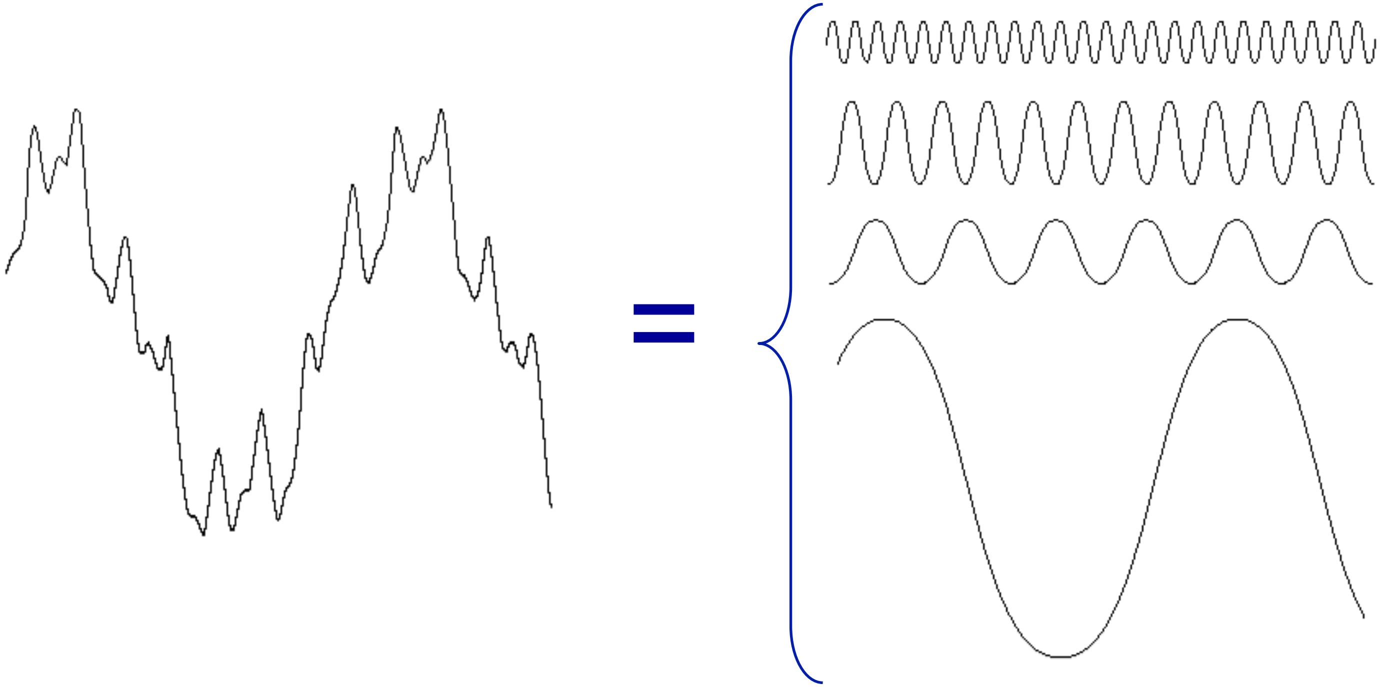
Image Enhancement in the frequency domain

GZ Chapter 4

Contents

- In this lecture we will look at image enhancement in the frequency domain
  - The Fourier series & the Fourier transform
  - Image Processing in the frequency domain
    - Image smoothing
    - Image sharpening
  - Fast Fourier Transform

Fourier representation



The Discrete Fourier Transform (DFT)

- Here we use the GW's notations
- The Discrete Time Fourier Transform of f(x,y)ß for

x = 0, 1, 2…M-1 and

y = 0,1,2…N-1, denoted by F(u, v), is given by the equation:

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

for u = 0, 1, 2…M-1 and v = 0, 1, 2…N-1.

The Inverse DFT

- It is really important to keep in mind that the Fourier transform is completely reversible
- The inverse DFT is given by:

$$f(x,y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) e^{j2\pi(ux/M + vy/N)}$$

for x = 0, 1, 2…M-1 and y = 0, 1, 2…N-1

Fourier spectra

- Fourier spectra in 1D
  - The absolute value of the complex function in the Fourier domain is the amplitude spectrum

$$F(u) = |F(u)|e^{-j\phi(u)} \tag{4.2-9}$$

$$|F(u)| = \left[R^2(u) + I^2(u)\right]^{1/2} \tag{4.2-10}$$

  - The angle defined by the inverse tg of the ration between the imaginary and the real components of the complex function is the phase spectrum

$$\phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$$

  - The square of the amplitude spectrum is the power spectrum

$$\begin{aligned} P(u) &= |F(u)|^2 \\ &= R^2(u) + I^2(u). \end{aligned}$$

Fourier spectra in 2D

We define the Fourier spectrum, phase angle, and power spectrum as in the previous section:

$$|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2} \tag{4.2-18}$$

$$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right] \tag{4.2-19}$$

and

$$\begin{aligned} P(u,v) &= |F(u,v)|^2 \\ &= R^2(u,v) + I^2(u,v) \end{aligned} \tag{4.2-20}$$

where $R(u,v)$ and $I(u,v)$ are the real and imaginary parts of $F(u,v)$, respectively.

Main properties

- The value of the DFT in the origin is the mean value of the function f(x,y)

$$F(0,0) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y),$$

- If f is real its DFT is conjugate symmetric

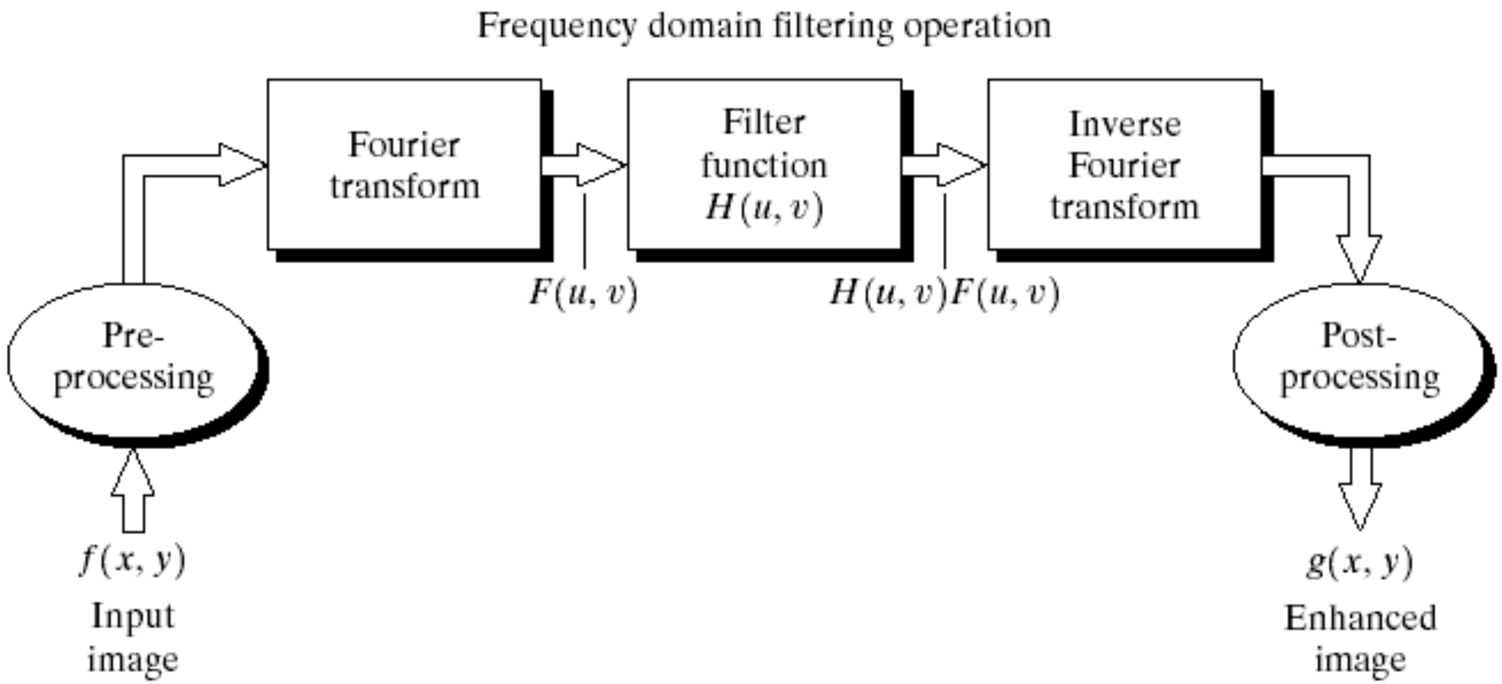
$$F(u,v) = F^*(-u,-v)$$

- Thus the Fourier spectrum is symmetric

$$|F(u,v)| = |F(-u,-v)|,$$

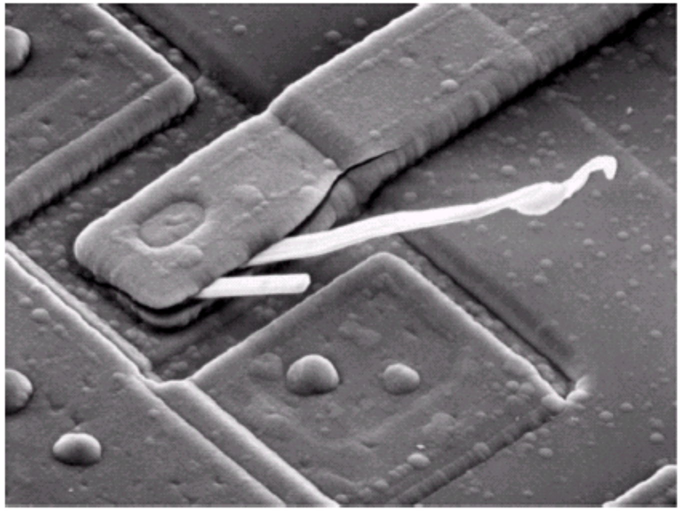
Basics of filtering in the frequency domain

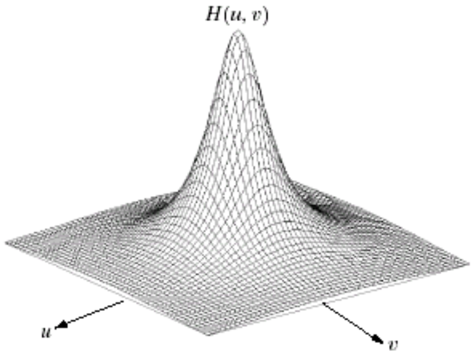
- To filter an image in the frequency domain:
  - Compute F(u,v) the DFT of the image
  - Multiply F(u,v) by a filter function H(u,v)
  - Compute the inverse DFT of the result

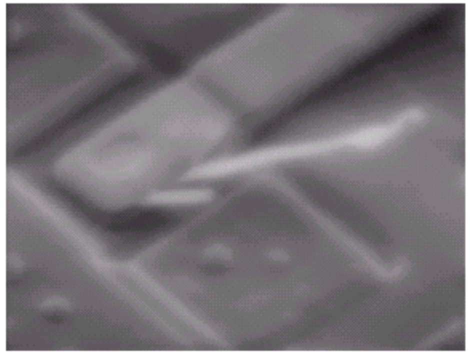


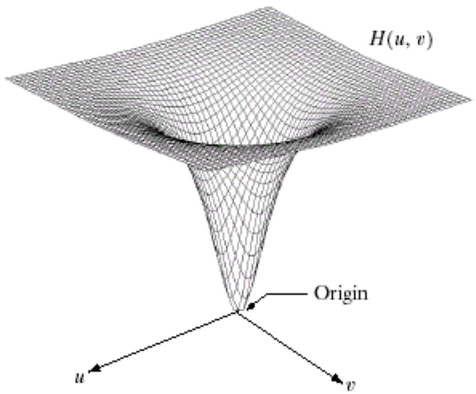
Some Basic Frequency Domain Filters

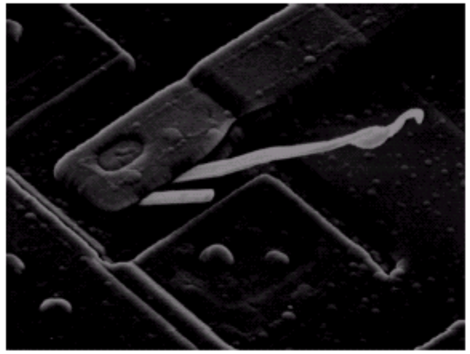
Low Pass Filter (smoothing)











High Pass Filter (edge detection)

Filtering in Fourier domain

1. Multiply the input image by $(-1)^{x+y}$ to center the transform, as indicated in Eq. (4.2-21).
2. Compute $F(u, v)$, the DFT of the image from (1).
3. Multiply $F(u, v)$ by a *filter* function $H(u, v)$.
4. Compute the inverse DFT of the result in (3).
5. Obtain the real part of the result in (4).
6. Multiply the result in (5) by $(-1)^{x+y}$.

$$G(u, v) = H(u, v)F(u, v).$$

- H(u,v) is the filter transfer function, which is the DFT of the filter impulse response
- The implementation consists in multiplying point-wise the filter H(u,v) with the function F(u,v)
- Real filters are called zero phase shift filters because they don't change the phase of F(u,v)

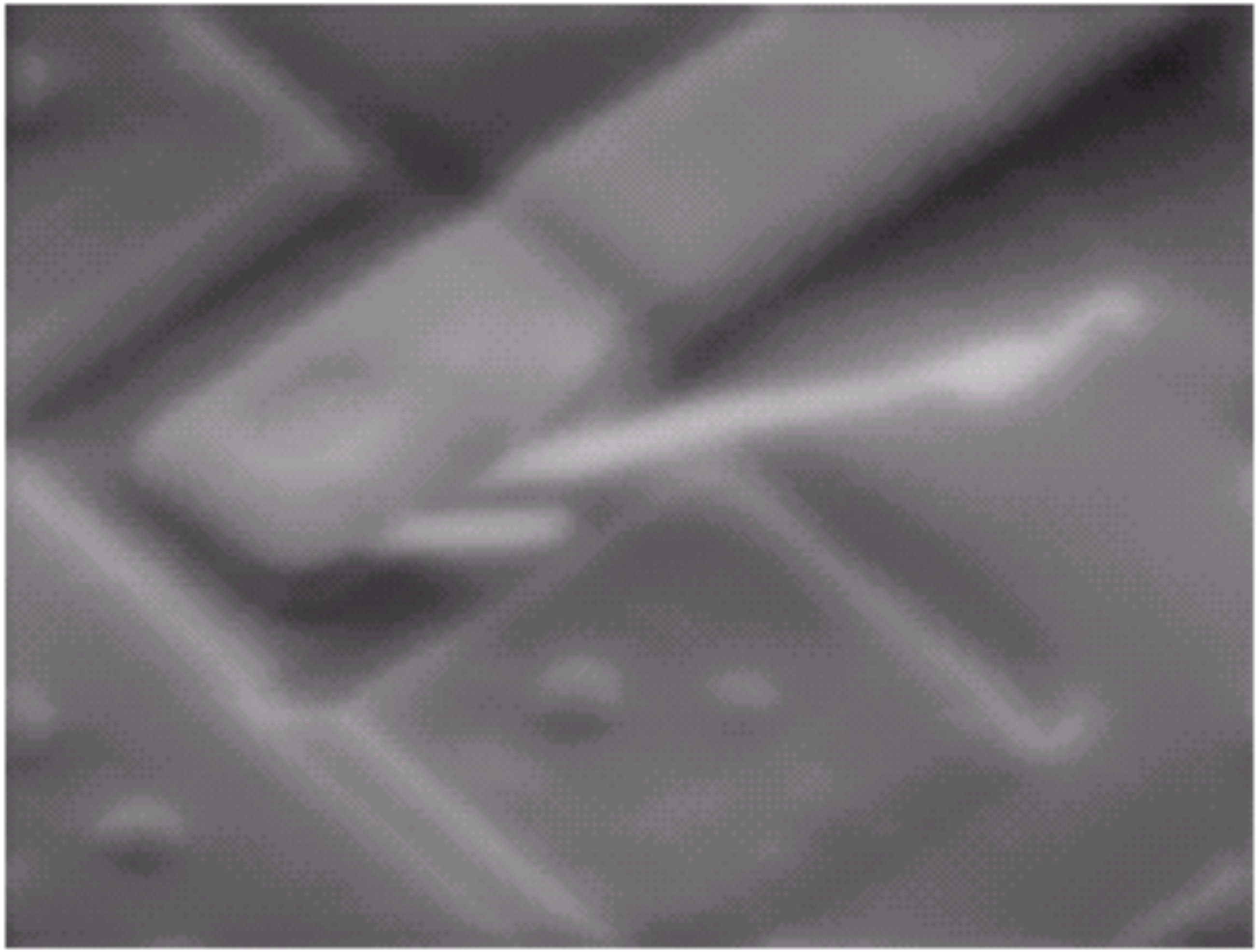
Filtered image

- The filtered image is obtained by taking the inverse DFT of the resulting image

$$\text{Filtered Image} = \Im^{-1}\left[G(u, v)\right].$$

- It can happen that the filtered image has spurious imaginary components even though the original image f(x,y) and the filter h(x,y) are real. These are due to numerical errors and are neglected
- The final result is thus the real part of the filtered image

Smoothing: low pass filtering

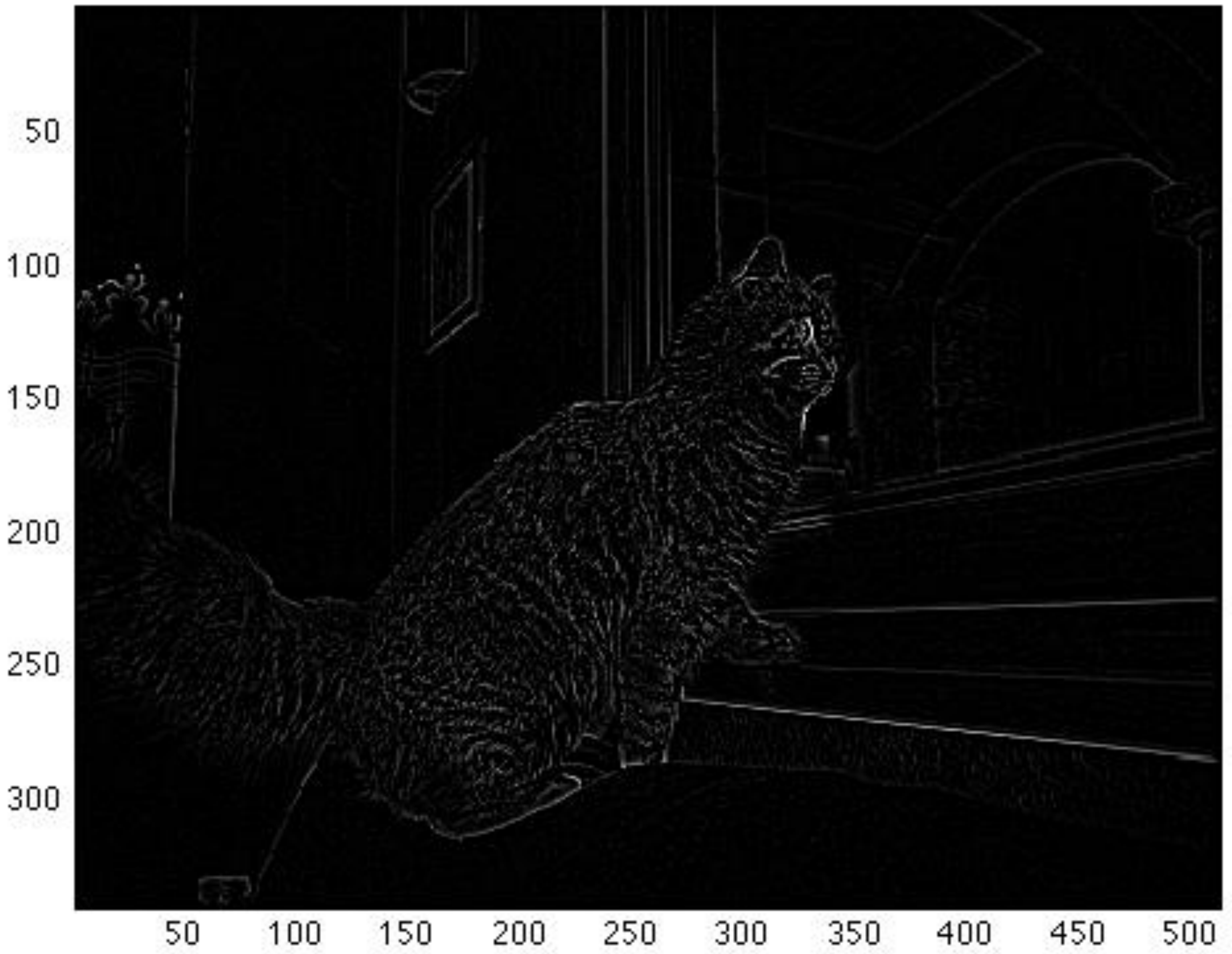


Edge detection: high-pass filtering

Original image

Edge detection: greylevel image

Filtered image



Hints for filtering

- Color images are usually converted to graylevel images before filtering. This is due to the fact that the information about the structure of the image (what is in the image) is represented in the luminance component
- Images are usually stored as “unsigned integers”. Some operations could require the explicit cast to double or float for being implemented
- The filtered image in general consists of double values, so a cast to unsigned integer could be required before saving it in a file using a predefined format. This could introduce errors due to rounding operations.

Frequency Domain Filters

- The basic model for filtering is:

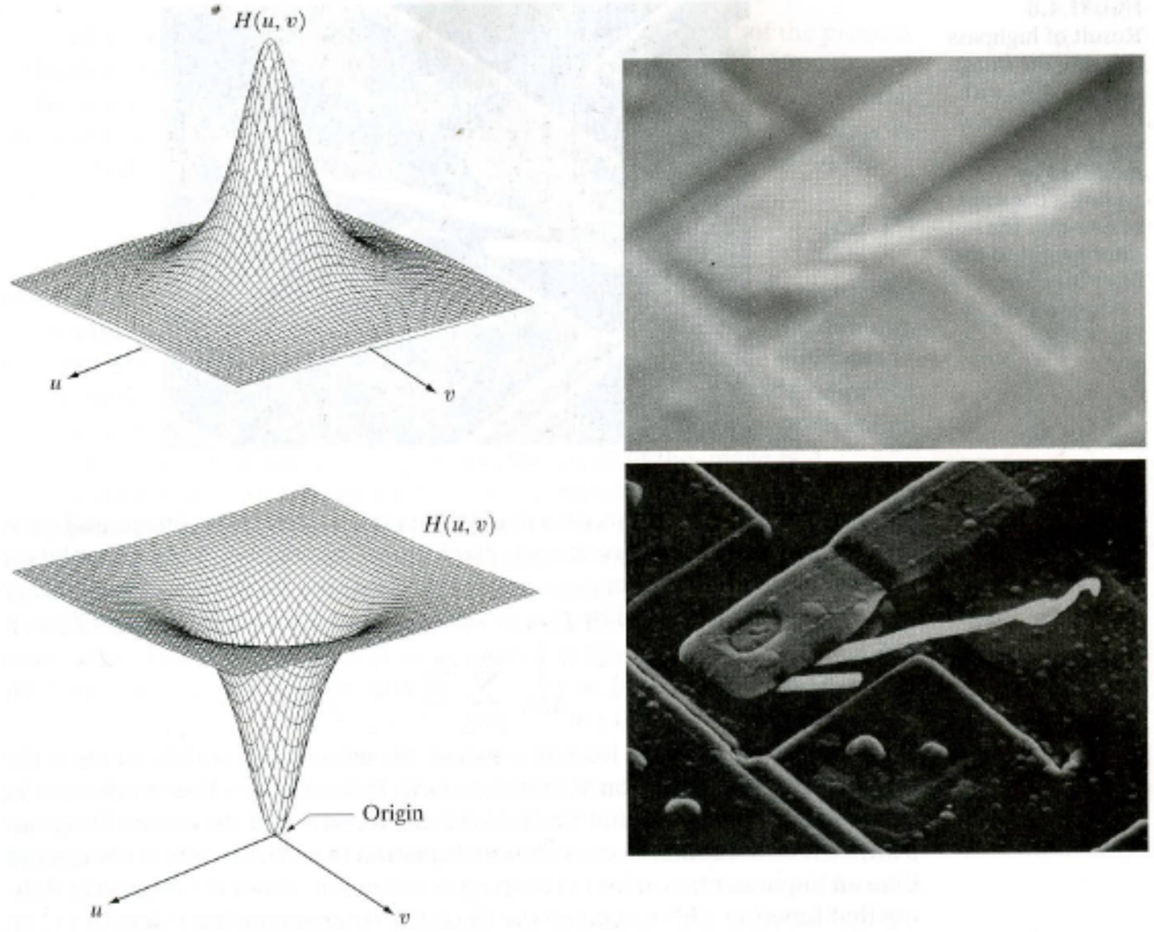
$$G(u,v) = H(u,v)F(u,v)$$

- where F(u,v) is the Fourier transform of the image being filtered and H(u,v) is the filter transform function
- Filtered image

$$f(x,y) = \Im^{-1}\left\{F(u,v)\right\}$$

- Smoothing is achieved in the frequency domain by dropping out the high frequency components
  - Low pass (LP) filters – only pass the low frequencies, drop the high ones
  - High-pass (HP) filters – olny pass the frequencies above a minimum value

LP and HP filtering



a b
c d

FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

Filtering in spatial and frequency domains

- The filtering operations in spatial and frequency domains are linked by the convolution theorem

$$f(x,y) * h(x,y) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x-m, y-n). \quad (4.2\text{-}30)$$

$$f(x,y) * h(x,y) \Leftrightarrow F(u,v)H(u,v).$$

- Modulation theorem (reminder)

$$f(x,y)h(x,y) \Leftrightarrow F(u,v) * H(u,v).$$

Another proof of the convolution theorem

- Starting from the digital delta function, we will prove that the filtering operation in the signal domain is obtained by the convolution of the signal with the *filter impulse response h(x,y)*
- Consider the digital delta function, so an impulse function of strength A located in (x_0,y_0)
- Shifting (or sampling) property

$$\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} s(x,y)A\delta(x-x_0,y-y_0) = As(x_0,y_0). \qquad (4.2\text{-}33)$$

$$\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} s(x,y)\delta(x,y) = s(0,0).$$

Getting to the impulse response

- FT of the delta function located in the origin

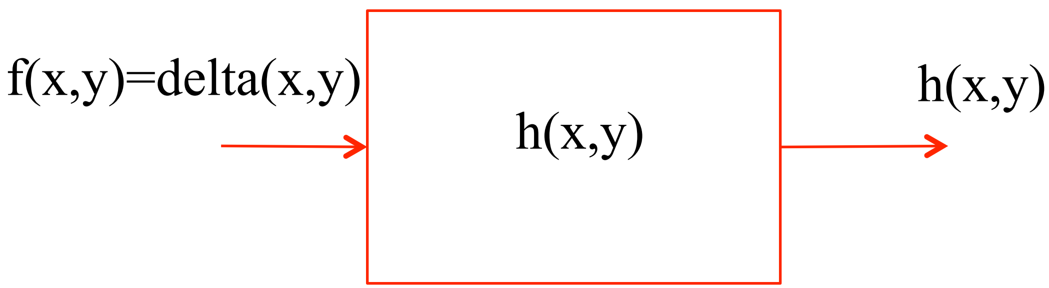
$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}\delta(x,y)e^{-j2\pi(ux/M+vy/N)} = \frac{1}{MN} \qquad (4.2\text{-}35)$$

- Now, let's set f(x,y)=δ(x,y) and calculate the convolution between f and a filter h(x,y)

$$f(x,y) * h(x,y) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\delta(m,n)h(x-m,y-n) = \frac{1}{MN}h(x,y)$$

Getting to the impulse response

- It can be observed that if f is a delta, then the result of the convolution is equal to the function h(x,y) apart from a change in the amplitude
- Then, h(x,y) is called impulse response because it represents the response (output) of the filter when the input is a delta



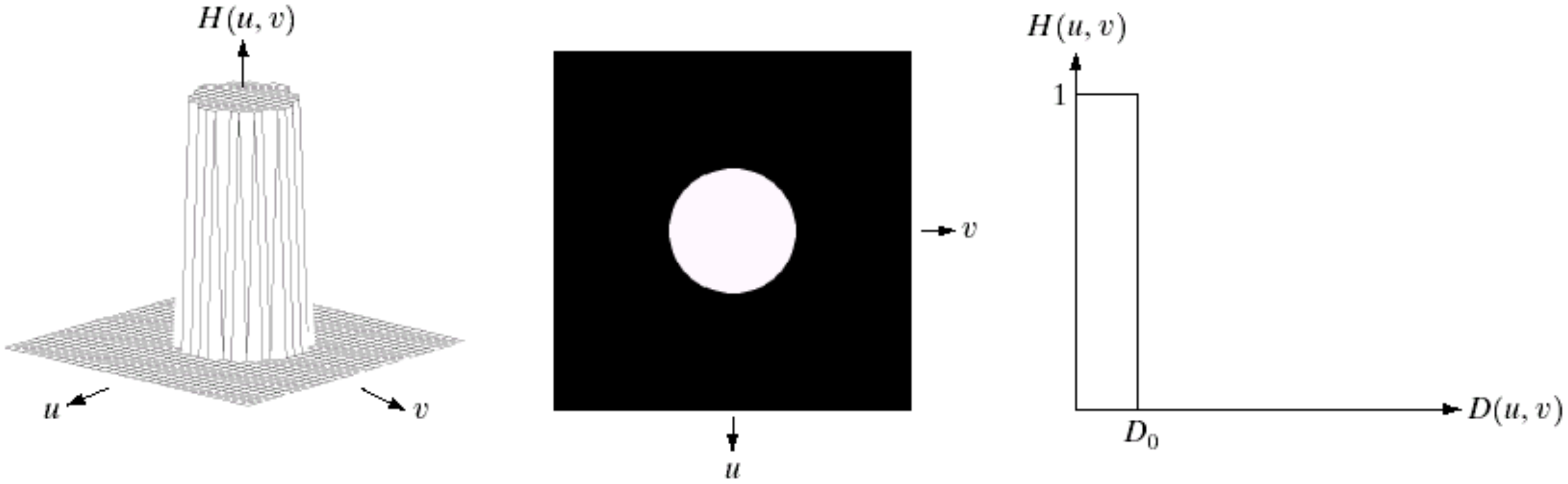
- So *filtering in the signal domain is performed by the convolution of the signal with the filter impulse response*

Filtering: summary

$$
\begin{aligned}
f(x, y) * h(x, y) &\Leftrightarrow F(u, v) H(u, v) \\
\delta(x, y) * h(x, y) &\Leftrightarrow \Im[\delta(x, y)] H(u, v) \\
h(x, y) &\Leftrightarrow H(u, v).
\end{aligned}
$$

Ideal Low Pass Filter

- Simply cut off all high frequency components that are a specified distance D_0 from the origin of the transform



Ideal Low Pass Filter (cont…)

- The transfer function for the ideal low pass filter can be given as:

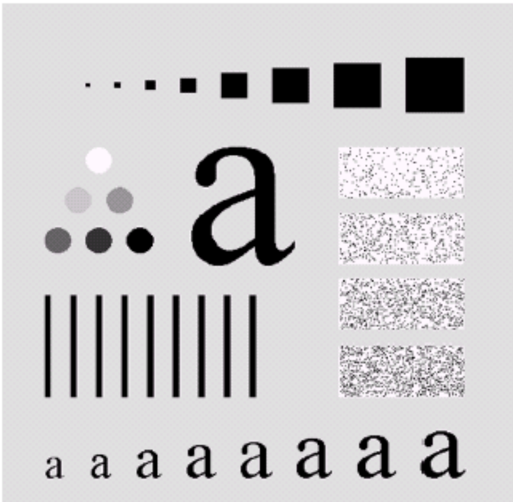
$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \leq D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$$

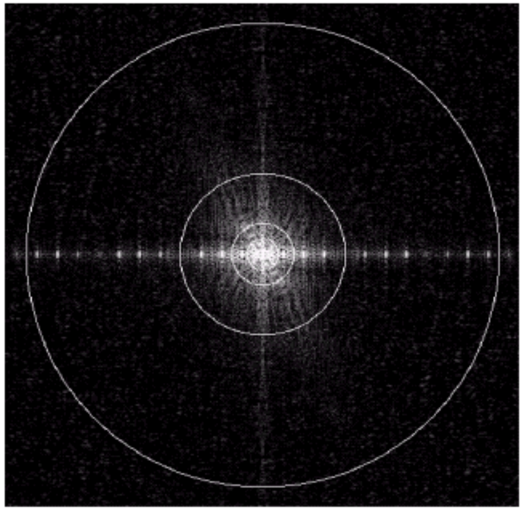
- where D(u,v) is given as:

$$D(u,v) = [(u - M/2)^2 + (v - N/2)^2]^{1/2}$$

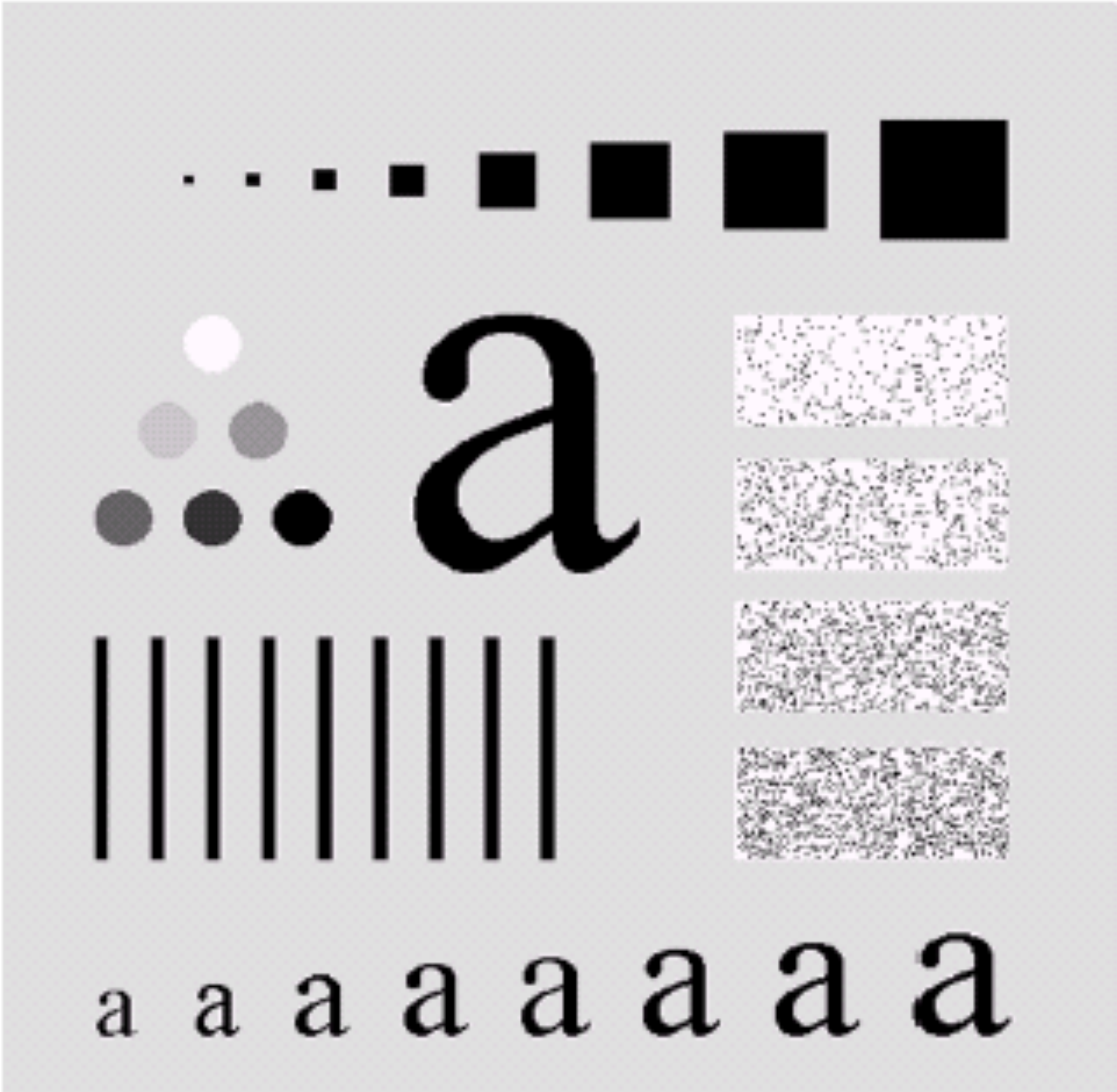
Ideal Low Pass Filter (cont…)

- Above we show an image, it's Fourier spectrum and a series of ideal low pass filters of radius 5, 15, 30, 80 and 230 superimposed on top of it

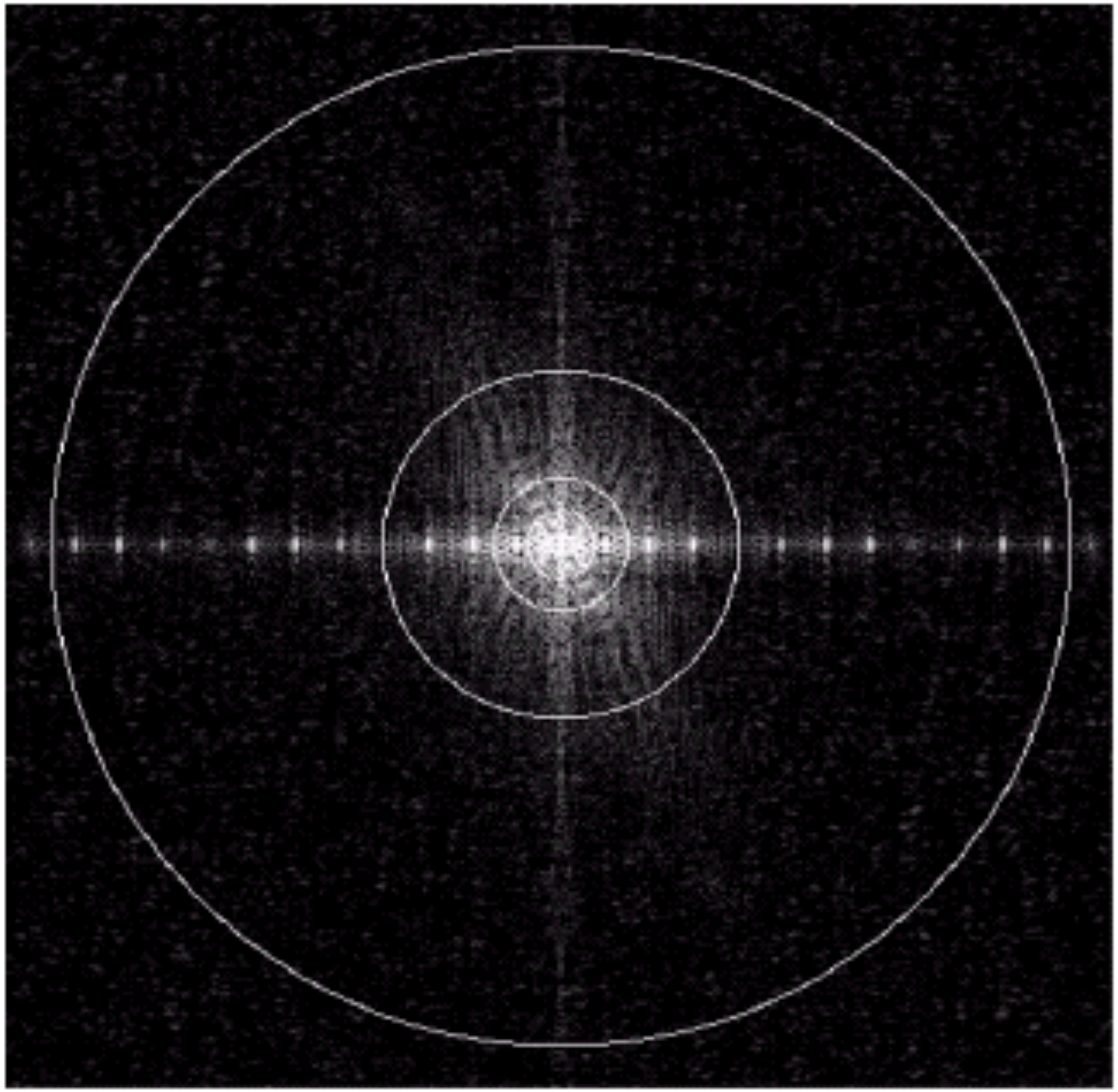




Ideal Low Pass Filter (cont…)



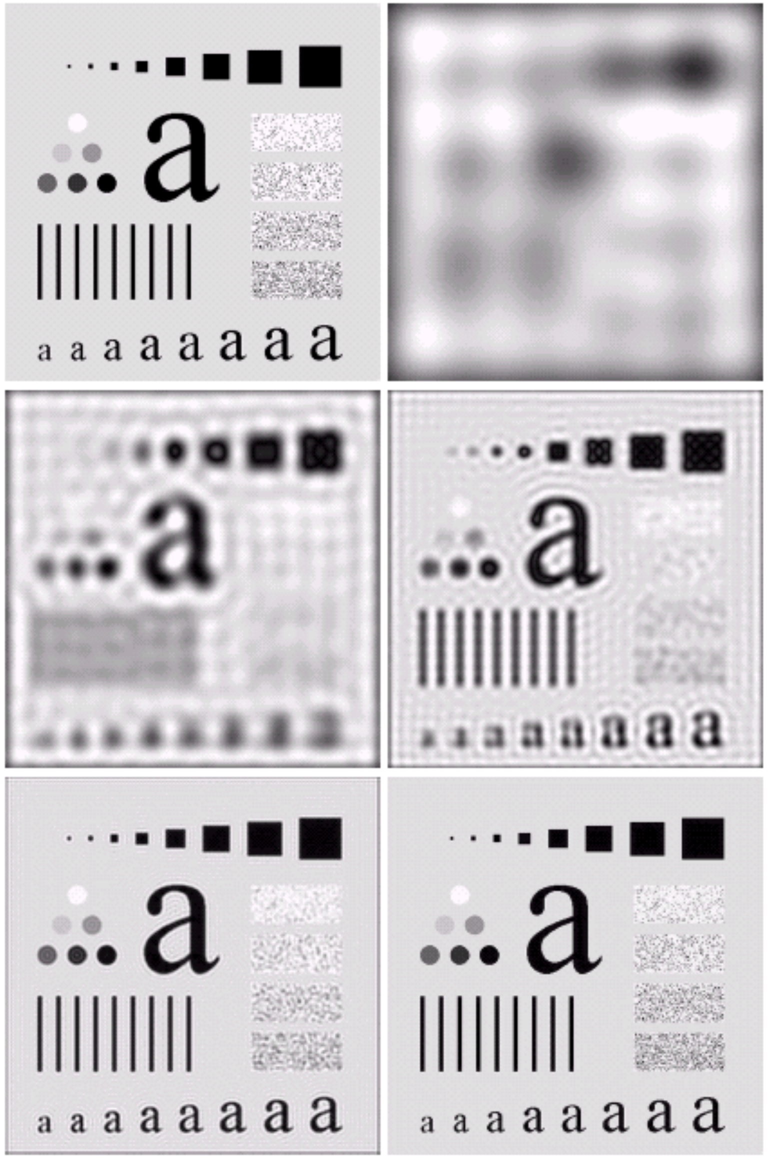
Ideal Low Pass Filter (cont…)



Ideal Low Pass Filter (cont…)

Original image

Result of filtering with ideal low pass filter of radius 5



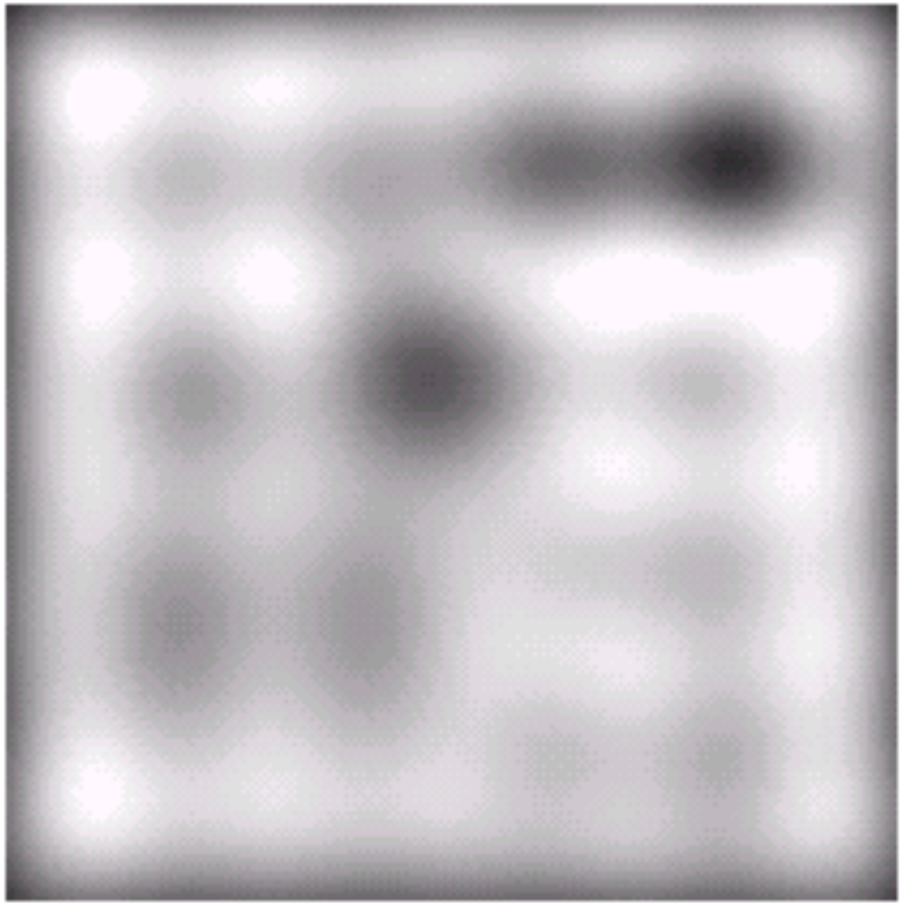
Result of filtering with ideal low pass filter of radius 15

Result of filtering with ideal low pass filter of radius 30

Result of filtering with ideal low pass filter of radius 80

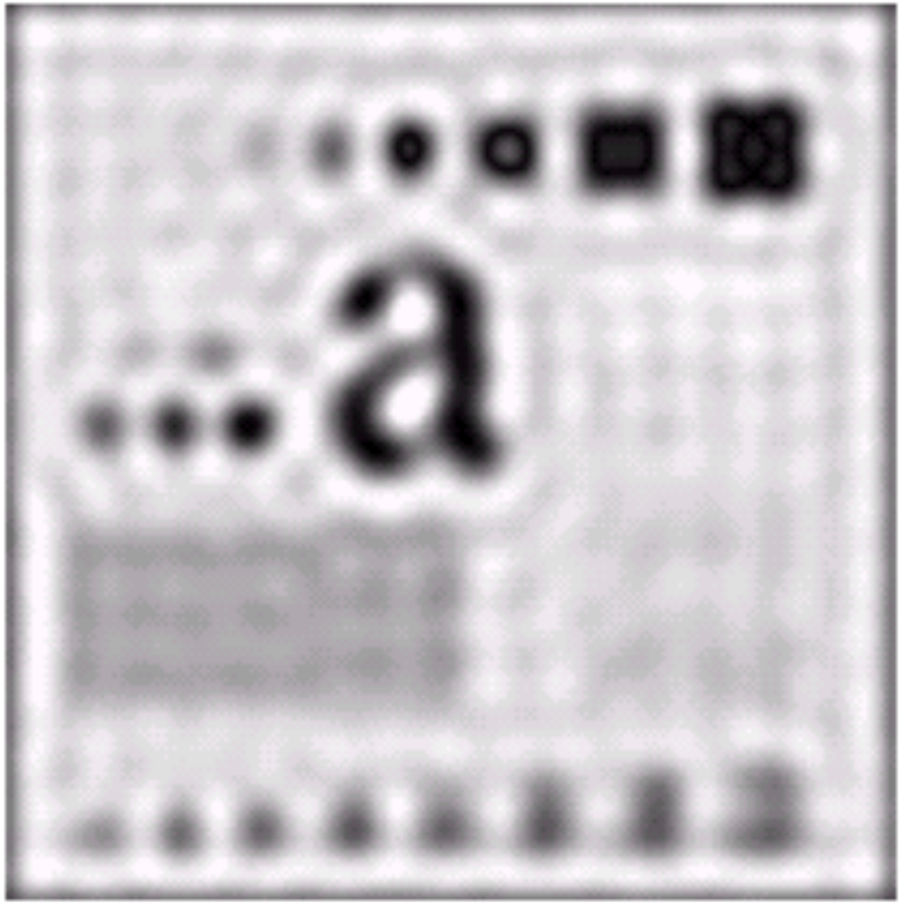
Result of filtering with ideal low pass filter of radius 230

Ideal Low Pass Filter (cont…)



Result of filtering with ideal low pass filter of radius 5

Ideal Low Pass Filter (cont…)

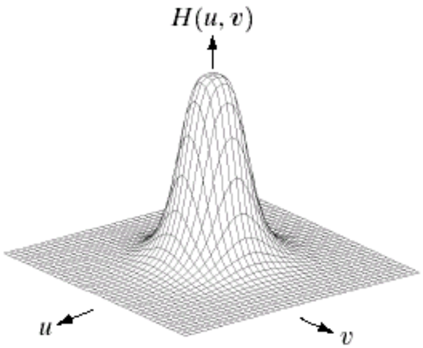


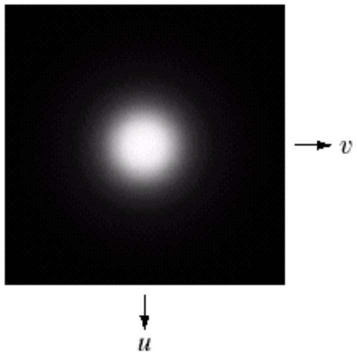
Result of filtering with ideal low pass filter of radius 15

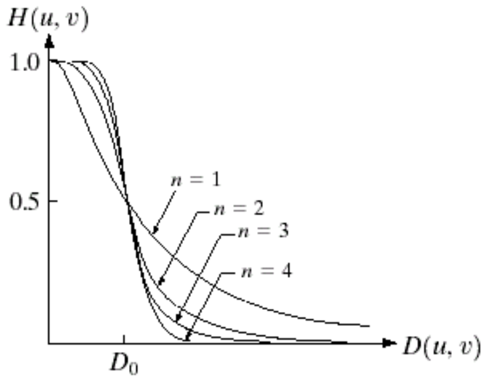
Butterworth Lowpass Filters

- The transfer function of a Butterworth lowpass filter of order n with cutoff frequency at distance D_0 from the origin is defined as:

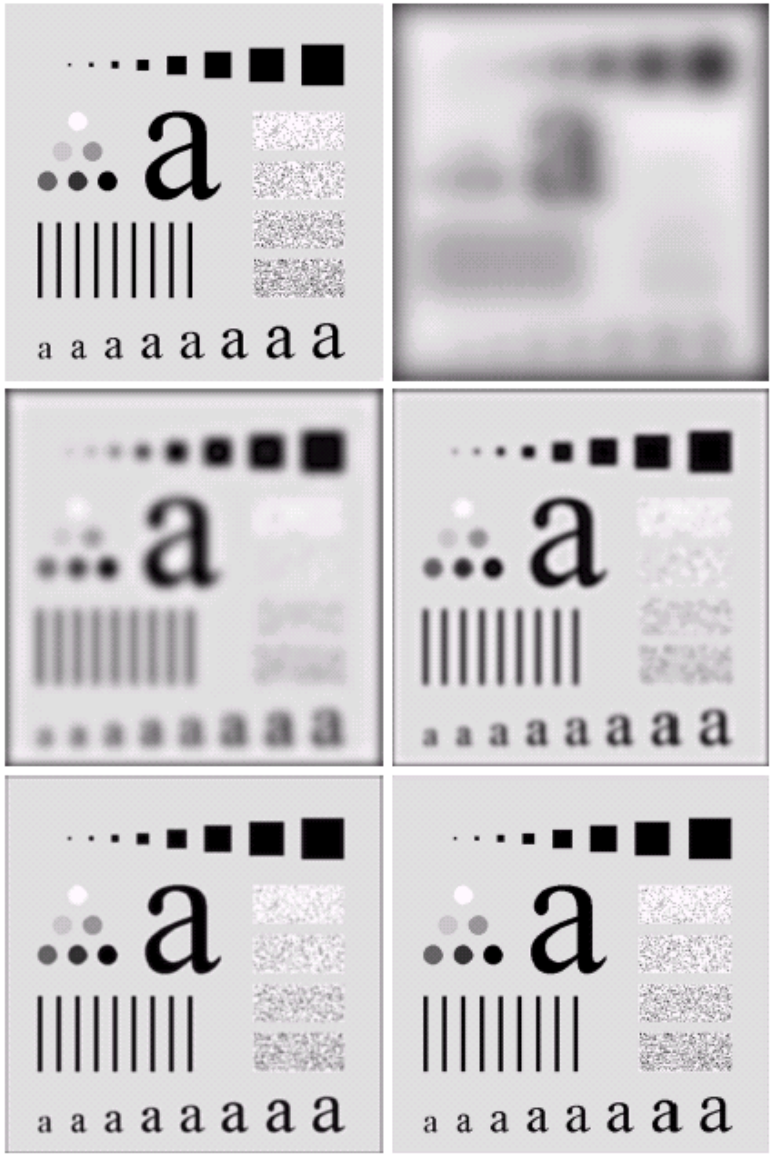
$$H(u,v) = \frac{1}{1 + [D(u,v) / D_0]^{2n}}$$







Butterworth Lowpass Filter (cont…)



Original image

Result of filtering with Butterworth filter of order 2 and cutoff radius 5

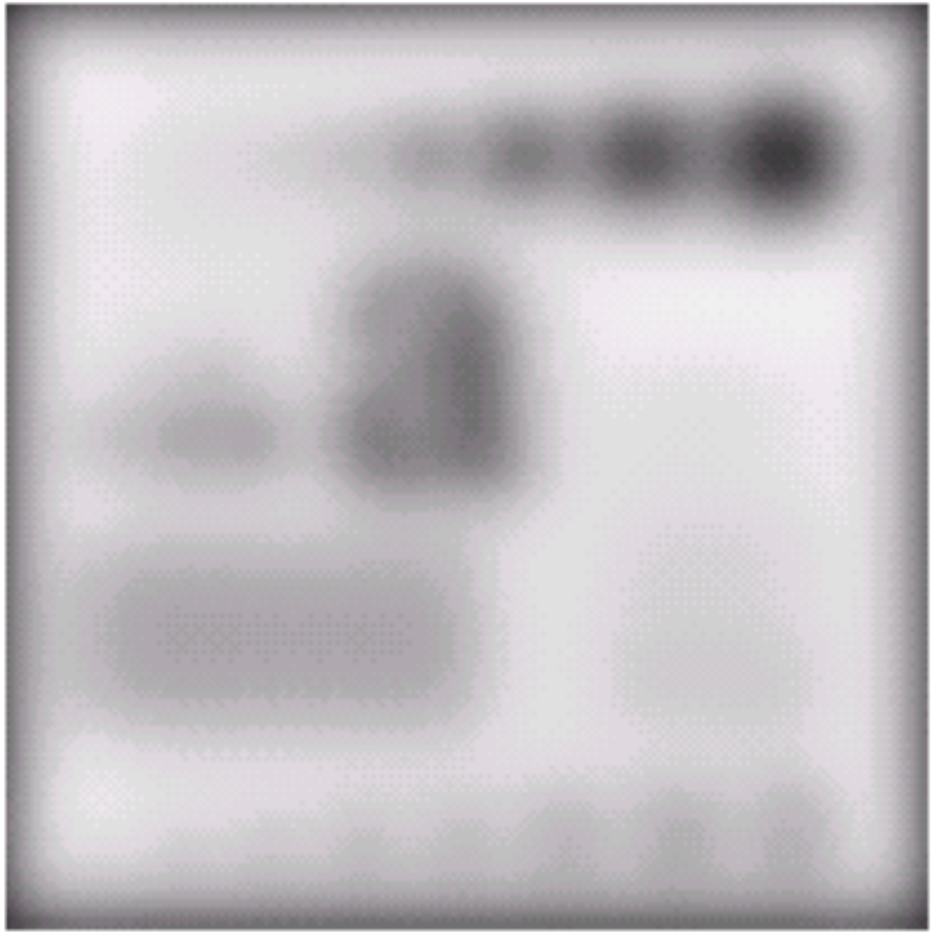
Result of filtering with Butterworth filter of order 2 and cutoff radius 15

Result of filtering with Butterworth filter of order 2 and cutoff radius 30

Result of filtering with Butterworth filter of order 2 and cutoff radius 80

Result of filtering with Butterworth filter of order 2 and cutoff radius 230

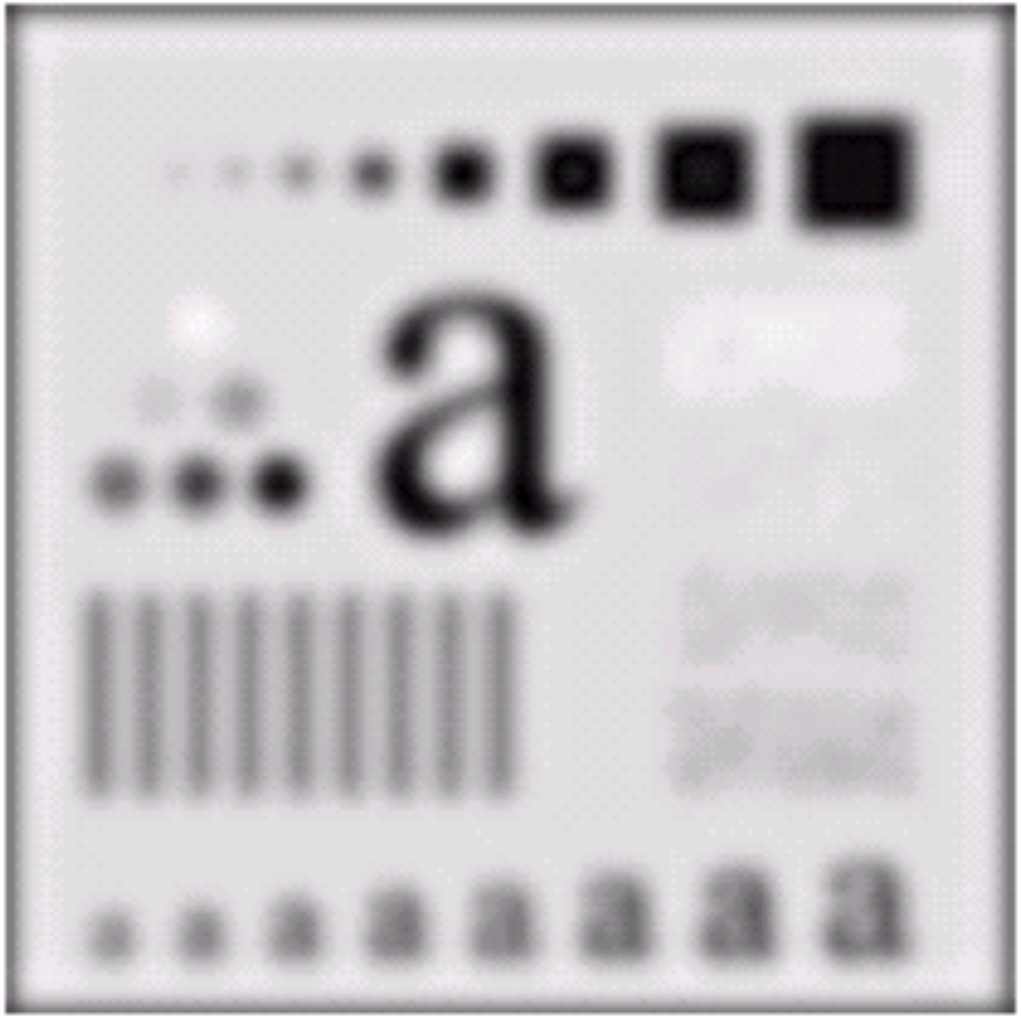
Butterworth Lowpass Filter (cont…)



Result of filtering with Butterworth filter of order 2 and cutoff radius 5

Butterworth Lowpass Filter (cont…)

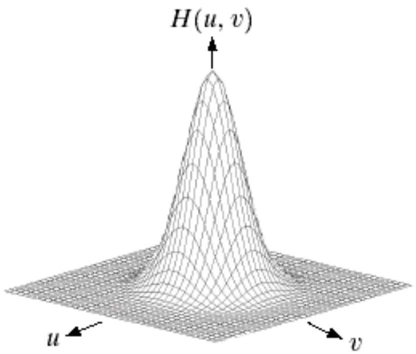
Result of filtering with Butterworth filter of order 2 and cutoff radius 15

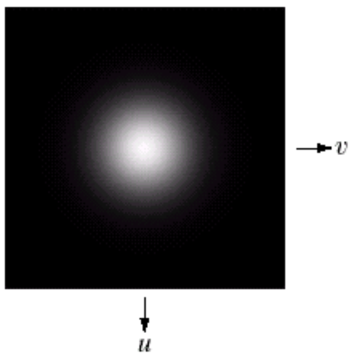


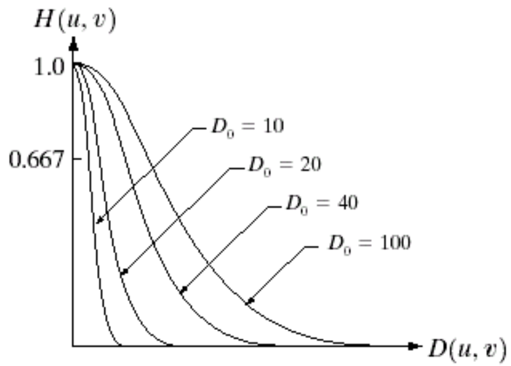
Gaussian Lowpass Filters

- The transfer function of a Gaussian lowpass filter is defined as:

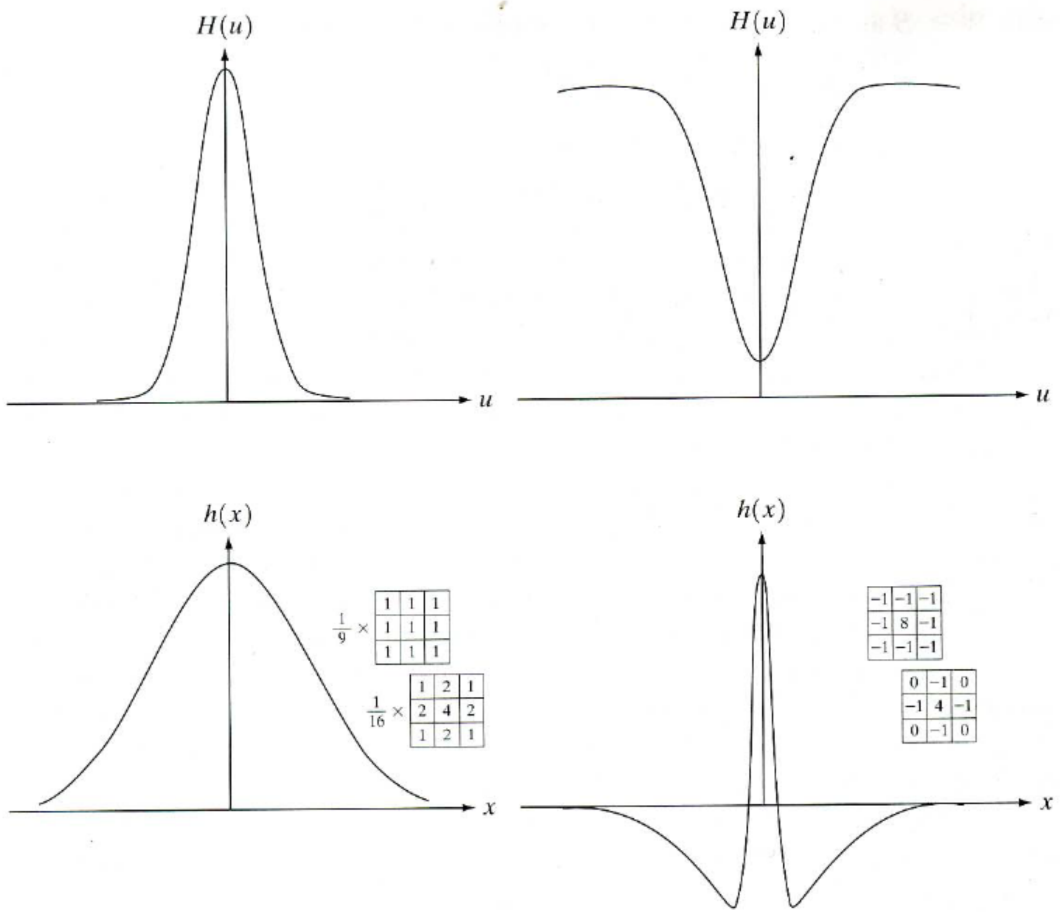
$$H(u,v) = e^{-D^2(u,v)/2D_0^2}$$







Gaussian LP



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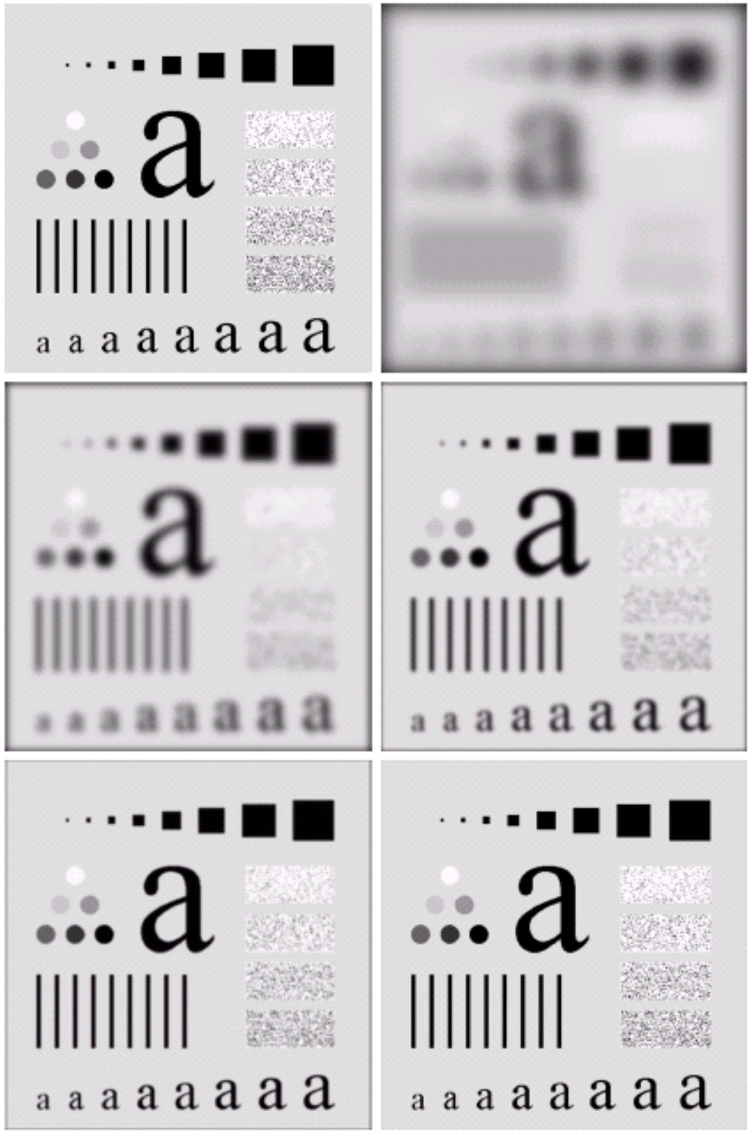
c d

FIGURE 4.9 (a) Gaussian frequency domain lowpass filter. (b) Gaussian frequency domain highpass filter. (c) Corresponding lowpass spatial filter. (d) Corresponding highpass spatial filter. The masks shown are used in Chapter 3 for lowpass and highpass filtering.

Gaussian Lowpass Filters (cont…)



Original image

Result of filtering with Gaussian filter with cutoff radius 5

Result of filtering with Gaussian filter with cutoff radius 15

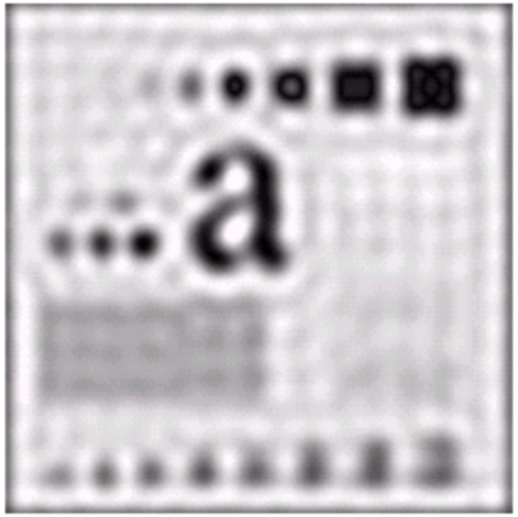
Result of filtering with Gaussian filter with cutoff radius 30

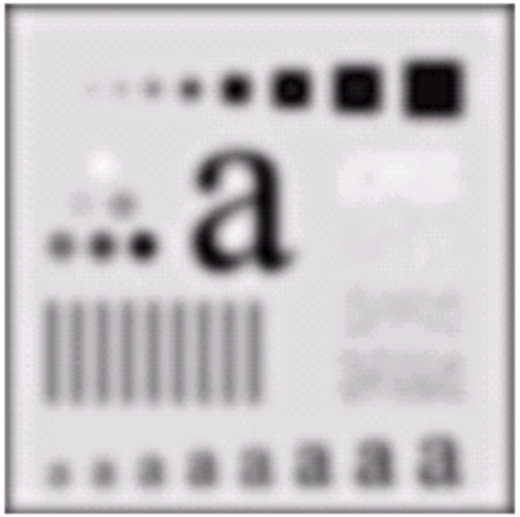
Result of filtering with Gaussian filter with cutoff radius 85

Result of filtering with Gaussian filter with cutoff radius 230

Lowpass Filters Compared

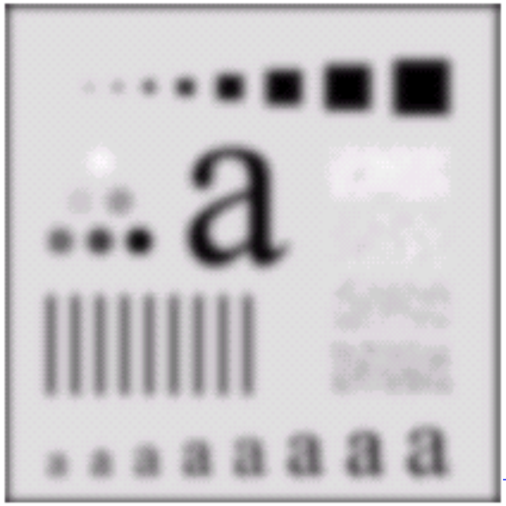
Result of filtering with ideal low pass filter of radius 15





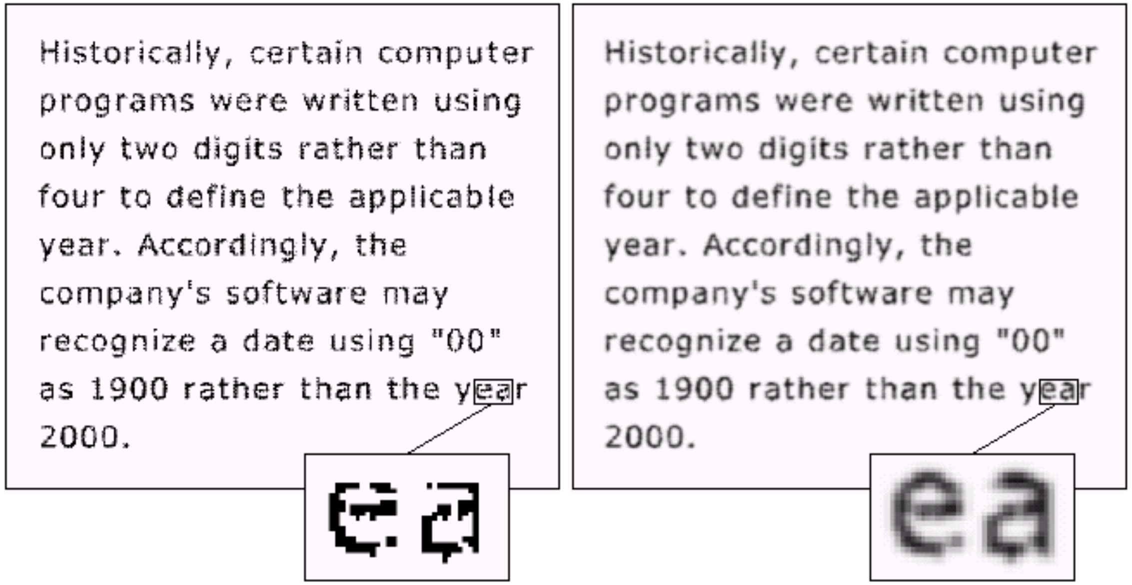
Result of filtering with Butterworth filter of order 2 and cutoff radius 15

Result of filtering with Gaussian filter with cutoff radius 15

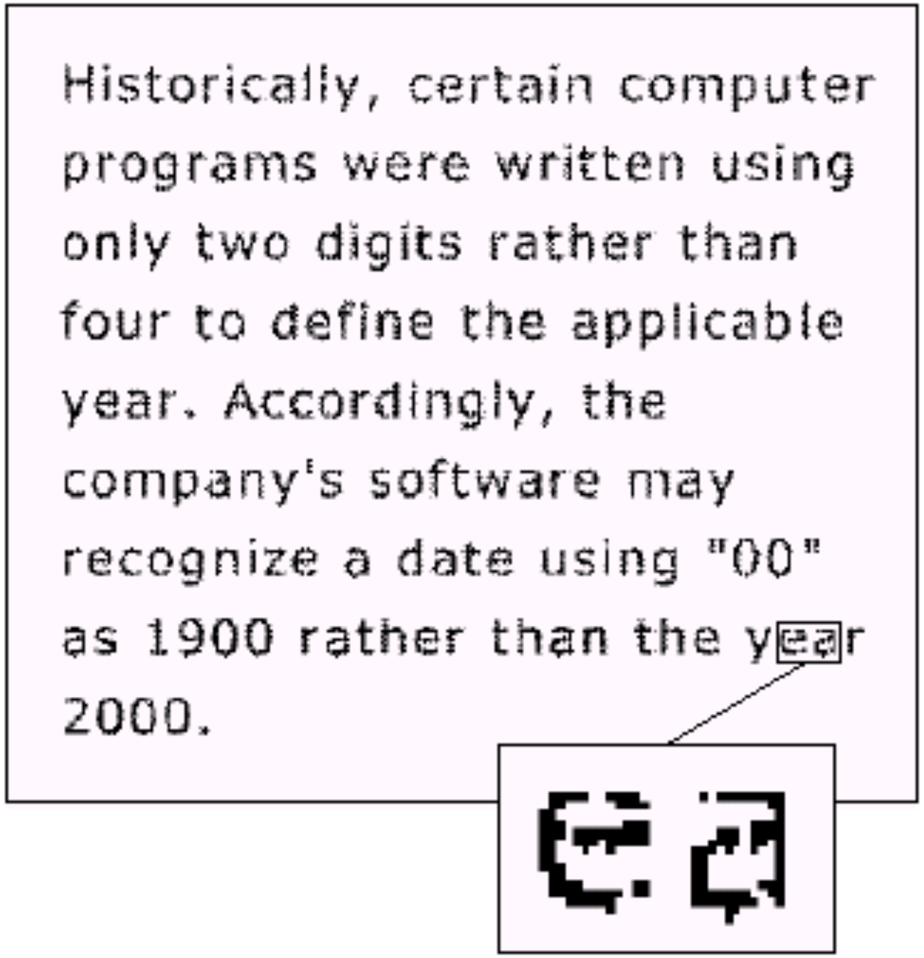


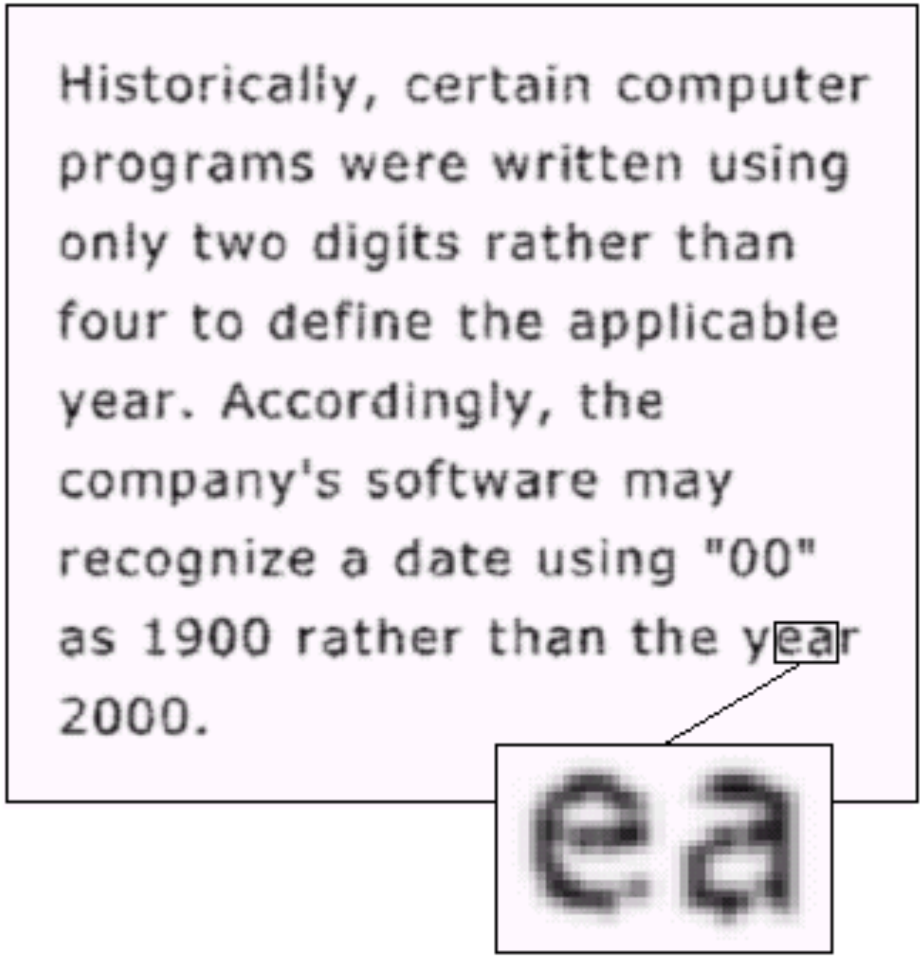
Lowpass Filtering Examples

- A low pass Gaussian filter is used to connect broken text



Lowpass Filtering Examples



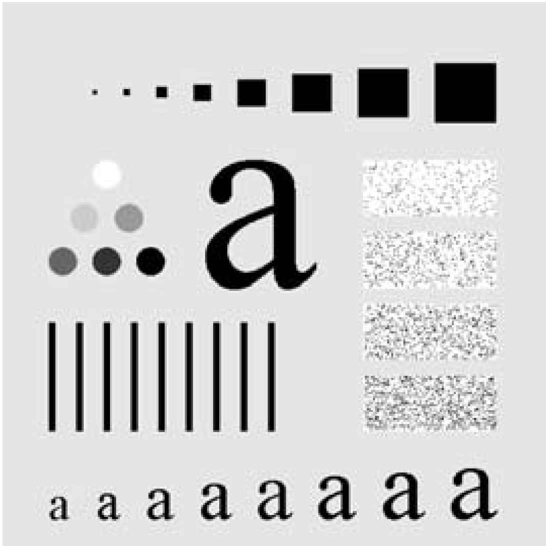


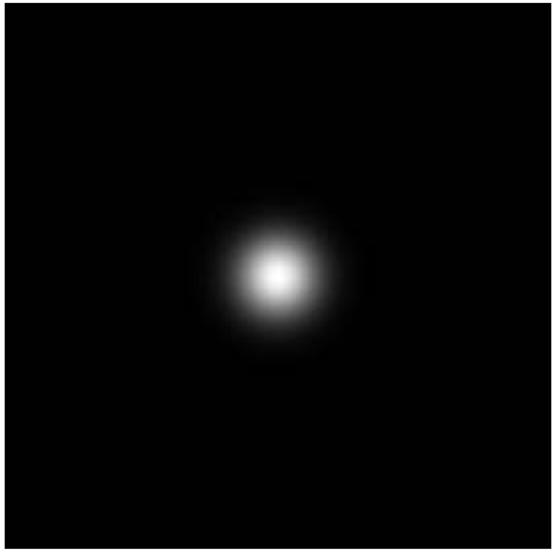
Lowpass Filtering Examples (cont…)

- Different lowpass Gaussian filters used to remove blemishes in a photograph

Lowpass Filtering Examples (cont…)

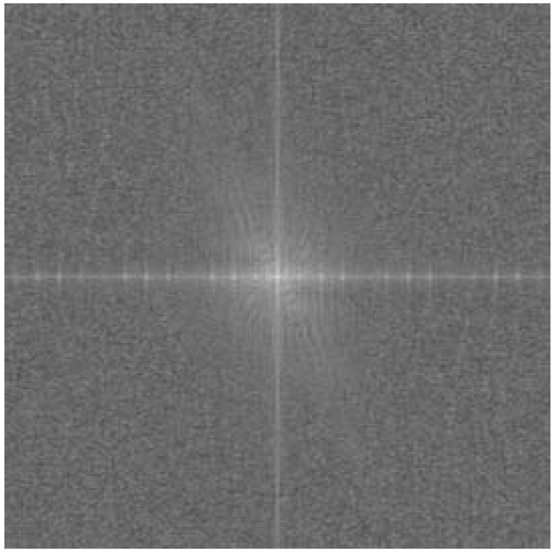
Original image





Gaussian lowpass filter

Spectrum of original image



Processed image

Sharpening in the Frequency Domain

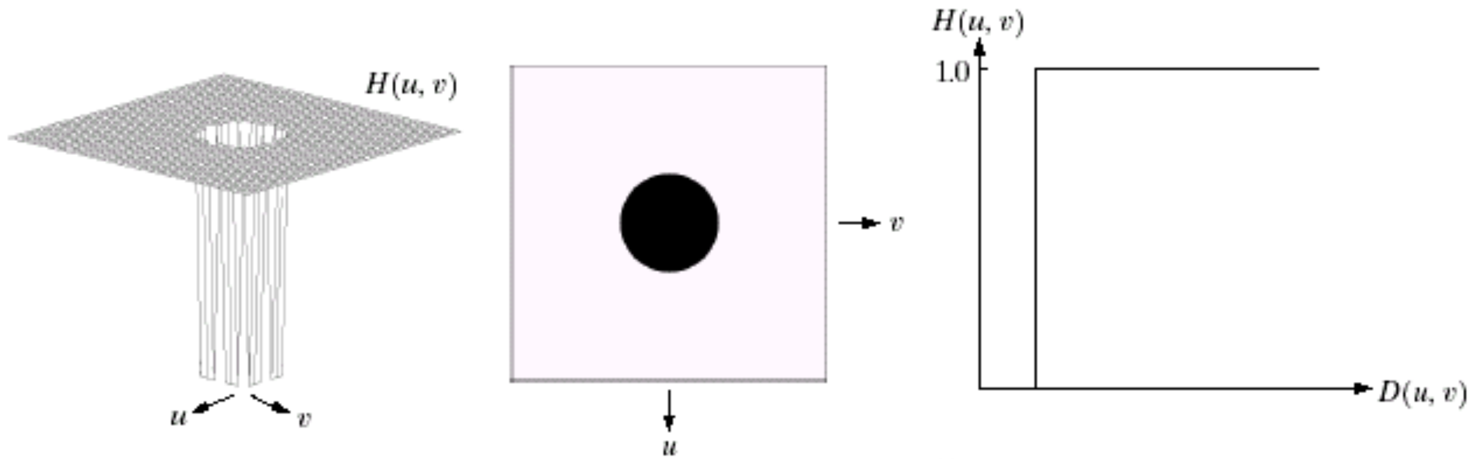
- Edges and fine details in images are associated with high frequency components
- High pass filters – only pass the high frequencies, drop the low ones
- High pass frequencies are precisely the reverse of low pass filters, so:
- $H_{hp}(u, v) = 1 – H_{lp}(u, v)$

Ideal High Pass Filters

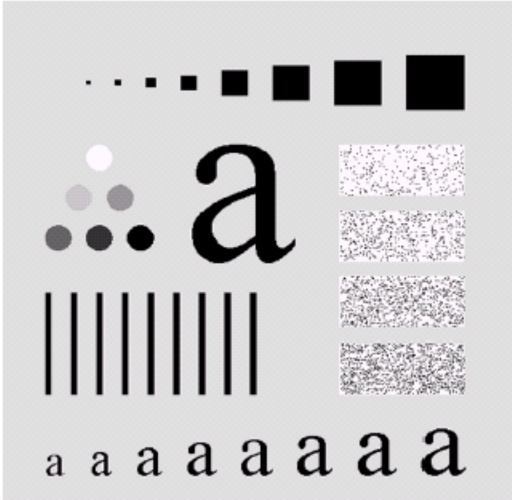
- The ideal high pass filter is given as:

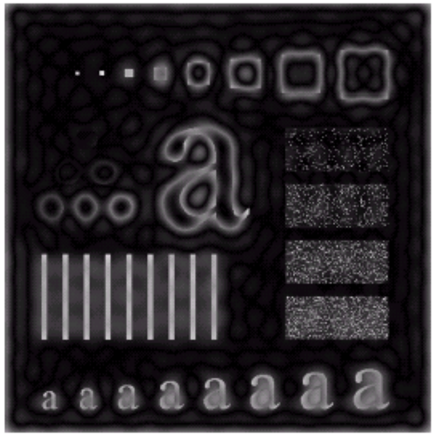
$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

- where D0 is the cut off distance as before

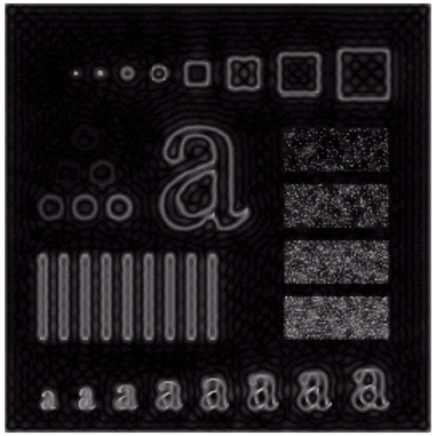


Ideal High Pass Filters (cont…)

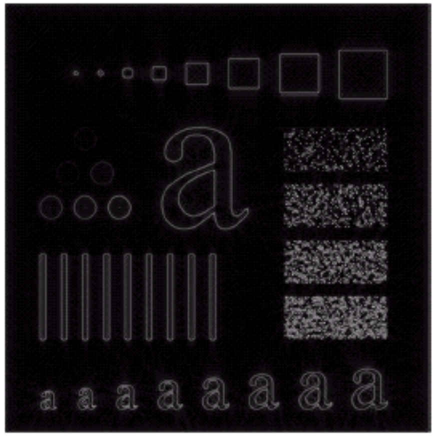




Results of ideal high pass filtering with $D_0 = 15$



Results of ideal high pass filtering with $D_0 = 30$



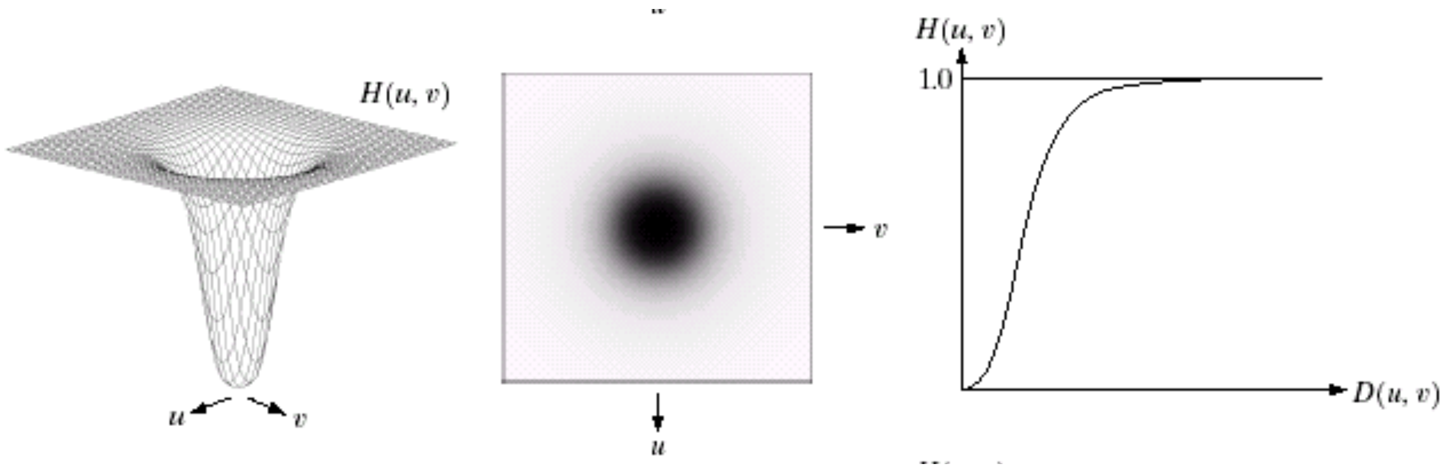
Results of ideal high pass filtering with $D_0 = 80$

Butterworth High Pass Filters

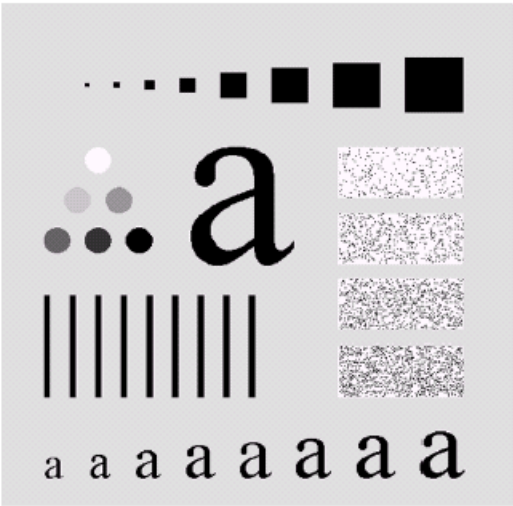
- The Butterworth high pass filter is given as:

$$H(u,v) = \frac{1}{1 + [D_0 / D(u,v)]^{2n}}$$

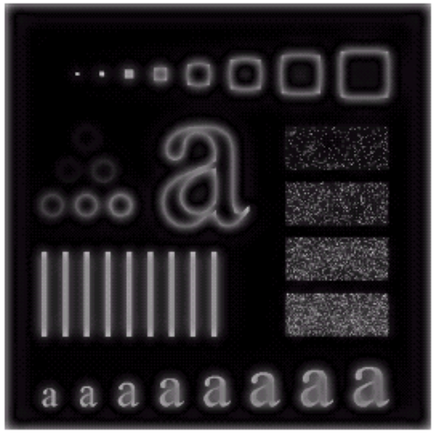
- where n is the order and D_0 is the cut off distance as before

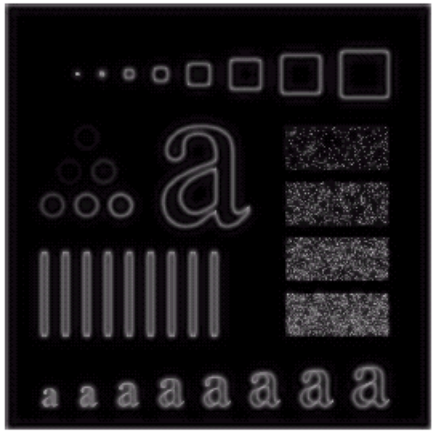


Butterworth High Pass Filters (cont…)



Butterworth
of order 2
$D_0 = 15$





Butterworth of order 2
$D_0 = 30$

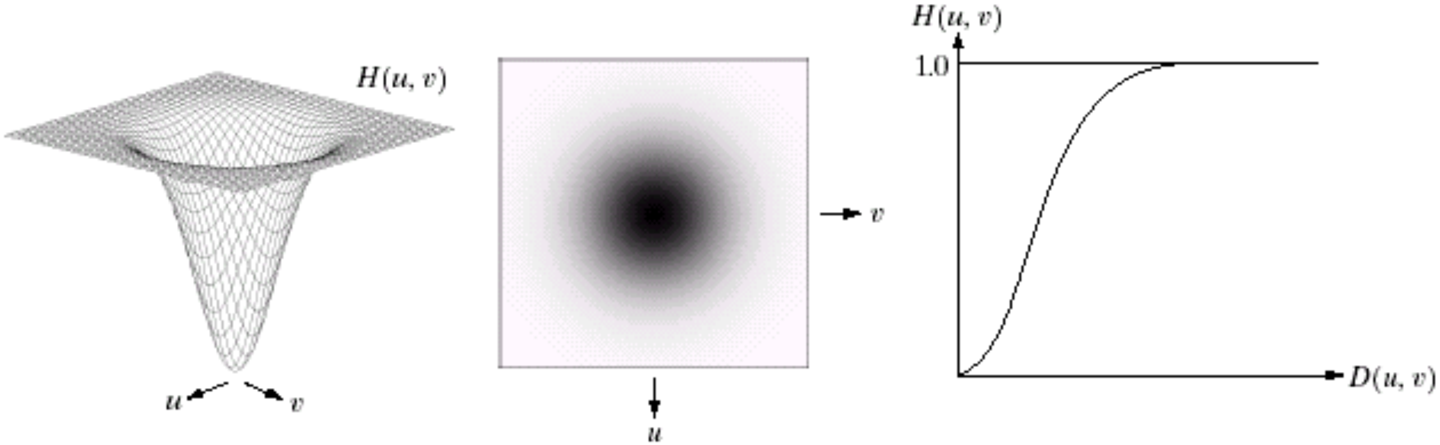
Butterworth
of order 2
$D_0 = 80$

Gaussian High Pass Filters

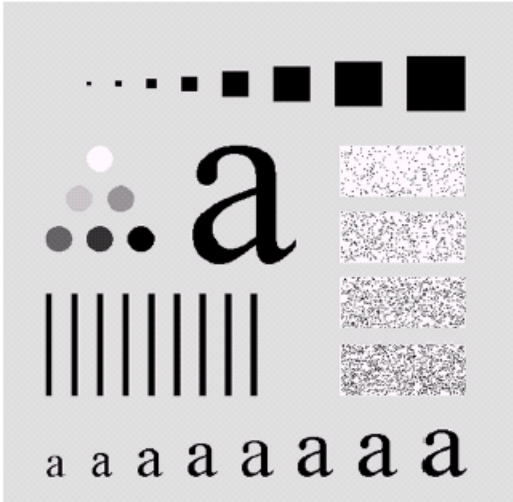
- The Gaussian high pass filter is given as:

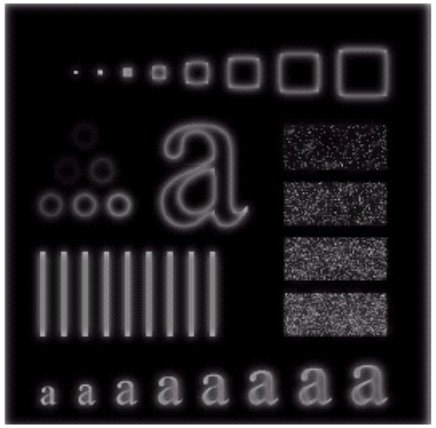
$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0{}^2}$$

- where D0 is the cut off distance as before



Gaussian High Pass Filters (cont…)



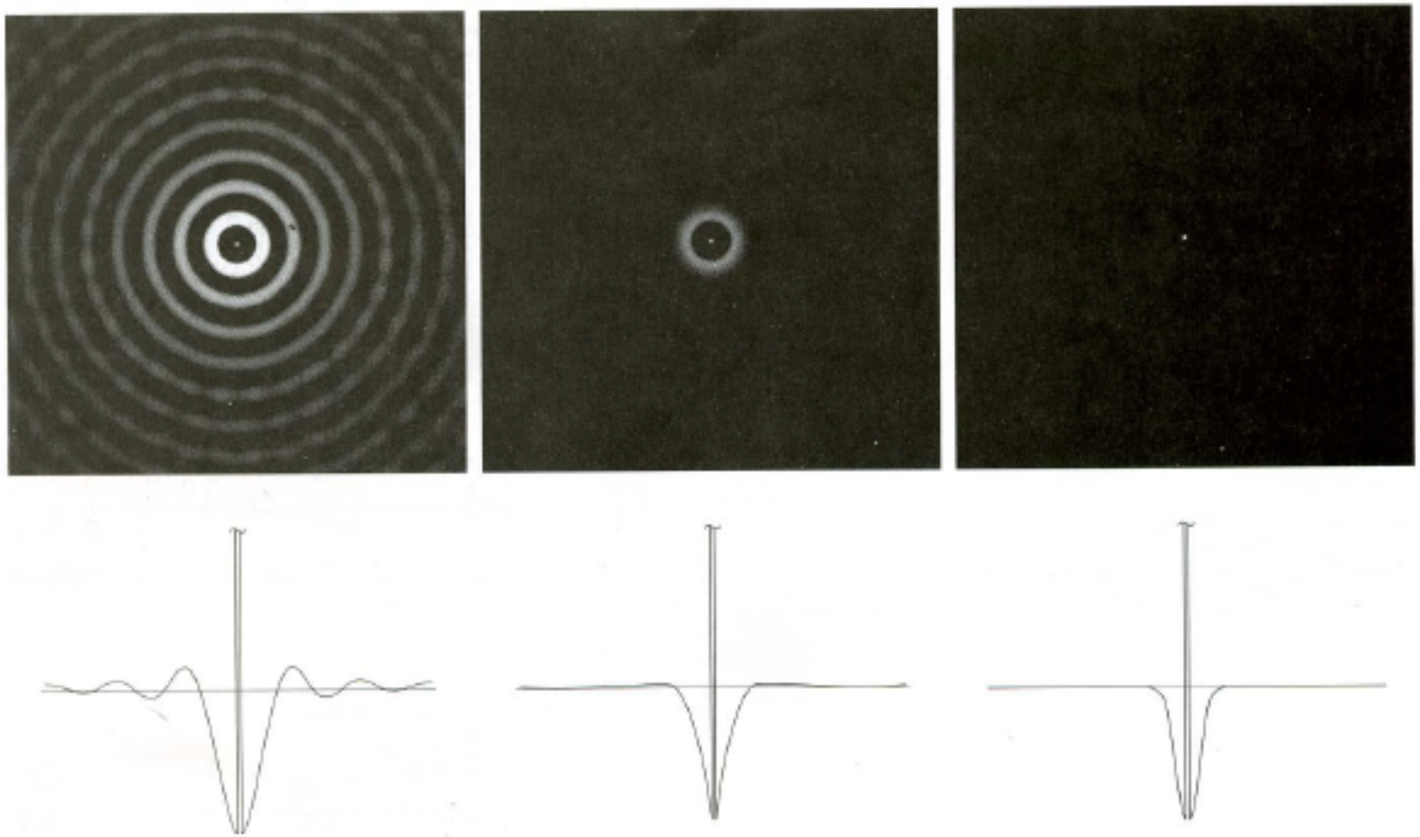


Gaussian high pass $D_0 = 15$

Gaussian high pass $D_0 = 30$

Gaussian high pass $D_0 = 80$

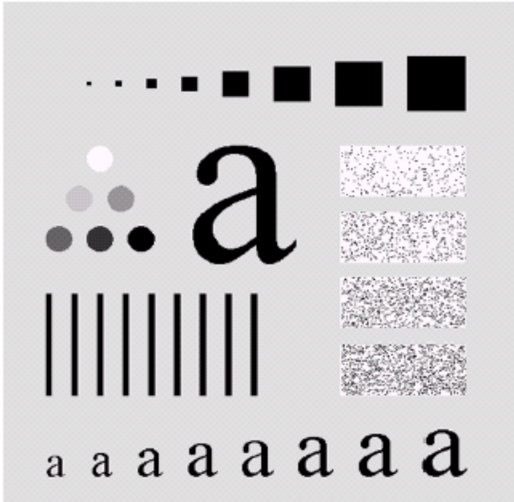
High pass filters comparison

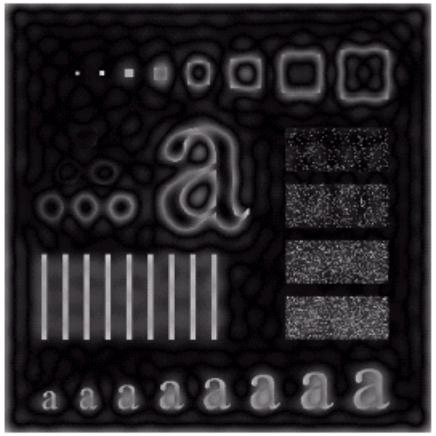


a b c

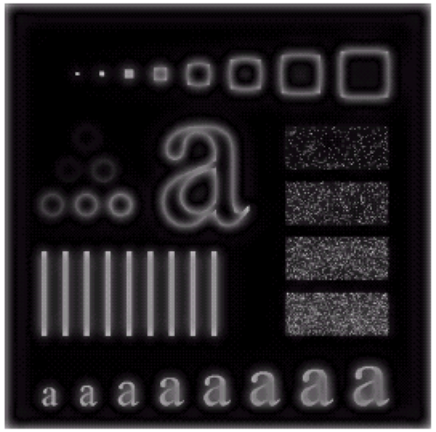
FIGURE 4.23 Spatial representations of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding gray-level profiles.

Highpass Filter Comparison

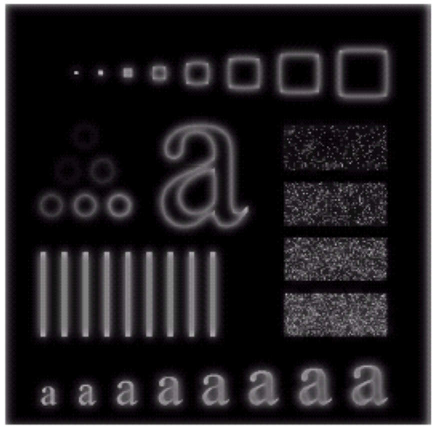




Results of ideal high pass filtering with D_0 = 15

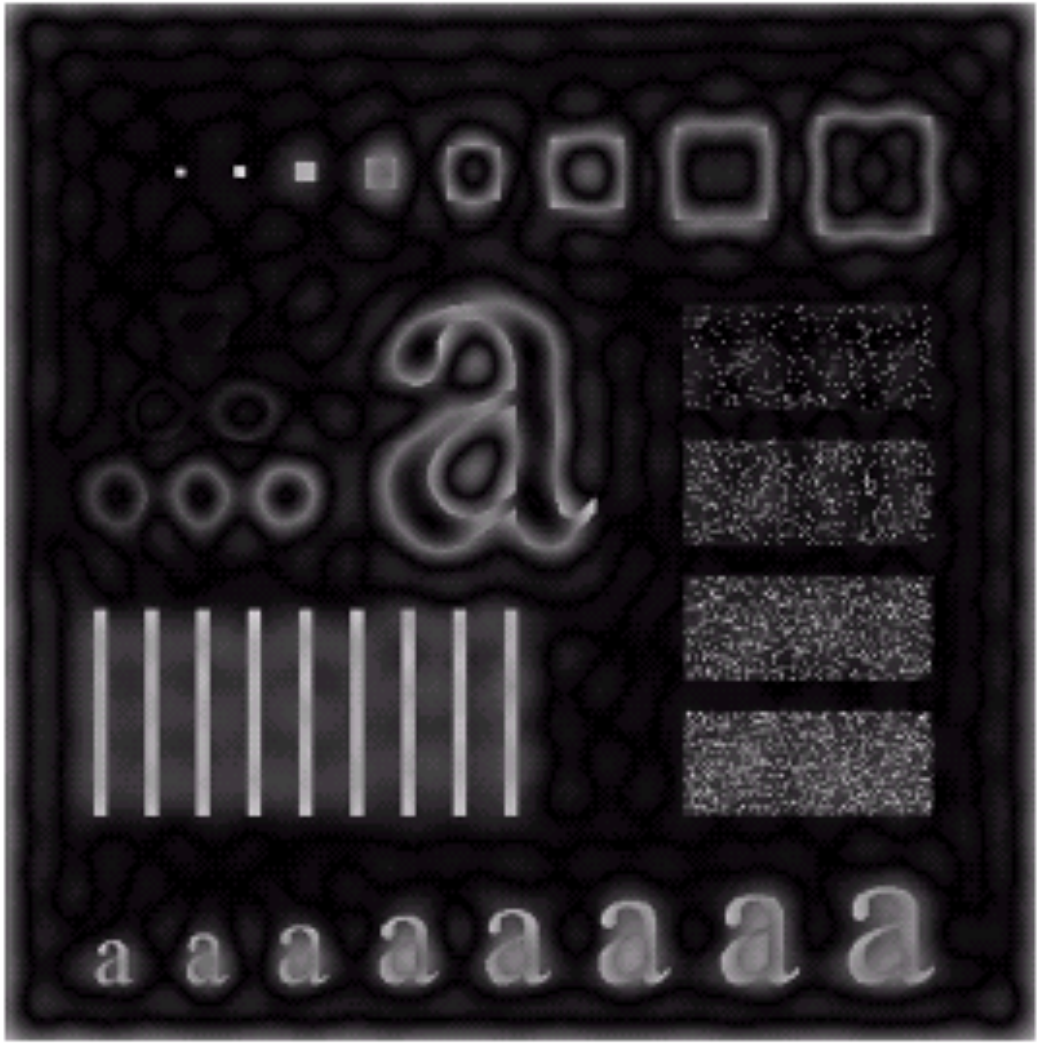


Results of Butterworth high pass filtering of order 2 with D_0 = 15



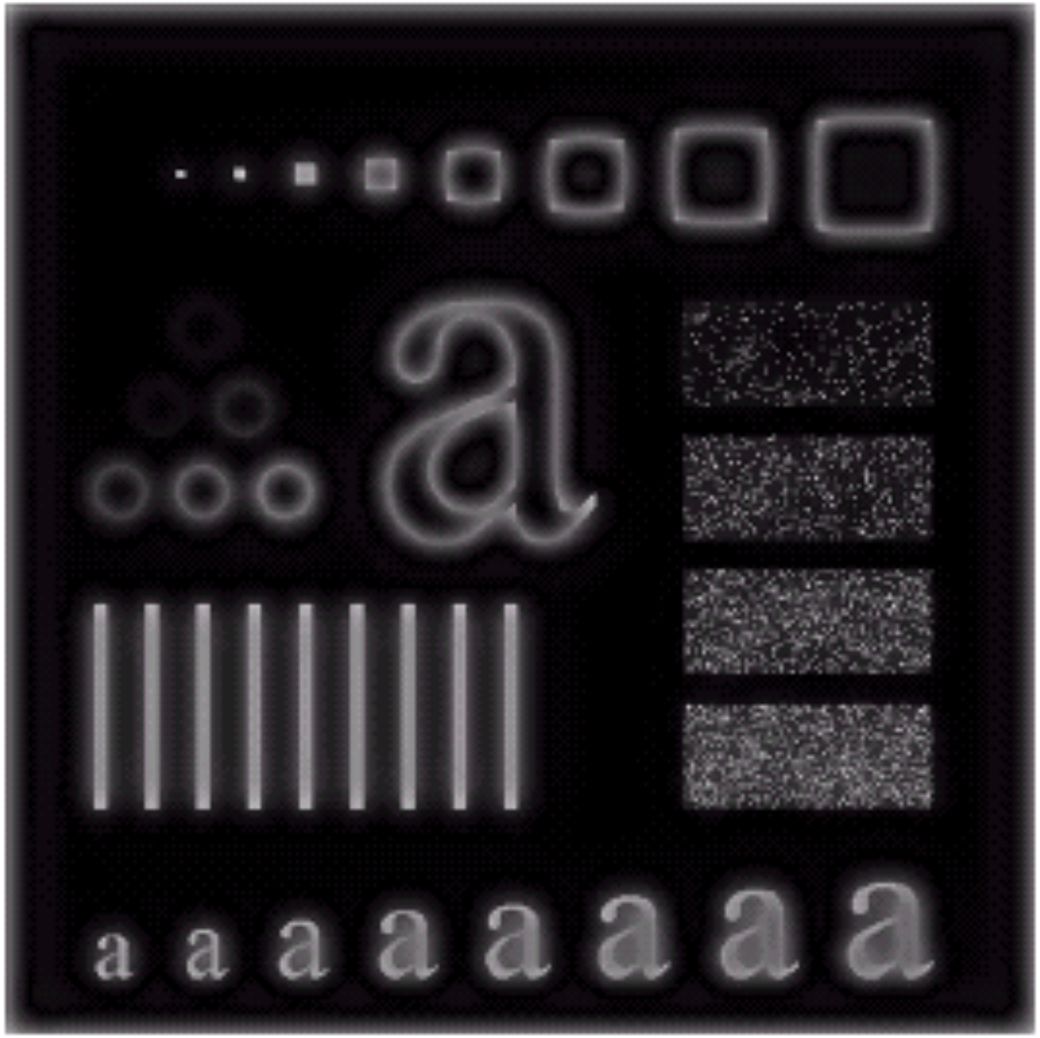
Results of Gaussian high pass filtering with D_0 = 15

Highpass Filter Comparison



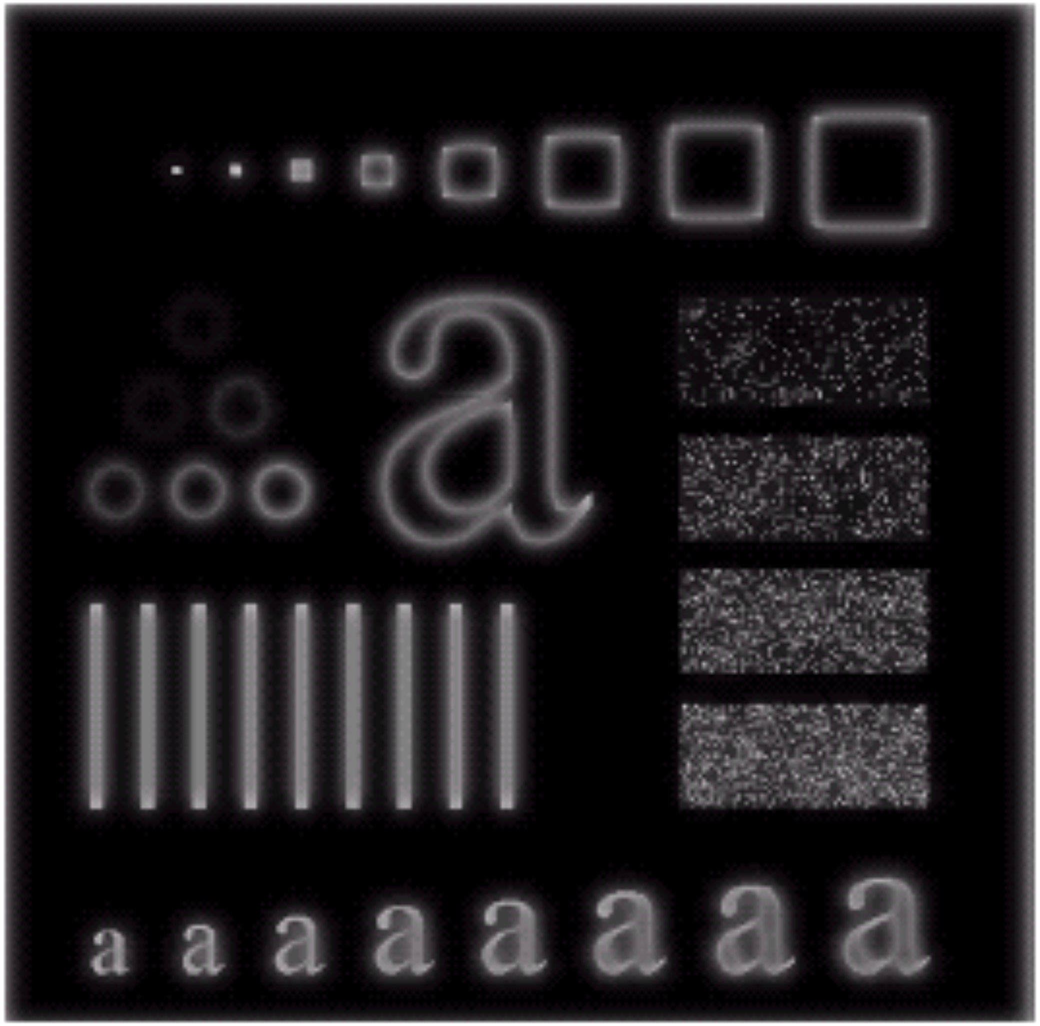
Results of ideal high pass filtering with D_0 = 15

Highpass Filter Comparison



Results of Butterworth high pass filtering of order 2 with D_0 = 15

Highpass Filter Comparison



Results of Gaussian high pass filtering with D_0 = 15

Laplacian filter

- It can be shown that

$$\Im\left[\frac{d^n f(x)}{dx^n}\right] = (ju)^n F(u).$$

From this it follows tnat

$$\Im\left[\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}\right] = (ju)^2 F(u,v) + (jv)^2 F(u,v)$$
$$= -(u^2 + v^2)F(u,v).$$

The expression on the left side is the laplacian of the function f, thus

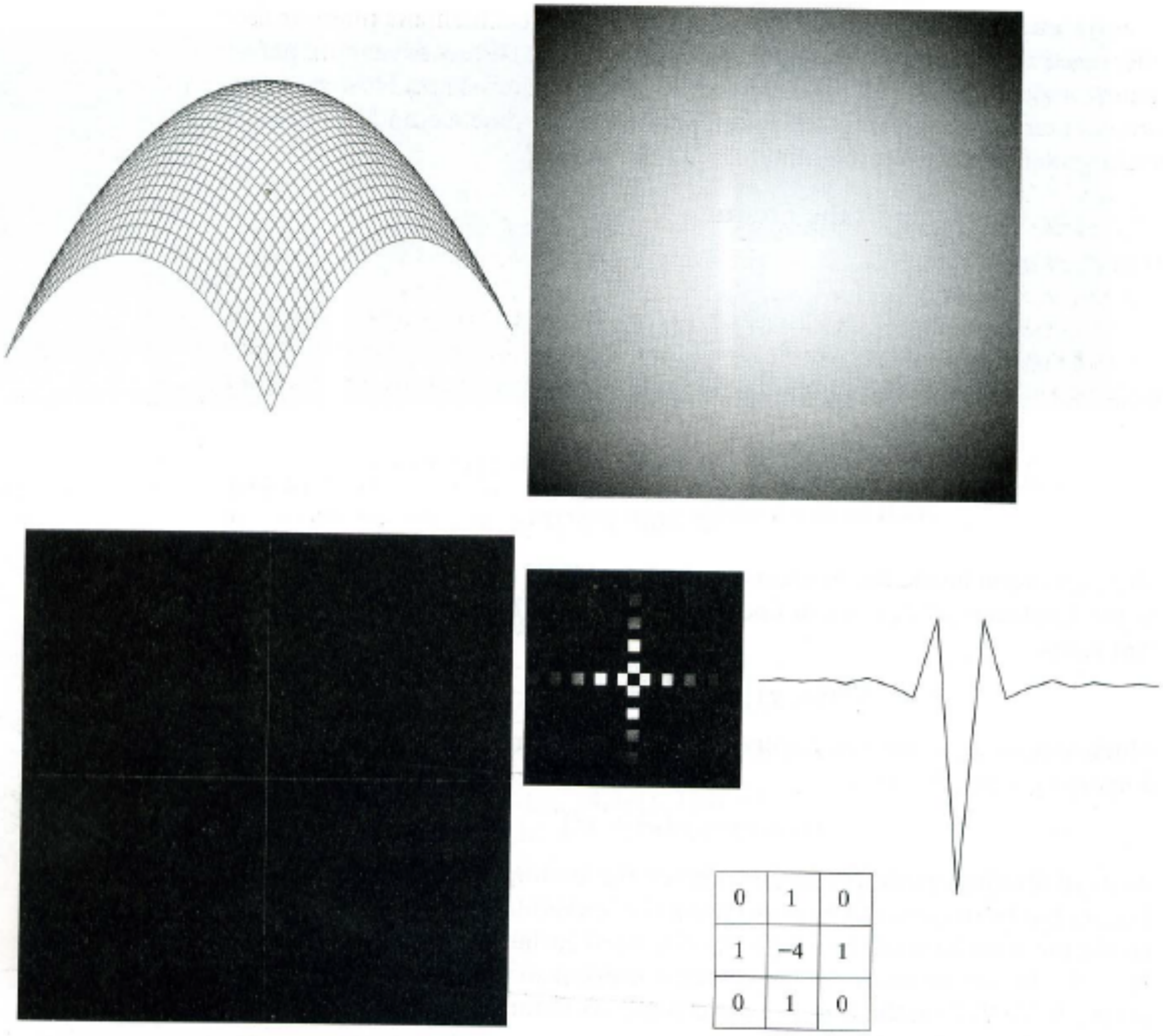
$$\Im\left[\nabla^2 f(x,y)\right] = -(u^2 + v^2)F(u,v),$$

Laplacian filter

- Thus, the Laplacian can be implemented in the Fourier domain by using the filter

$$H(u, v) = -(u^2 + v^2).$$

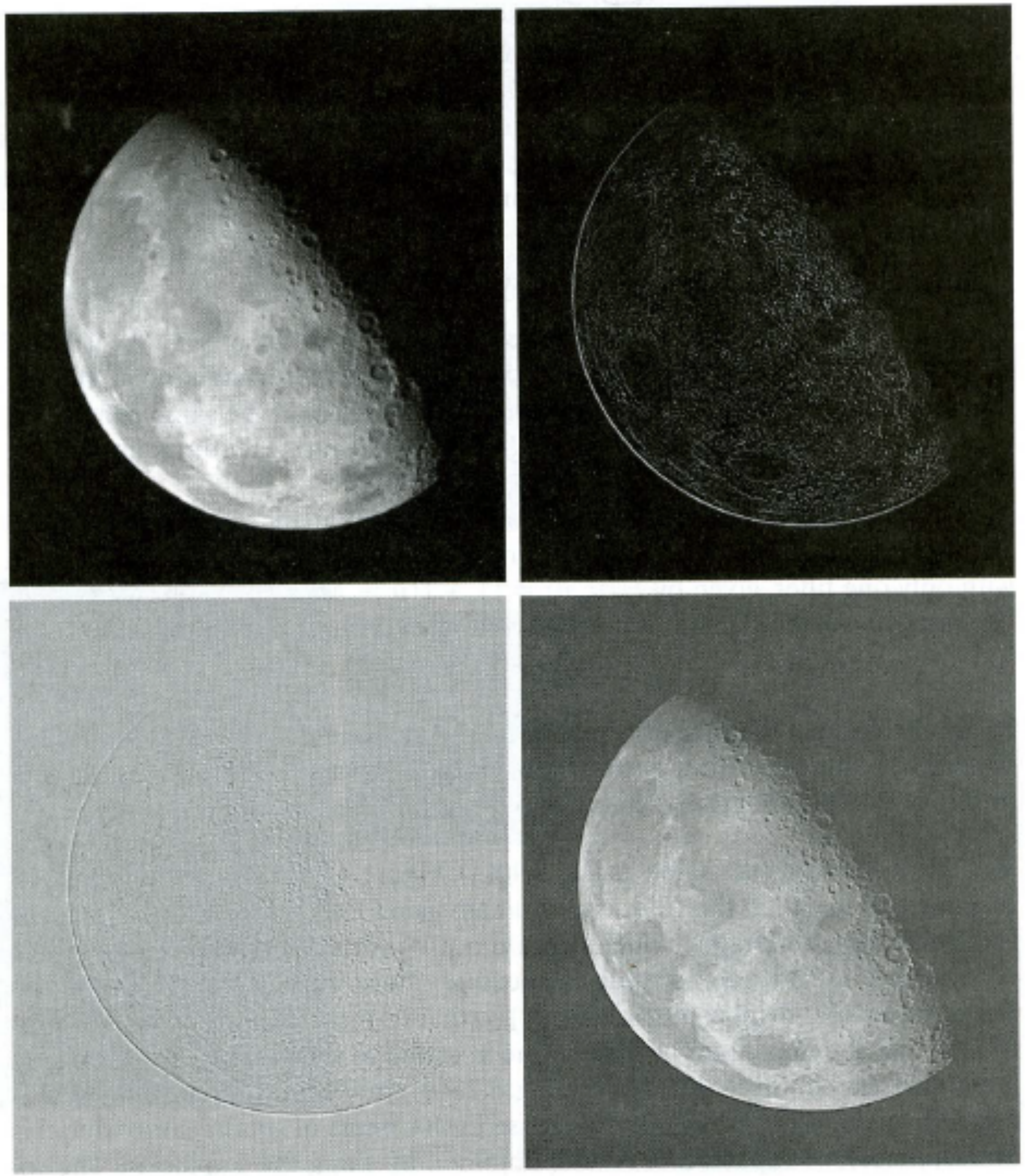
Laplacian filter



a b
c d e
f

FIGURE 4.27 (a) 3-D plot of Laplacian in the frequency domain. (b) Image representation of (a). (c) Laplacian in the spatial domain obtained from the inverse DFT of (b). (d) Zoomed section of the origin of (c). (e) Gray-level profile through the center of (d). (f) Laplacian mask used in Section 3.7.

Laplacian filter



a b
c d

FIGURE 4.28
(a) Image of the North Pole of the moon.
(b) Laplacian filtered image.
(c) Laplacian image scaled.
(d) Image enhanced by using Eq. (4.4-12).
(Original image courtesy of NASA.)

High boost filtering

- A special case of unsharp masking
- Idea: HP filters cut the zero frequency component, namely the mean value. The resulting image is zero mean and looks very dark
- High boost filtering "sums" the original image to the result of HPF in order to get an image with sharper (emphasized) edges but with same range of gray values as the original one
- In formulas
  - High pass $$f_{\mathrm{hp}}(x, y) = f(x, y) - f_{\mathrm{lp}}(x, y).$$
  - High boost $$f_{\mathrm{hb}} = A f(x, y) - f_{\mathrm{lp}}(x, y).$$ Signal domain

$$H_{\mathrm{hp}}(u, v) = 1 - H_{\mathrm{lp}}(u, v).$$ Frequency domain

High boost filtering

Reworking the formulas

$$f_{hb}(x,y) = Af(x,y) - f_{lp}(x,y)$$

$$f_{hb}(x,y) = Af(x,y) - f(x,y) + f(x,y) - f_{lp}(x,y) =$$

$$= (A-1)f(x,y) + f_{hp}(x,y)$$

For A=1 the high-boost corresponds to the HP
For A>1 the contribution of the original image becomes larger

High boost filtering

- Note: high pass filtering is also called unsharp filtering
- In the Fourier domain

$$H_{hb}(u,v) = (A-1)H(u,v) + H_{hp}(u,v)$$

High boost filtering

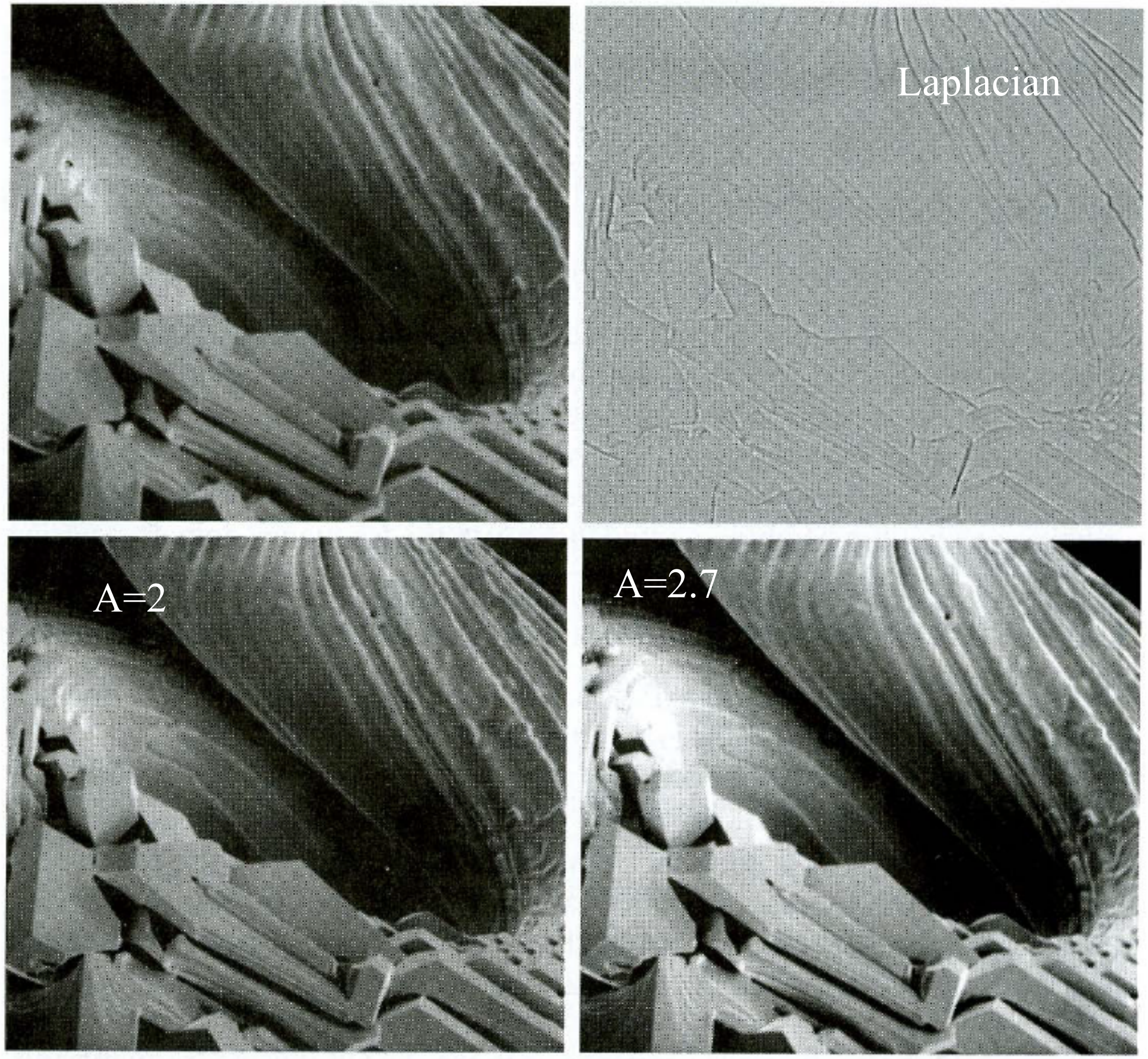
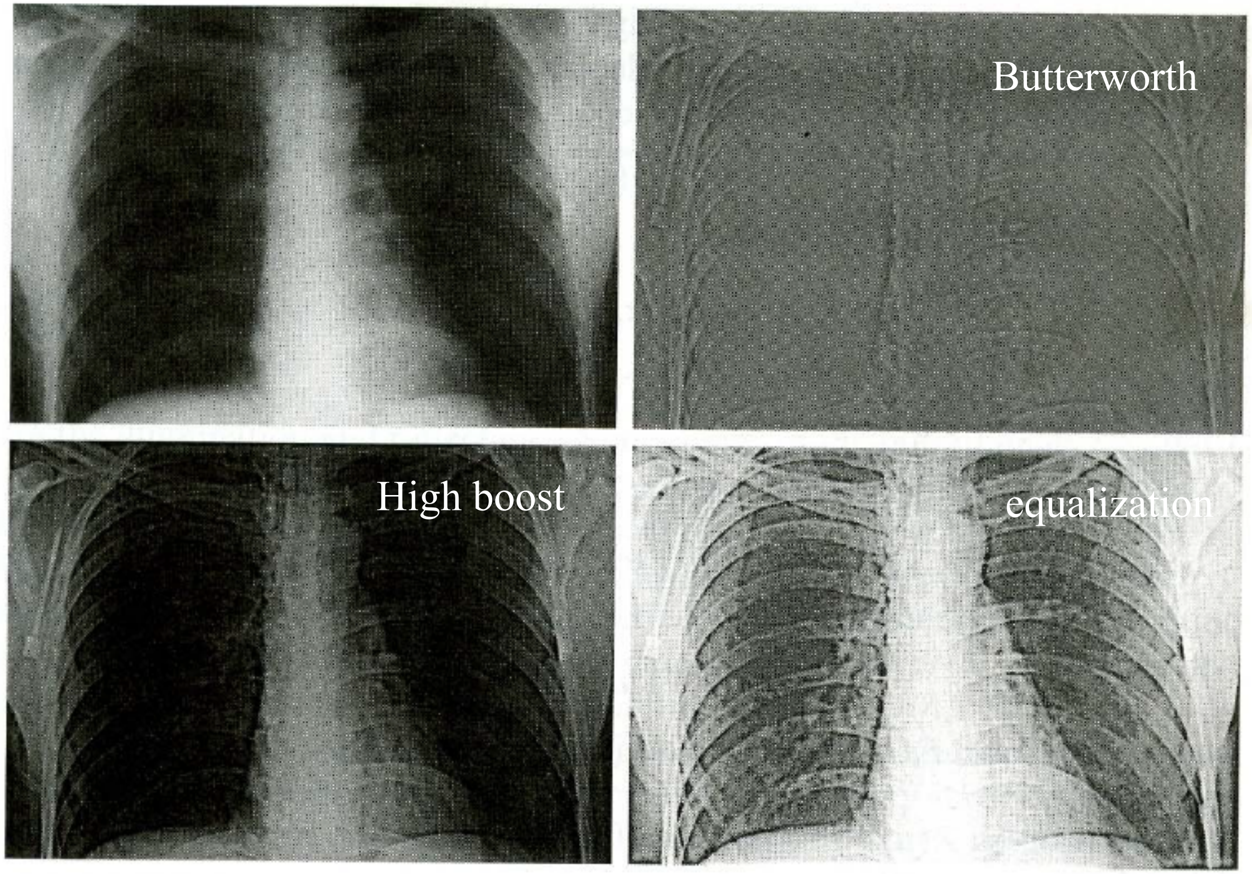


FIGURE 4.29 Same as Fig. 3.43, but using frequency domain filtering. (a) Input image. (b) Laplacian of (a). (c) Image obtained using Eq. (4.4-17) with $A = 2$. (d) Same as (c), but with $A = 2.7$. (Original image courtesy of Mr. Michael Shaffer, Department of Geological Sciences, University of Oregon, Eugene.)

High boost + histogram equalization



a b
c d

FIGURE 4.30 (a) A chest X-ray image. (b) Result of Butterworth highpass filtering. (c) Result of high-frequency emphasis filtering. (d) Result of performing histogram equalization on (c). (Original image courtesy Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

Rest of Chapter 4

- 2D Fourier transform and properties
- Convolution and correlation
- Need for padding
- Fast Fourier Transform (FFT)