{array}{ll}
(i) & \models \neg\forall x\varphi \leftrightarrow \exists x\neg\varphi \\
(ii) & \models \neg\exists x\varphi \leftrightarrow \forall x\neg\varphi \\
(iii) & \models \forall x\varphi \leftrightarrow \neg\exists x\neg\varphi \\
(iv) & \models \exists x\varphi \leftrightarrow \neg\forall x\neg\varphi
\end{array}
$$

exercises

$$(i)\ \models \neg\forall x\varphi \leftrightarrow \exists x\neg\varphi$$
$$(ii)\ \models \neg\exists x\varphi \leftrightarrow \forall x\neg\varphi$$
$$(iii)\ \models \forall x\varphi \leftrightarrow \neg\exists x\neg\varphi$$
$$(iv)\ \models \exists x\varphi \leftrightarrow \neg\forall x\neg\varphi$$

$\models \alpha \vee \neg\alpha \iff \forall \mathfrak{A}\, \forall \rho\ \ \mathfrak{A}, \rho \models \alpha \vee \neg\alpha$

$\mathfrak{A}, \rho \models \alpha \vee \neg\alpha \iff \mathfrak{A}, \rho \models \alpha$ OR $\mathfrak{A}, \rho \models \neg\alpha$

$\iff \mathfrak{A}, \rho \models \alpha$ OR $\mathfrak{A}, \rho \not\models \alpha$

exercises

$$(i)\ \models \neg\forall x\varphi \leftrightarrow \exists x\neg\varphi$$
$$(ii)\ \models \neg\exists x\varphi \leftrightarrow \forall x\neg\varphi$$
$$(iii)\ \models \forall x\varphi \leftrightarrow \neg\exists x\neg\varphi$$
$$(iv)\ \models \exists x\varphi \leftrightarrow \neg\forall x\neg\varphi$$

i-1) $\models \neg\forall x\varphi \to \exists x\neg\varphi$

$\forall\mathfrak{A}.\forall\rho.(\mathfrak{A},\rho \models \neg\forall x\varphi \Rightarrow \mathfrak{A},\rho \models \exists x\neg\varphi)$

$\mathfrak{A},\rho \models \neg\forall x\varphi \Leftrightarrow \mathfrak{A},\rho \not\models \forall x\varphi \Leftrightarrow$

$\Leftrightarrow \exists a \in |\mathfrak{A}|\ \ \mathfrak{A},\rho[x\mapsto a] \not\models \varphi \Leftrightarrow$

$\Leftrightarrow \exists a \in |\mathfrak{A}|\ \ \mathfrak{A},\rho[x\mapsto a] \models \neg\varphi \Leftrightarrow$

$\Leftrightarrow\ \mathfrak{A},\rho \models \exists x\neg\varphi$

$$\models (\forall x\, \alpha \land \forall x\, \beta) \to \forall x\, (\alpha \land \beta)$$

$$\forall \mathfrak{A}, \forall \rho \left(\mathfrak{A}, \underbrace{\rho \models \forall x\, \alpha \land \forall x\, \beta}_{} \Rightarrow \mathfrak{A}, \rho \models \forall x\, (\alpha \land \beta) \right)$$

$$\mathfrak{A}, \rho \models \forall x\, \alpha \land \forall x\, \beta \iff$$

$$\mathfrak{A}, \rho \models \forall x\, \alpha \;\;\&\;\; \mathfrak{A}, \rho \models \forall x\, \beta \iff$$

$$(\forall \mathfrak{a}\;\; \mathfrak{A}, \rho[x \mapsto \mathfrak{a}] \models \alpha) \;\&\; (\forall \mathfrak{a}\;\; \mathfrak{A}, \rho[x \mapsto \mathfrak{a}] \models \beta) \iff$$

$$\forall \mathfrak{a} \left[\mathfrak{A}, \rho[x \mapsto \mathfrak{a}] \models \alpha \;\;\&\;\; \mathfrak{A}, \rho[x \mapsto \mathfrak{a}] \models \beta \right] \iff$$

$$\forall \mathfrak{a} \left[\mathfrak{A}, \rho[x \mapsto \mathfrak{a}] \models \alpha \land \beta \right] \iff \mathfrak{A}, \rho \models \forall x\, (\alpha \land \beta)$$

$$\boxed{\models \forall x\, \alpha \vee \forall x\, \beta \rightarrow \forall x(\alpha \vee \beta) \quad \text{ESERCIZIO CASA}}$$

$$\not\models \forall x(\alpha \vee \beta) \rightarrow (\forall x\, \alpha \vee \forall x\, \beta) \equiv \gamma$$

SI ESIBISCE UN'ISTANZA $\bar{\gamma}$ DI γ

&

SI ESIBISCE UN CONTROMODELLO DI $\bar{\gamma}$

$$\mathfrak{A} \not\models \bar{\gamma}$$

$$\exists \mathfrak{A} \quad \mathfrak{A} \not\models \forall x(P(x) \vee Q(x)) \rightarrow (\forall x\, P(x) \vee \forall x\, Q(x))$$

$\exists \mathfrak{A} \quad \mathfrak{A} \not\models \forall x(P(x) \vee Q(x)) \to (\forall x\, P(x) \vee \forall x\, Q(x))$

$\langle 1, 1; -; - \rangle \quad \langle P, Q; -; - \rangle$ ALFAB.

$\mathfrak{A} = \langle \mathbb{N}, \mathcal{P}, \mathcal{Q} \rangle \qquad \mathcal{P} = \{ n \mid n$ è pari $\}$

$I(P) = \mathcal{P} \qquad I(Q) = \mathcal{Q} \qquad \mathcal{Q} = \{ n \mid n$ è dispari $\}$

$\mathfrak{A} \not\models \forall x(P(x) \vee Q(x)) \to (\forall x\, P(x) \vee \forall x\, Q(x)) \Leftrightarrow$

$\exists \rho \quad \mathfrak{A}, \rho \not\models \forall x(P(x) \vee Q(x)) \to (\forall x\, P(x) \vee \forall x\, Q(x)) \Leftrightarrow$

$\exists \rho \left[\mathfrak{A}, \rho \models \forall x (P(x) \vee Q(x)) \;\&\; \mathfrak{A}, \rho \not\models \forall x\, P(x) \vee \forall x\, Q(x) \right]$

1) $\mathfrak{A}, \rho \models \forall x (P(x) \vee Q(x))$

2) $\mathfrak{A}, \rho \not\models \forall x\, P(x) \vee \forall x\, Q(x)$

$$\boxed{\models \forall x\, \alpha \vee \forall x\, \beta \rightarrow \forall x(\alpha \vee \beta) \quad \text{ESERCIZIO CASA}}$$

$$\not\models \forall x(\alpha \vee \beta) \rightarrow (\forall x\, \alpha \vee \forall x\, \beta) \equiv \gamma$$

SI ESIBISCE UN'ISTANZA $\bar{\gamma}$ DI γ
&
SI ESIBISCE UN CONTROMODELLO DI $\bar{\gamma}$

$$\mathfrak{A} \not\models \bar{\gamma}$$

$$\exists \mathfrak{A} \quad \mathfrak{A} \not\models \forall x(P(x) \vee Q(x)) \rightarrow (\forall x\, P(x) \vee \forall x\, Q(x))$$

Change of Bound Variables

If x, y are free for z in φ and $x, y \notin FV(\varphi))$,

(or simply: if x and y does not occur in φ) then

$\models \exists x(\varphi[x/z]) \leftrightarrow \exists y(\varphi[y/z])$,

$\models \forall x(\varphi[x/z]) \leftrightarrow \forall y(\varphi[y/z])$.

Every formula is equivalent to one in which no variable occurs both free and bound.

IDENTITY

1. $\forall x(x = x)$,
2. $\forall xy(x=y \rightarrow y=x)$,
3. $\forall xyz(x=y \wedge y=z \rightarrow x=z)$,
4. $\forall x_1 \ldots x_n y_1 \ldots y_n(\bigwedge_{i=1,n} x_i = y_i \rightarrow t(x_1,\ldots,x_n) = t(y_1,\ldots,y_n))$
5. $\forall x_1 \ldots x_n y_1 \ldots y_n(\bigwedge_{i=1,n} x_i = y_i \rightarrow (\phi(x_1,\ldots,x_n) \rightarrow \phi(y_1,\ldots,y_n)))$

exercise:

$\models \forall x \exists y(x = y)$

Soundness

$$\Gamma \vdash \sigma \Rightarrow \Gamma \vDash \sigma$$

$\mathbf{hp}\mathcal{D} \subseteq \Gamma$ $x \notin FV(\mathbf{hp}\mathcal{D})$

$$\frac{\begin{array}{c}\mathcal{D}\\ \varphi(x)\end{array}}{\forall x \varphi(x)}$$

by Induction hypothesis

$\Gamma \vDash \varphi$ i.e.

$\forall\, \mathfrak{U}, \forall \rho, (\mathfrak{U}, \rho \vDash \mathbf{hp}\mathcal{D} \Rightarrow \mathfrak{U}, \rho \vDash \phi\ ) \Rightarrow$

$\forall\, \mathfrak{U}, \forall \rho \forall, a\ (\mathfrak{U}, \rho[x \mapsto a] \vDash \mathbf{hp}\mathcal{D} \Rightarrow \mathfrak{U}, \rho[x \mapsto a] \vDash \varphi\ ) \Rightarrow$

$\Rightarrow \forall\, \mathfrak{U}, \forall \rho, (\ (\forall a\ \mathfrak{U}, \rho[x \mapsto a] \vDash \mathbf{hp}\mathcal{D}) \Rightarrow (\forall a\ \mathfrak{U}, \rho[x \mapsto a] \vDash \varphi)\ \ )\ \Rightarrow$

(because $\mathfrak{U}, \rho[x \mapsto a] \vDash \mathrm{hp}\mathcal{D} \Leftrightarrow \mathfrak{U}, \rho \vDash \mathrm{hp}\mathcal{D}$)

$\forall\, \mathfrak{U}, \forall \rho, (\mathfrak{U}, \rho \vDash \mathbf{hp}\mathcal{D} \Rightarrow \mathfrak{U}\ , \rho \vDash \forall x.\varphi) \Rightarrow$

$\Gamma \vDash \forall x.\varphi$

$(\forall E)$ $\quad \mathcal{D}$

$$\frac{\forall x \varphi(x)}{\varphi(t)}$$

$[\![s[t/x]]\!]_\rho = [\![s]\!]_{\rho[x\mapsto[\![t]\!]_\rho]}$

$\mathfrak{U},\rho \models \phi[t/x]$ iff $\quad \mathfrak{U},\rho[x\mapsto [\![t]\!]_\rho] \models \phi$

t free for x in ϕ

by IH: $\Gamma \models \forall x.\varphi$

i.e. $\forall \mathfrak{U}\ \forall\ \rho,.\ \mathfrak{U},\rho \models \Gamma \Rightarrow \mathfrak{U},\rho \models \forall x.\varphi$

$\mathfrak{U},\rho \models \forall x.\varphi \Rightarrow \forall a\ \mathfrak{U},\rho[x\mapsto a]\models\varphi \Rightarrow$

$\forall t\ \mathfrak{U},\rho[x\mapsto[\![t]\!]]\models\varphi \Leftrightarrow \forall t\ \mathfrak{U},\rho\models\varphi[t/x]$

and therefore $\forall t(\mathfrak{U},\rho \models \forall x.\varphi(\rho) \Rightarrow \mathfrak{U},\rho\models\varphi[t/x])$

Adding the Existential Quantifier

$$\frac{\varphi(t)}{\exists x\ \varphi(x)}\ \exists I$$

t free for x in φ

$$\frac{\exists x \varphi(x) \qquad \begin{array}{c}[\varphi] \\ \mathcal{D} \\ \psi\end{array}}{\psi}\ \exists E$$

x∉FV(hp$\mathcal{D}$-{φ})∪FV(ψ)

$$
\dfrac{\dfrac{[\exists x\varphi(x)]^2 \qquad \dfrac{\dfrac{[\forall x(\varphi(x)\to\psi)]^3}{\varphi(x)\to\psi}\,\forall E \qquad [\varphi(x)]^1}{\psi}\,\to E}{\dfrac{\psi}{\exists x\varphi(x)\to\psi}\,\to I_2}\,\exists E_1}{\forall x(\varphi(x)\to\psi)\to(\exists x\varphi(x)\to\psi)}\,\to I_3
$$

$$\exists x(\varphi(x) \vee \psi(x)) \to \exists x \varphi(x) \vee \exists x \psi(x)$$

$$\dfrac{\dfrac{[\exists x(\varphi(x) \vee \psi(x))]^3 \qquad \dfrac{[\varphi(x) \vee \psi(x)]^2 \qquad \dfrac{\dfrac{[\varphi(x)]^1}{\exists x \varphi(x)}}{\exists x \varphi(x) \vee \exists x \psi(x)} \qquad \dfrac{\dfrac{[\psi(x)]^1}{\exists x \psi(x)}}{\exists x \varphi(x) \vee \exists x \psi(x)}}{\exists x \varphi(x) \vee \exists x \psi(x)} \vee E_1}{\exists x \varphi(x) \vee \exists x \psi(x)} \exists E_2}{\exists x(\varphi(x) \vee \psi(x)) \rightarrow \exists x \varphi(x) \vee \exists x \psi(x)} \rightarrow I_3$$

$$\vdash \exists x \varphi(x) \leftrightarrow \neg \forall x \neg \varphi(x).$$

$$\forall I \ \frac{\varphi}{\forall x \varphi} \qquad \forall E \ \frac{\forall x \varphi}{\varphi[t/x]}$$

$$\exists I \ \frac{\varphi[t/x]}{\exists x \varphi} \qquad \exists E \ \frac{\exists x \varphi \qquad \begin{matrix} [\varphi] \\ \vdots \\ \psi \end{matrix}}{\psi}$$

$$\dfrac{\dfrac{\forall x(x = x)}{x = x}\,\forall E}{\exists y(x = y))}\,\exists I$$

?

$$\dfrac{\dfrac{\forall x(x = x)}{x = x}\,\forall E}{\exists y(x = y))}\,\exists I$$

yes!

$$\frac{\forall x.\phi}{\phi[t/x]} \equiv \psi[u/y]$$

$$\frac{\forall x(x = x)}{(x{=}x)[x/x]} \forall E$$

$$\frac{\forall x.\phi}{\phi[t/x]} \equiv \psi[u/y]$$

$$\frac{\forall x(x = x)}{(x{=}x)[x/x]} \equiv (x{=}y)[x/y]$$

$$\frac{\forall x.\phi}{\dfrac{\psi[u/y]}{\exists y.\psi}}$$

$$\dfrac{\dfrac{\forall x(x = x)}{(x=y)[x/y]}\,\forall E}{\exists y(x = y))}\,\exists I$$

1. $\vdash \exists x(\varphi(x) \wedge \psi) \leftrightarrow \exists x\varphi(x) \wedge \psi$ if $x \notin FV(\psi)$,
2. $\vdash \forall x(\varphi(x) \vee \psi) \leftrightarrow \forall x\varphi(x) \vee \psi$ if $x \notin FV(\psi)$,
3. $\vdash \forall x\varphi(x) \leftrightarrow \neg\exists x\neg\varphi(x)$,
4. $\vdash \neg\forall x\varphi(x) \leftrightarrow \exists x\neg\varphi(x)$,
5. $\vdash \neg\exists x\varphi(x) \leftrightarrow \forall x\neg\varphi(x)$,
6. $\vdash \exists x(\varphi(x) \rightarrow \psi) \leftrightarrow (\forall x\varphi(x) \rightarrow \psi)$ if $x \notin FV(\psi)$,
7. $\vdash \exists x(\varphi \rightarrow \psi(x)) \leftrightarrow (\varphi \rightarrow \exists x\psi(x))$ if $x \notin FV(\varphi)$,
8. $\vdash \exists x\exists y\varphi \leftrightarrow \exists y\exists x\varphi$,
9. $\vdash \exists x\varphi \leftrightarrow \varphi$ if $x \notin FV(\varphi)$.

Natural Deduction and Identity

$$\frac{x = y}{y = x}\,\mathrm{RI}_2 \qquad \frac{}{x=x}\,\mathrm{RI}_1$$

$$\frac{x = y \quad y = z}{x = z}\,\mathrm{RI}_3$$

$$\frac{x_1 = y_1, \ldots, x_n = y_n}{t(x_1, \ldots, x_n) = t(y_1, \ldots, y_n)}\,\mathrm{RI}_4$$

$$\frac{x_1 = y_1, \ldots, x_n = y_n \quad \varphi(x_1, \ldots, x_n)}{\varphi(y_1, \ldots, y_n)}\,\mathrm{RI}_4$$

$$\frac{x_1 = y_1, \ldots, x_n = y_n}{t[x_1, \ldots, x_n/z_1, \ldots, z_n] = t[y_1, \ldots, y_n/z_1, \ldots, z_n]}$$

$$\frac{x_1 = y_1, \ldots, x_n = y_n \quad \varphi[x_1, \ldots, x_n/z_1, \ldots, z_n]}{\varphi[y_1, \ldots, y_n/z_1, \ldots, z_n]}$$

$$\frac{x = y \quad x^2 + y^2 > 12x}{2y^2 > 12x} \qquad \frac{x = y \quad x^2 + y^2 > 12x}{x^2 + y^2 > 12y} \qquad \frac{x = y \quad x^2 + y^2 > 12x}{2y^2 > 12y}$$

Lemma 2.10.2 *Let L be of type $\langle r_1, \ldots, r_n; a_1, \ldots, a_m; k\rangle$. If the rules*

$$\frac{x_1 = y_1, \ldots, x_{r_i} = y_{r_i} \quad \boldsymbol{P_i}(x_1, \ldots, x_{r_i})}{\boldsymbol{P_i}\,(y_1, \ldots, y_{r_i})} \text{ for all } i \leq n$$

and

$$\frac{x_1 = y_1, \ldots, x_{a_j} = y_{a_j}}{f_j(x_1, \ldots, x_{a_j}) = f_j(y_1, \ldots, y_{a_j})} \text{ for all } j \leq m$$

are given, then the rules RI_4 are derivable.