Università degli Studi di Verona
Corso di Laurea Magistrale in Matematica Applicata

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Some exercises of functional analysis - A.A. 2012/13 - N.1

Pb 1. Let $C_n =]2^{-n}, 2^{1-n}\lambda[$, $n \geq 1$. For which $\lambda > \frac{1}{2}$ it holds $\mathcal{L}^1(\bigcup_{n=1}^{\infty} C_n) < 1$?

Pb 2. Let (q_n) a sequence having $\mathbb{Q} \cap [0,1]$ as image and let

$$A = \bigcup_{n=1}^{\infty}]q_n - 4^{-n}, q_n + 4^{-n}[,$$

and let $K = [0,1] \setminus A$. Prove that K is compact, $\mathcal{L}^1(K) > 0$ and $\mathrm{int}(K) = \emptyset$.

Pb 3. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function, strictly increasing, with $\varphi(1) = 1$. Let $f : [0,1] \to \mathbb{R}$ be integrable with $\int_0^1 \varphi \circ f \leq 1$. Justify the fact that $\int_0^1 f \leq 1$ (Jensen inequality was stated).

Pb 4. Let $f : E \to \mathbb{R}^+$ be integrable with $\int_E f \leq 9$. Show that $\mathcal{L}^1(\{x \in E : f(x) \geq 3\}) \leq 3$.

Pb 5. Study the sequences of functions $f_n : \mathbb{R} \to \mathbb{R}$

$$f_n(x) = \begin{cases} 1 & \text{for } n \leq x \leq n+1, \\ 0 & \text{elsewhere,} \end{cases} \qquad f_n(x) = \begin{cases} 0 & \text{for } x \leq -\frac{1}{n}, \\ n + n^2 x & \text{for } -\frac{1}{n} \leq x \leq 0, \\ n - n^2 x & \text{for } 0 \leq x \leq \frac{1}{n}, \\ 0 & \text{for } x \geq \frac{1}{n}. \end{cases}$$

Pb 6. Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = -\chi_{]-\infty,0[}(x), \qquad g(x) = \chi_{]0,+\infty[}(x).$$

Prove that f, g are $\mathcal{L}^1$-integrable but the sum $(f+g)$ is not.

Pb 7. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = -\chi_{]n,+\infty[}(x)$$

Prove that the sequence (f_n) converges pointwise (increasing) to zero, but for every $n \in \mathbb{N}$

$$\int_{\mathbb{R}} f_n(x)dx = -\infty.$$

Compare this with the statement of the Monotone Convergence Theorem.

Pb 8. Let μ be an outer measure on $\mathbb{R}^n$, (f_n) a sequence of integrable functions from $\mathbb{R}^n$ to $\bar{\mathbb{R}}$ such that $f_n \leq f_{n+1}$ for all $h \in \mathbb{N}$. Let $f : \mathbb{R}^n \to \bar{\mathbb{R}}$ denote the pointwise limit of (f_n) and assume that f_0 is summable. Prove that

$$\lim_n \int f_n d\mu = \int f d\mu$$

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