2D Fourier Transform

Overview

- Signals as functions (1D, 2D)
  - Tools
- 1D Fourier Transform
  - Summary of definition and properties in the different cases
    - CTFT, CTFS, DTFS, DTFT
    - DFT
- 2D Fourier Transforms
  - Generalities and intuition
  - Examples
  - A bit of theory
- Discrete Fourier Transform (DFT)
- Discrete Cosine Transform (DCT)

Signals as functions

1. Continuous functions of real independent variables
   - 1D: $f=f(x)$
   - 2D: $f=f(x,y)$ x,y
   - Real world signals (audio, ECG, images)
2. Real valued functions of discrete variables
   - 1D: $f=f[k]$
   - 2D: $f=f[i,j]$
   - *Sampled* signals
3. Discrete functions of discrete variables
   - 1D: $y=y[k]$
   - 2D: $y=y[i,j]$
   - *Sampled and quantized* signals
   - For ease of notations, we will use the same notations for 2 and 3

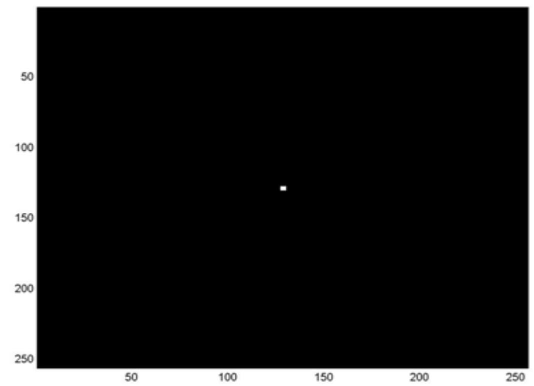
Images as functions

- Gray scale images: 2D functions
  - Domain of the functions: set of (x,y) values for which $f(x,y)$ is defined : 2D lattice $[i,j]$ defining the pixel locations
  - Set of values taken by the function : gray levels

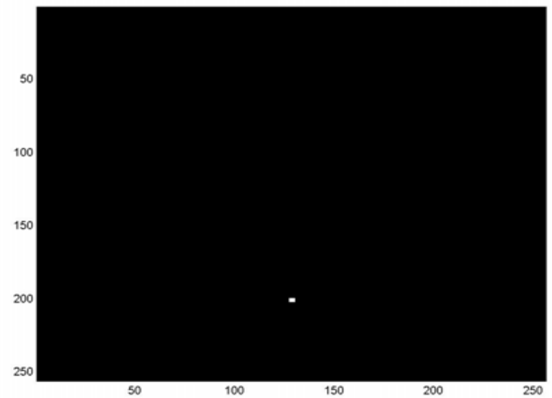
- Digital images can be seen as functions defined over a discrete domain $\{i,j: 0<i<I, 0<j<J\}$
  - I,J: number of rows (columns) of the matrix corresponding to the image
  - $f=f[i,j]$: gray level in position $[i,j]$

Example 1: δ function

$$\delta[i, j] = \begin{cases} 1 & i = j = 0 \\ 0 & i, j \neq 0; i \neq j \end{cases}$$



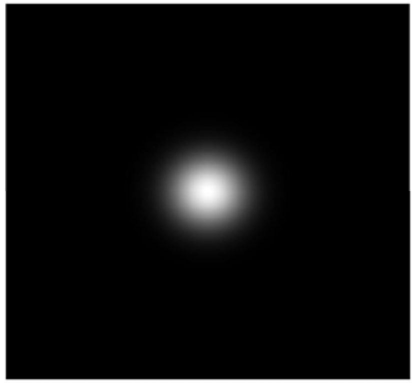
$$\delta[i, j - J] = \begin{cases} 1 & i = 0; j = J \\ 0 & otherwise \end{cases}$$



Example 2: Gaussian

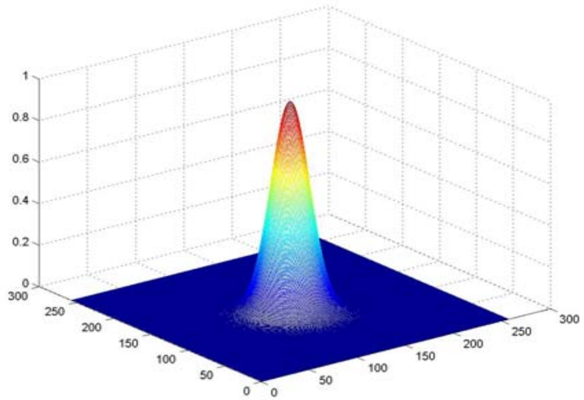
Continuous function

$$f(x, y) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{x^2+y^2}{2\sigma^2}}$$

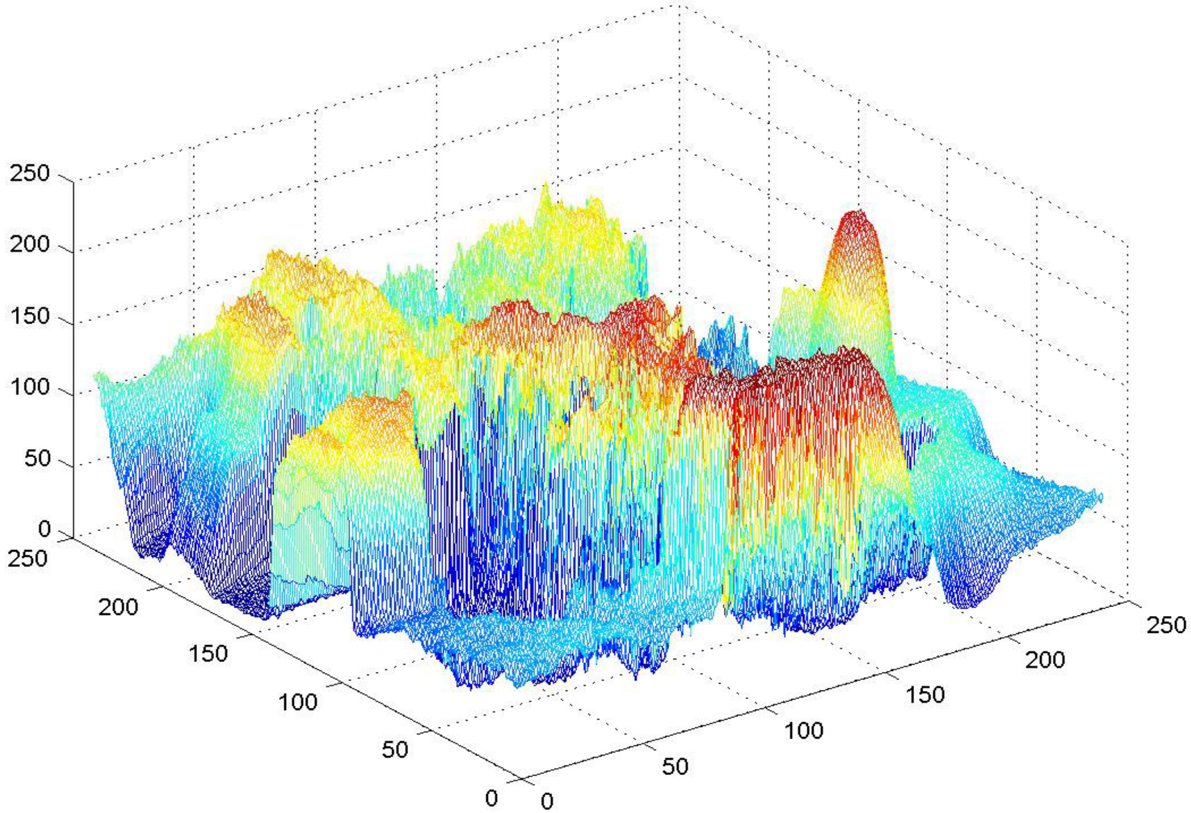


Discrete version

$$f[i, j] = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{i^2+j^2}{2\sigma^2}}$$



Example 3: Natural image



Example 3: Natural image

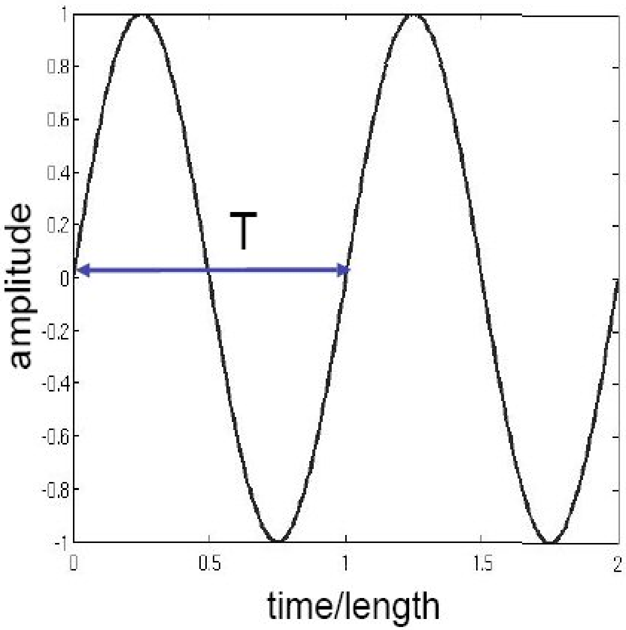
Fourier Transform

- Different formulations for the different classes of signals
  - Summary table: Fourier transforms with various combinations of continuous/discrete time and frequency variables.
  - Notations:
    - CTFT: continuous time FT
    - DTFT: Discrete Time FT
    - CTFS: CT Fourier Series (summation synthesis)
    - DTFS: DT Fourier Series (summation synthesis)
    - P: periodical signals
    - T: sampling period
    - ω_s: sampling frequency ($\omega_s=2\pi/T$)
    - For DTFT: $T=1 \rightarrow \omega_s=2\pi$

1D FT: basics

- Define frequency
  = 1/T
  cycles per unit time
  cycles per unit distance

- Here f = 1



Fourier Transform: Concept

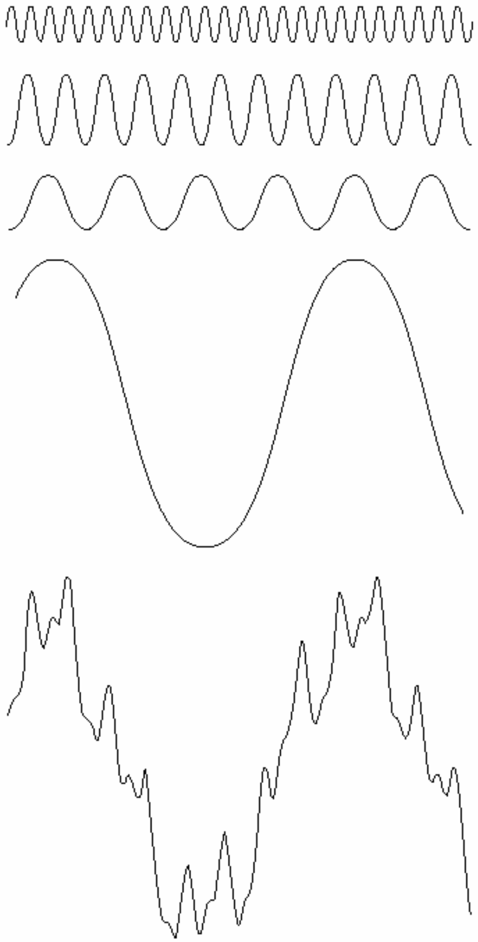


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

- A signal can be represented as a weighted sum of sinusoids.

- Fourier Transform is a change of basis, where the basis functions consist of sines and cosines (complex exponentials).

Fourier Transform

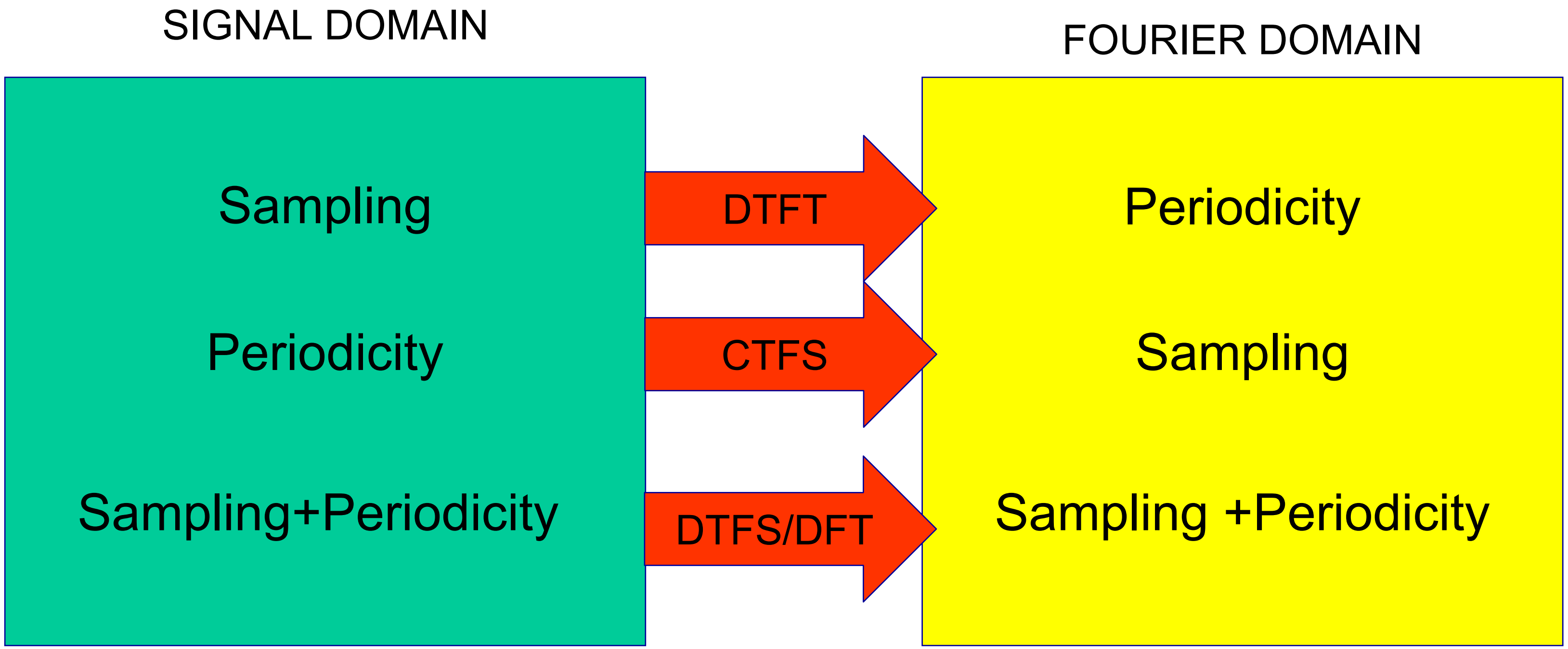
- Cosine/sine signals are easy to define and interpret.
- However, it turns out that the analysis and manipulation of sinusoidal signals is greatly simplified by dealing with related signals called complex exponential signals.
- A complex number has real and imaginary parts: $z = x + j\,y$
- A complex exponential signal:

$$r\,e^{j\alpha} = r\left(\cos\alpha + j\sin\alpha\right)$$

Overview

Transform	Time	Frequency	Analysis/Synthesis	Duality
(Continuous Time) Fourier Transform (CTFT)	C	C	$F(\omega) = \int_t f(t) e^{-j\omega t} dt$ $f(t) = \frac{1}{2\pi} \int_\omega F(\omega) e^{j\omega t} d\omega$	Self-dual
(Continuous Time) Fourier Series (CTFS)	C P	D	$F[k] = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-j2\pi kt/T} dt$ $f(t) = \sum_k F[k] e^{j2\pi kt/T}$	Dual with DTFT
Discrete Time Fourier Transform (DTFT)	D	C P	$F\left(e^{j\omega t}\right) = \sum_n f[n] e^{-j2\pi\omega n/\omega_s}$ $f[n] = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} F\left(e^{j\omega t}\right) e^{j2\pi\omega n/\omega_s} d\omega$	Dual with CTFS
Discrete Time Fourier Series (DTFS)	D P	D P	$F[k] = \frac{1}{N} \sum_{n=0}^{N-1} f[n] e^{-j2\pi kn/T}$ $f[n] = \sum_{n=0}^{N-1} F[k] e^{j2\pi kn/T}$	Self dual

Dualities


SIGNAL DOMAIN
FOURIER DOMAIN
Sampling
DTFT
Periodicity
Periodicity
CTFS
Sampling
Sampling+Periodicity
DTFS/DFT
Sampling +Periodicity

Discrete time signals

- Sequences of samples
- f[k]: sample values
- Assumes a unitary spacing among samples (T_s=1)
- Normalized frequency Ω
- Transform
  - DTFT for NON periodic sequences
  - CTFS for periodic sequences
  - DFT for *periodized* sequences
- All transforms are 2π periodic

- *Sampled* signals
- f(kT_s): sample values
- The sampling interval (or period) is T_s
- Non normalized frequency ω
- Transform
  - DTFT
  - CSTF
  - DFT
  - BUT accounting for the fact that the sequence values have been generated by sampling a real signal → $f_k = f(kT_s)$
- All transforms are periodic with period ω_s

$$\Omega = \omega T_s$$

CTFT

- Continuous Time Fourier Transform
- Continuous time *a-periodic* signal
- Both time (space) and frequency are continuous variables
  - NON normalized frequency ω is used
- Fourier integral can be regarded as a Fourier series with fundamental frequency approaching zero
- Fourier spectra are continuous
  - A signal is represented as a sum of sinusoids (or exponentials) of all frequencies over a continuous frequency interval

Fourier *integral*

$$F(\omega) = \int_t f(t) e^{-j\omega t} dt$$ analysis

$$f(t) = \frac{1}{2\pi} \int_\omega F(\omega) e^{j\omega t} d\omega$$ synthesis

CTFT: change of notations

- Fourier Transform of a 1D continuous signal

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-j\omega x} dx$$

"Euler's formula"

$$e^{-j\omega x} = \cos(\omega x) - j\sin(\omega x)$$

- Inverse Fourier Transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega x} d\omega$$

Change of notations:

$$\omega \rightarrow 2\pi u$$

$$\begin{cases} \omega_x \rightarrow 2\pi u \\ \omega_y \rightarrow 2\pi v \end{cases}$$

Then CTFT becomes

- Fourier Transform of a 1D continuous signal

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$$

"Euler's formula" $\quad e^{-j2\pi ux} = \cos(2\pi ux) - j\sin(2\pi ux)$

- Inverse Fourier Transform

$$f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

CTFS

- Continuous Time Fourier Series
- Continuous time periodic signals
  - The signal is periodic with period T_0
  - The transform is “sampled” (it is a series)

our notations

$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f_{T_o}(t) e^{-jn\omega_0 t} dt$$

$$f_{T_0}(t) = \sum_n D_n e^{jn\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T_0}$$

fundamental frequency

table notations

$$F[k] = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-j2\pi kt/T} dt$$

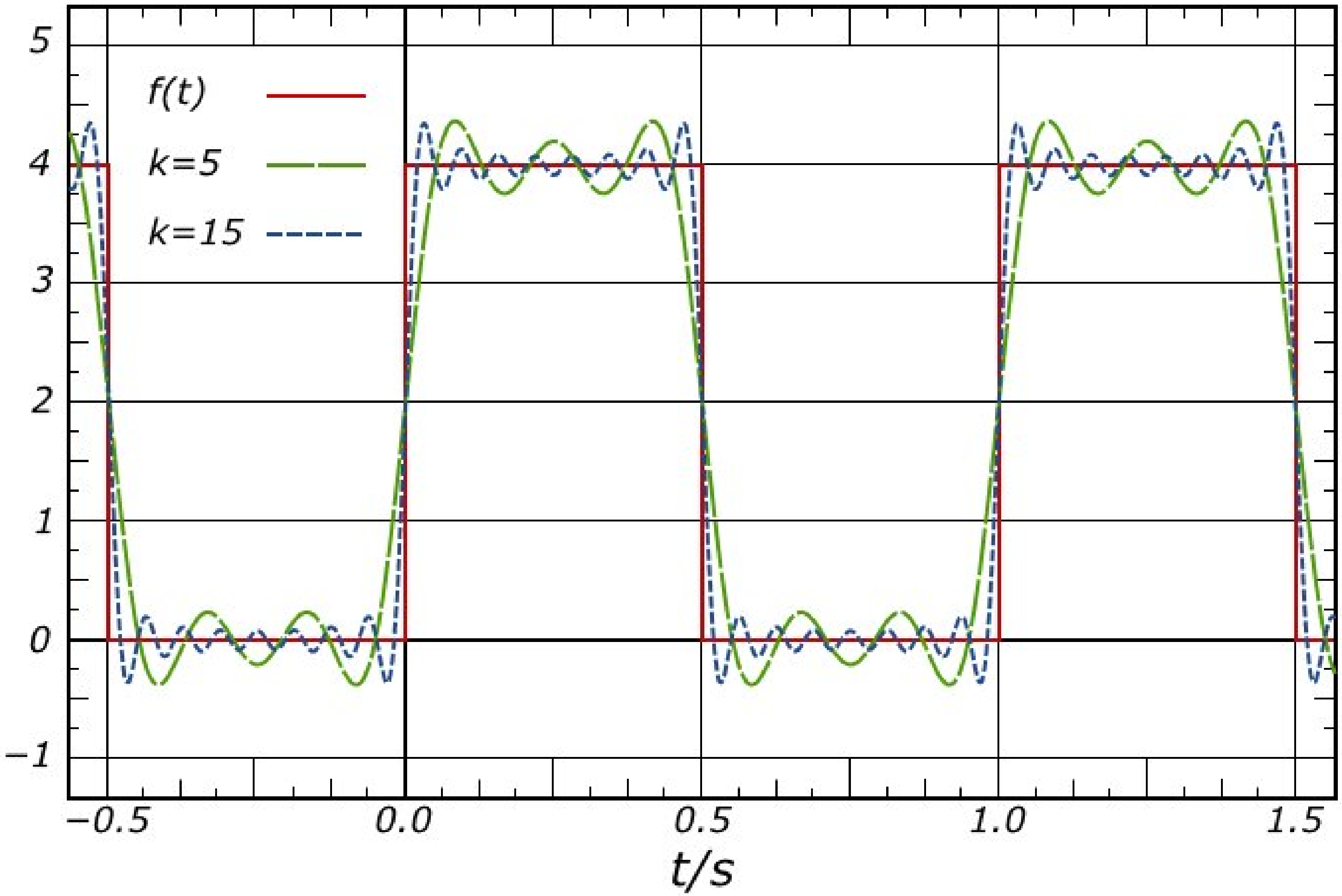
$$f(t) = \sum_k F[k] e^{j2\pi kt/T}$$

$T_0 \leftrightarrow T$
$D_n \leftrightarrow F[k]$

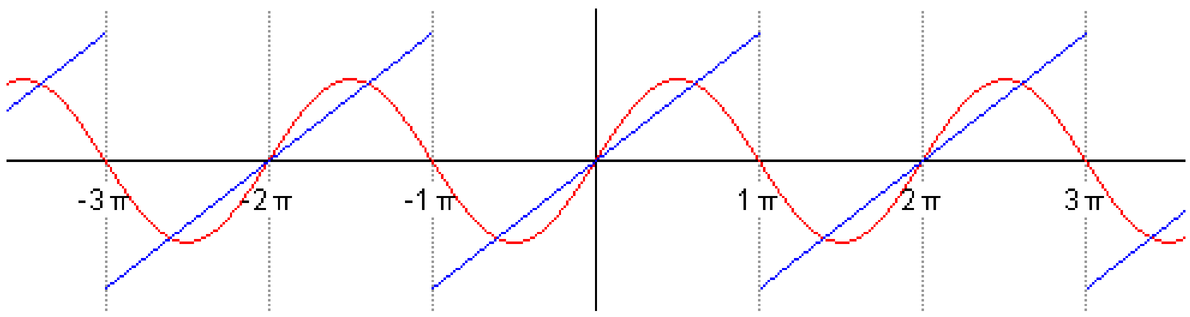
CTFS

- Representation of a continuous time signal as a sum of orthogonal components in a complete orthogonal signal space
  - The exponentials are the basis functions
- Fourier series are periodic with period equal to the fundamental in the set ($2\pi/T_0$)
- Properties
  - even symmetry → only cosinusoidal components
  - odd symmetry → only sinusoidal components

CTFS: example 1



CTFS: example 2



From sequences to discrete time signals

- Looking at the sequence as to a set of samples obtained by sampling a real signal with frequency ω_s we can still use the formulas for calculating the transforms as derived for the sequences by
  - Stratching the time axis (and thus squeezing the frequency axis if $T_s>1$)

normalized frequency (sample series) → $\Omega = \omega T_s$ ← frequency (sampled signal)

spectral periodicity in Ω $\quad 2\pi \rightarrow \omega_s = \dfrac{2\pi}{T_s} \quad$ spectral periodicity in ω

  - Enclosing the sampling interval T_s in the value of the sequence samples (DFT)

$$f_k = T_s f\left(kT_s\right)$$

DTFT

- Discrete Time Fourier Transform
- *Discrete time a-periodic* signal
- The transform is periodic and continuous with period

$$\Omega_0 = 2\pi$$

our notations

$$F(\Omega) = \sum_{k=-\infty}^{+\infty} f[k] e^{-j\Omega k}$$

$$f[k] = \frac{1}{2\pi} \int_{2\pi} F(\Omega) e^{j\Omega k} d\Omega$$

table notations

$$F\left(e^{j\omega}\right) = \sum_{n} f[n] e^{-j2\pi\omega n/\omega_s}$$

$$f[n] = \frac{1}{\omega_s} \int_{-\omega_s/2}^{\omega_s/2} F\left(e^{j\omega t}\right) e^{j2\pi\omega n/\omega_s} d\omega$$

normalized frequency

$$F(\Omega) = \bar{F}_c\left(\frac{\Omega}{T_s}\right)$$

$$\boxed{\Omega = \omega T_s}$$

$$T_s = 2\pi / \omega_s \qquad T_s = 2\pi / \omega_s$$

non normalized frequency

Discrete Time Fourier Transform (DTFT)

- $F(\Omega)$ can be obtained from $F_c(\bar{\omega})$ by replacing ω with Ω/T_s. Thus $F(\Omega)$ is identical to $F_c(\omega)$ frequency scaled by a factor $1/T_s$
  - T_s is the sampling interval in time domain
- Notations

$$F(\Omega) = \bar{F}_c\left(\frac{\Omega}{T_s}\right)$$

$$\omega_s = \frac{2\pi}{T_s} \rightarrow T_s = \frac{2\pi}{\omega_s}$$ periodicity of the spectrum

$$\omega = \frac{\Omega}{T_s} \rightarrow \Omega = \omega T_s$$ normalized frequency (the spectrum is 2π-periodic)

$$F(\Omega) \rightarrow F\left(\omega T_s\right) = F\left(2\pi\omega / \omega_s\right)$$

$$F(\Omega) = \sum_{k=-\infty}^{+\infty} f[k]e^{-j\Omega k} \rightarrow F\left(\omega T_s\right) = F(\omega) = \sum_{k=-\infty}^{+\infty} f[k]e^{-j2k\pi\omega/\omega_s}$$

DTFT: *unitary* frequency

$$\Omega = 2\pi u \qquad (\omega = 2\pi f)$$

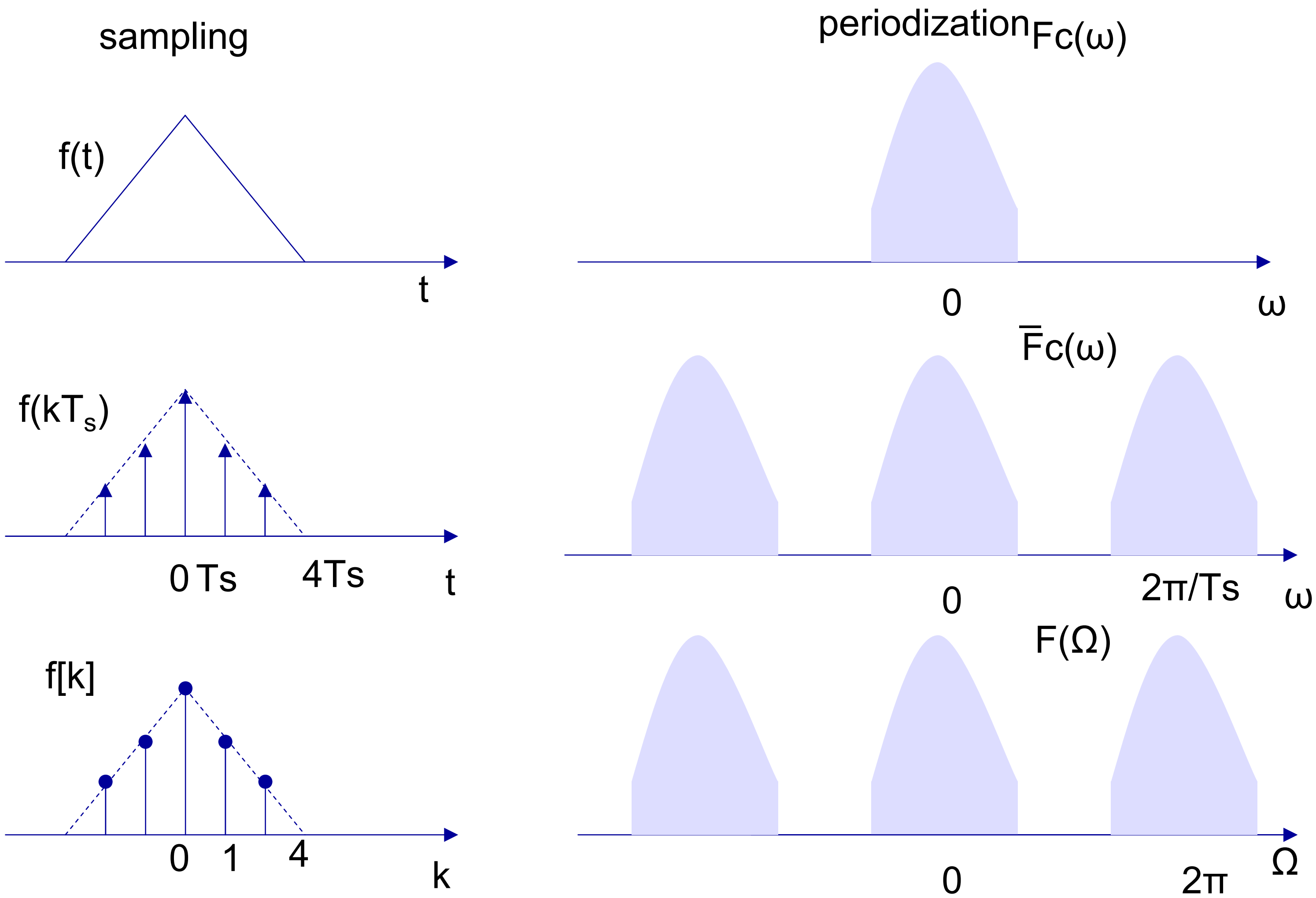
$$F(\Omega) = \sum_{k=-\infty}^{\infty} f[k]e^{-j\Omega k} \rightarrow F(u) = \sum_{k=-\infty}^{\infty} f[k]e^{-j2\pi ku}$$

$$f[k] = \frac{1}{2\pi}\int_{2\pi} F(\Omega)e^{j\Omega k}d\Omega \rightarrow f[k] = \int_{1} F(u)e^{j2\pi ku}du = \int_{-\frac{1}{2}}^{\frac{1}{2}} F(u)e^{j2\pi ku}du$$

$$\begin{cases} F(u) = \sum_{k=-\infty}^{\infty} f[k]e^{-j2\pi ku} \\ f[k] = \int_{-\frac{1}{2}}^{\frac{1}{2}} F(u)e^{j2\pi ku}du \end{cases}$$

NOTE: when $T_s=1$, $\Omega=\omega$ and the spectrum is 2π-periodic. The unitary frequency $u=2\pi/\ \Omega$ corresponds to the signal frequency $f=2\pi/\omega$. This could give a better intuition of the transform properties.

Connection DTFT-CTFT



Differences DTFT-CTFT

- The DTFT is periodic with period $\Omega_s=2\pi$ (or $\omega_s=2\pi/T_s$)
- The discrete-time exponential $e^{j\Omega k}$ has a unique waveform only for values of Ω in a continuous interval of 2π
- *Numerical computations can be conveniently performed with the Discrete Fourier Transform (DFT)*

DTFS

- Discrete Time Fourier Series
- Discrete time periodic sequences of period N_0
  - Fundamental frequency

$$\Omega_0 = 2\pi / N_0$$

our notations

$$D_r = \frac{1}{N_0} \sum_{k=0}^{N_0 - 1} f[k] e^{-jr\Omega_0 k}$$

$$f[k] = \sum_{r=0}^{N_0 - 1} D_r e^{jr\Omega_0 k}$$

table notations

$$F[k] = \frac{1}{N} \sum_{n=0}^{N-1} f[n] e^{-j2\pi kn/T}$$

$$f[k] = \sum_{n=0}^{N-1} F[k] e^{j2\pi kn/T}$$

Discrete Fourier Transform (DFT)

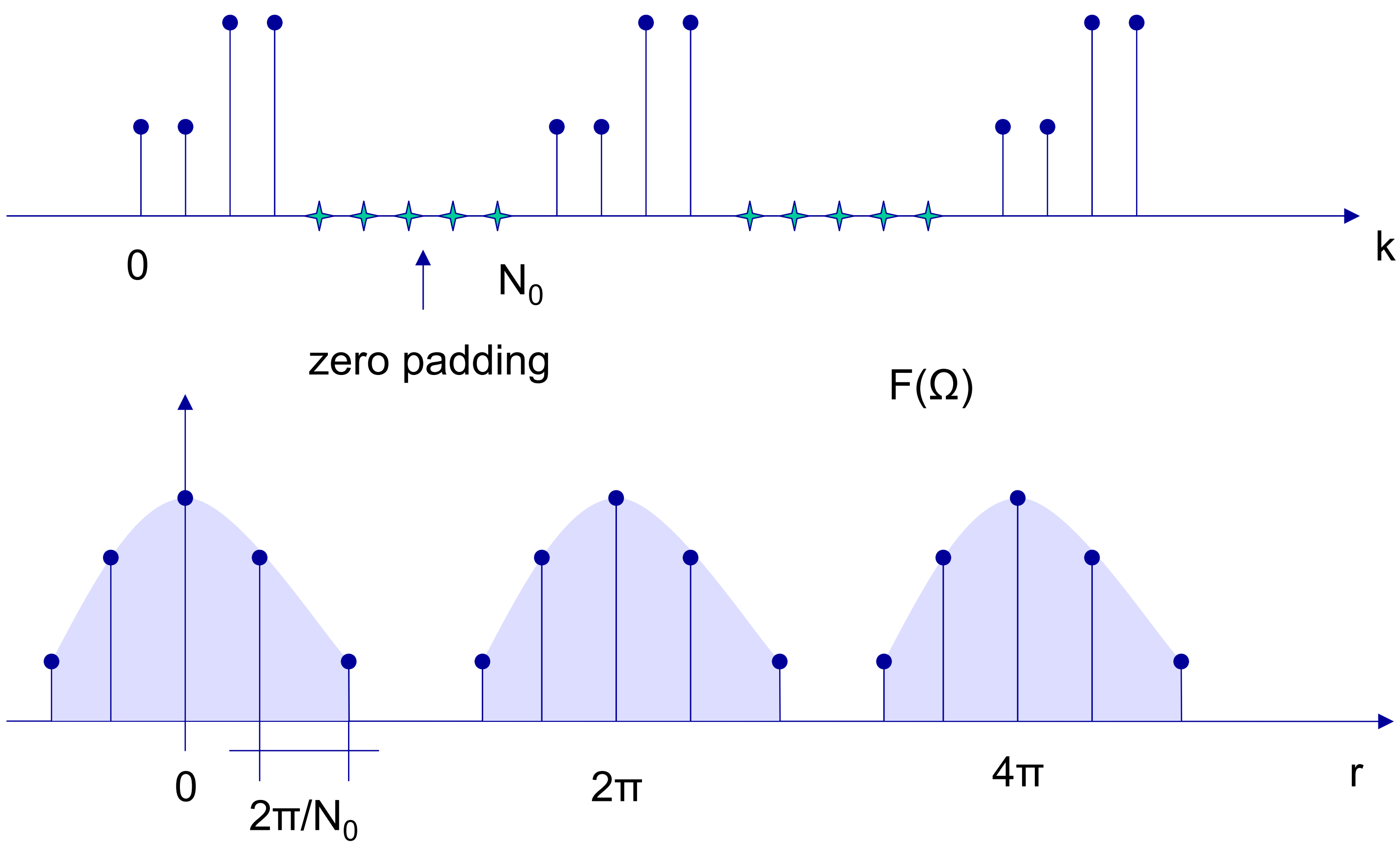
$$F_r = \sum_{k=0}^{N_0-1} f_k e^{-jr\Omega_0 k} = \sum_{k=0}^{N_0-1} f_k e^{-j\frac{2\pi}{N_0}rk}$$

$$f_k = \frac{1}{N_0}\sum_{k=0}^{N_0-1} F_r e^{jr\Omega_0 k} = \frac{1}{N_0}\sum_{k=0}^{N_0-1} F_r e^{jr\frac{2\pi}{N_0}k}$$

$$\Omega_0 = \frac{2\pi}{N_0}$$

- The DFT transforms N_0 samples of a discrete-time signal to the same number of discrete frequency samples
- The DFT and IDFT are a *self-contained*, one-to-one transform pair for a length-N_0 discrete-time signal (that is, the DFT is not merely an approximation to the DTFT as discussed next)
- However, the DFT is very often used as a practical approximation to the DTFT

DFT



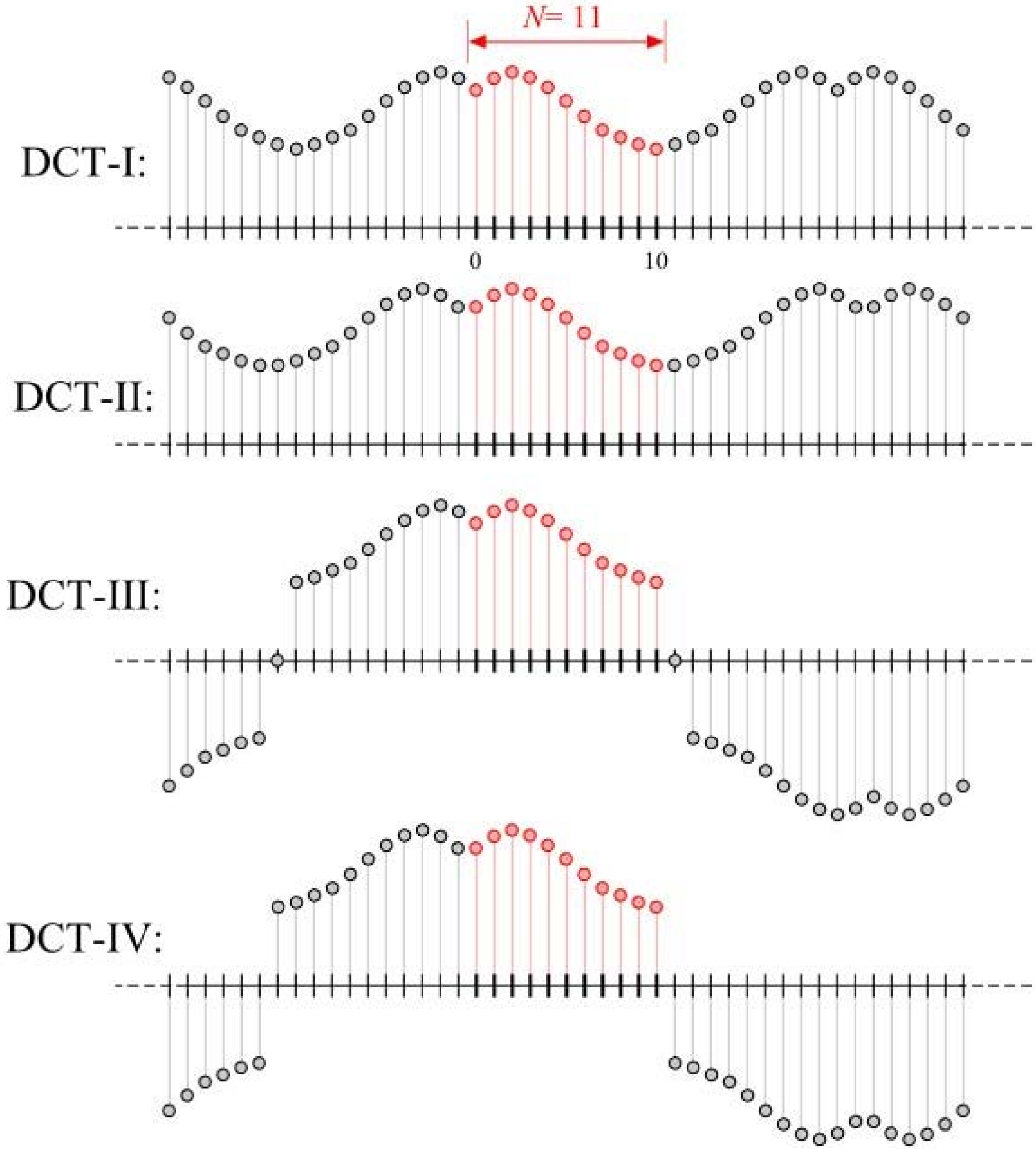
Discrete *Cosine* Transform (DCT)

- Operate on finite discrete sequences (as DFT)
- A **discrete cosine transform** (**DCT**) expresses a sequence of finitely many data points in terms of a sum of cosine functions oscillating at different frequencies
- DCT is a Fourier-related transform similar to the DFT but using only real numbers
- DCT is equivalent to DFT of roughly twice the length, operating on real data with even symmetry (since the Fourier transform of a real and even function is real and even), where in some variants the input and/or output data are shifted by half a sample
- There are eight standard DCT variants, of which four are common
- Strong connection with the Karunen-Loeven transform
  - VERY important for signal compression

DCT

- DCT implies different boundary conditions than the DFT or other related transforms
- A DCT, like a cosine transform, implies an *even periodic* extension of the original function
- Tricky part
  - First, one has to specify whether the function is even or odd at *both* the left and right boundaries of the domain
  - Second, one has to specify around *what point* the function is even or odd
    - In particular, consider a sequence *abcd* of four equally spaced data points, and say that we specify an even *left* boundary. There are two sensible possibilities: either the data is even about the sample *a*, in which case the even extension is ***dcbabcd***, or the data is even about the point *halfway* between *a* and the previous point, in which case the even extension is ***dcbaabcd*** (*a* is repeated).

Symmetries



DCT

$$X_k = \sum_{n=0}^{N_0-1} x_n \cos\left[\frac{\pi}{N_0}\left(n+\frac{1}{2}\right)k\right] \qquad k = 0,\ldots., N_0 - 1$$

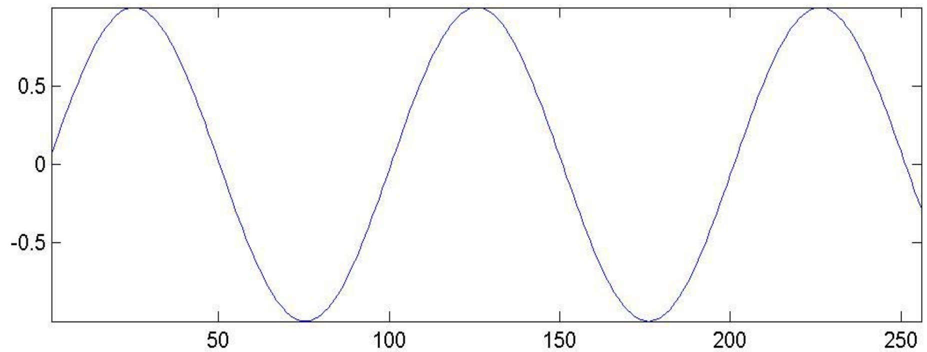
$$x_n = \frac{2}{N_0}\left\{\frac{1}{2}X_0 + \sum_{k=0}^{N_0-1} X_k \cos\left[\frac{\pi k}{N_0}\left(k+\frac{1}{2}\right)\right]\right\}$$

- *Warning*: the normalization factor in front of these transform definitions is merely a convention and differs between treatments.
  - Some authors multiply the transforms by $(2/N_0)^{1/2}$ so that the inverse does not require any additional multiplicative factor.
    - Combined with appropriate factors of √2 (see above), this can be used to make the transform matrix orthogonal.

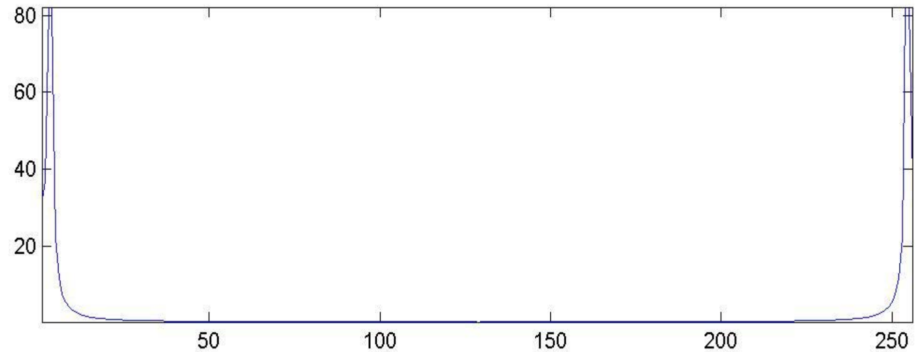
Sinusoids

- Frequency domain characterization of signals

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt$$

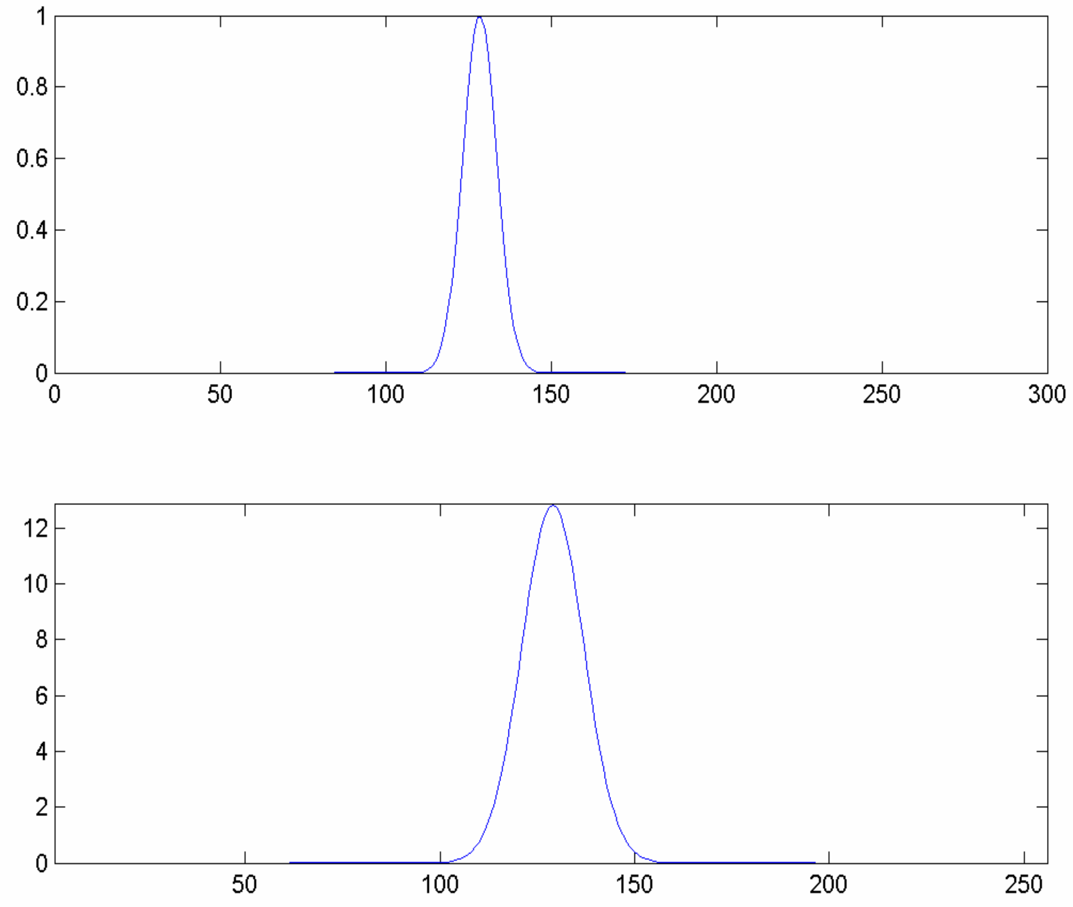


Signal domain



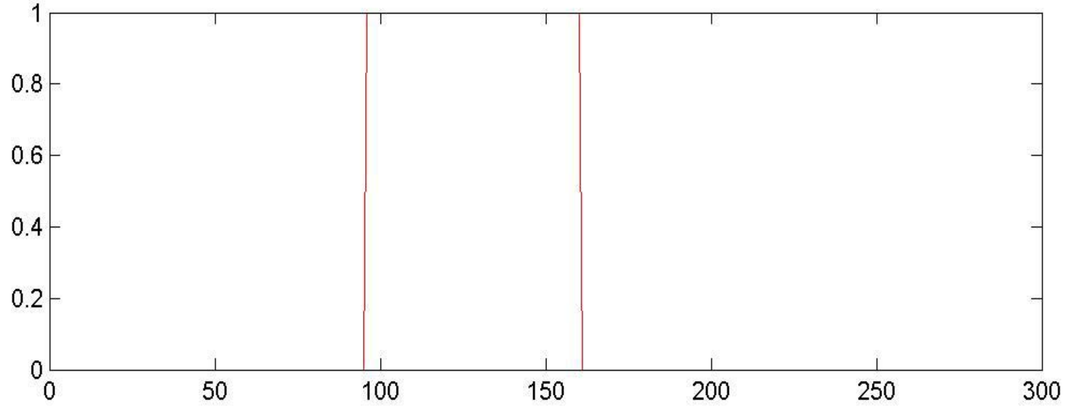
Frequency domain

Gaussian

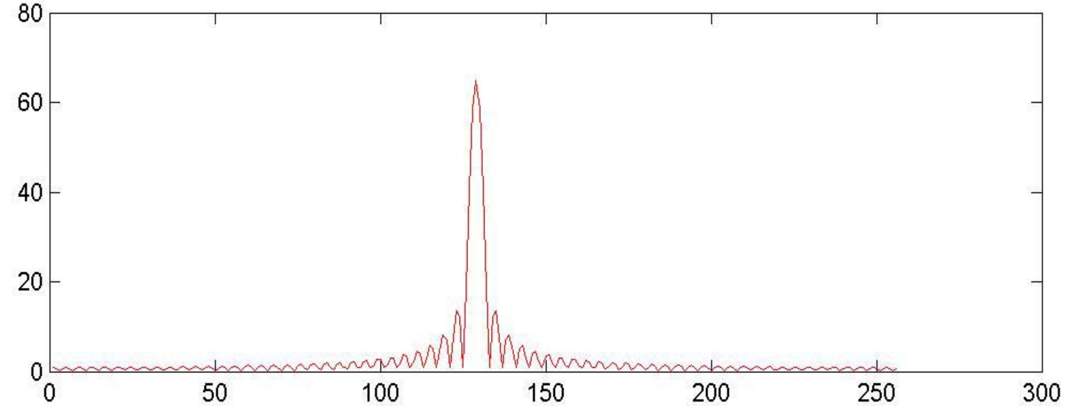


1
0.8
0.6
0.4
0.2
0
0
50
100
150
200
250
300
12
10
8
6
4
2
0
50
100
150
200
250

rect



sinc function



Images vs Signals

1D

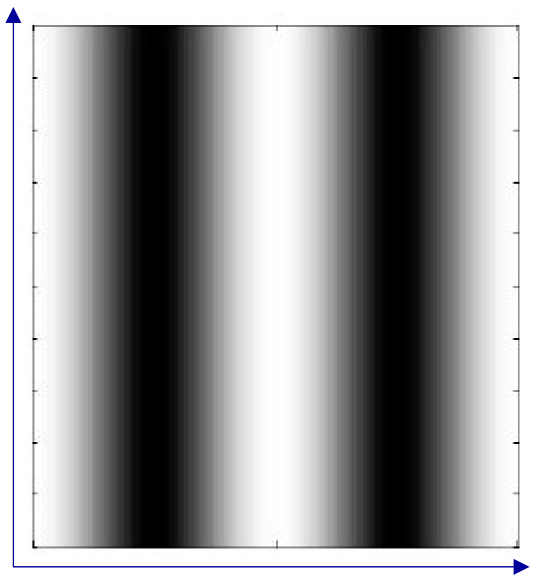
- Signals
- Frequency
  - Temporal
  - Spatial
- Time (space) frequency characterization of signals
- Reference space for
  - Filtering
  - Changing the sampling rate
  - Signal analysis
  - ….

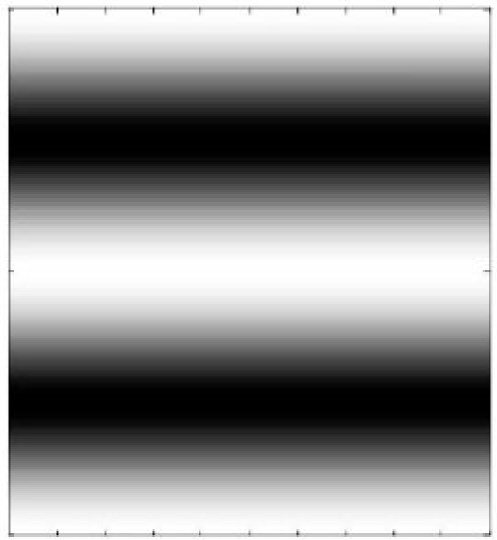
2D

- Images
- Frequency
  - Spatial
- Space/frequency characterization of 2D signals
- Reference space for
  - Filtering
  - Up/Down sampling
  - Image analysis
  - Feature extraction
  - Compression
  - ….

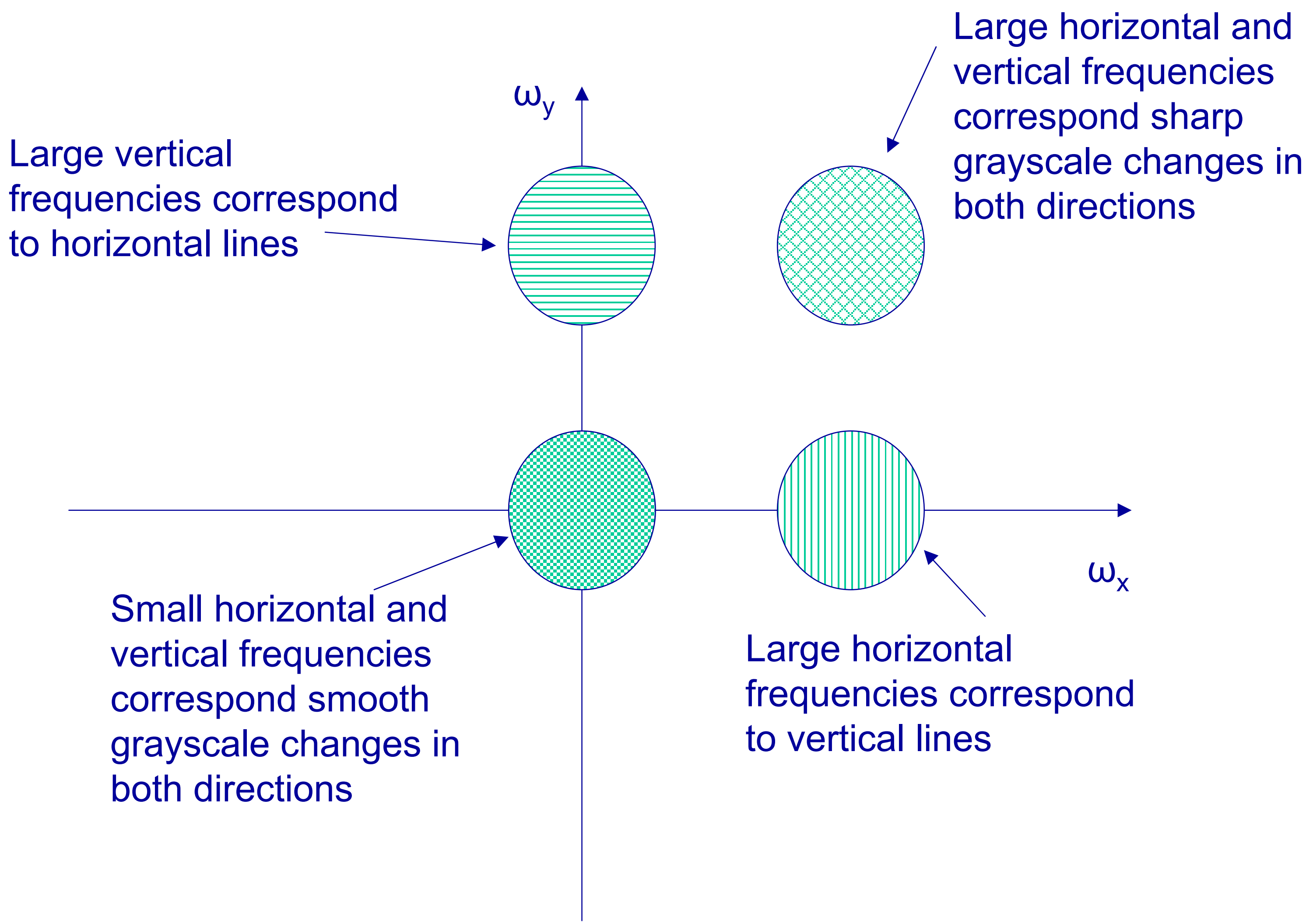
2D spatial frequencies

- 2D spatial frequencies characterize the image spatial changes in the horizontal (x) and vertical (y) directions
  - Smooth variations -> low frequencies
  - Sharp variations -> high frequencies

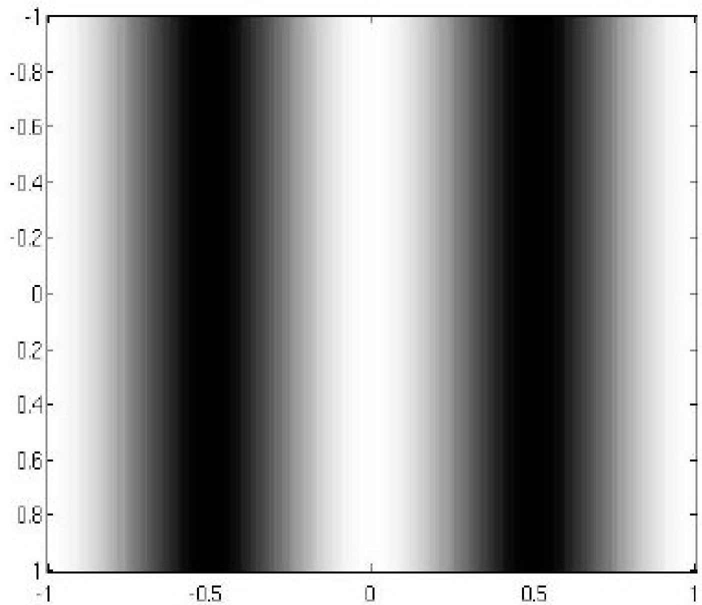


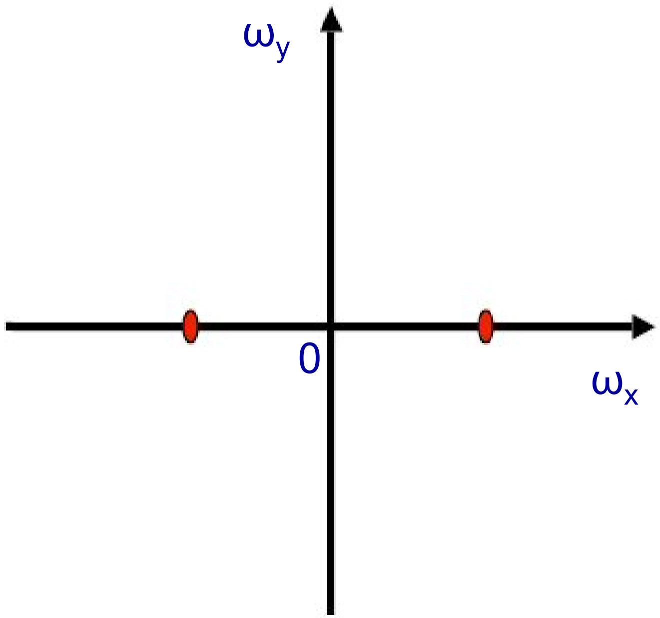


2D Frequency domain

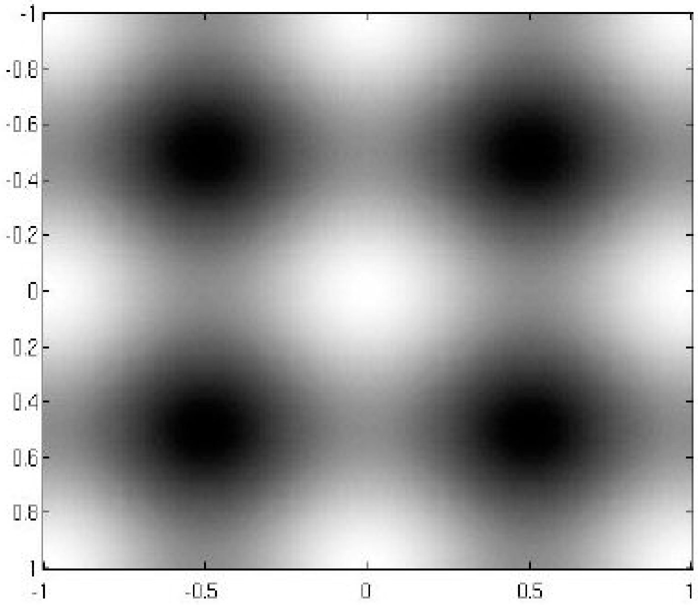


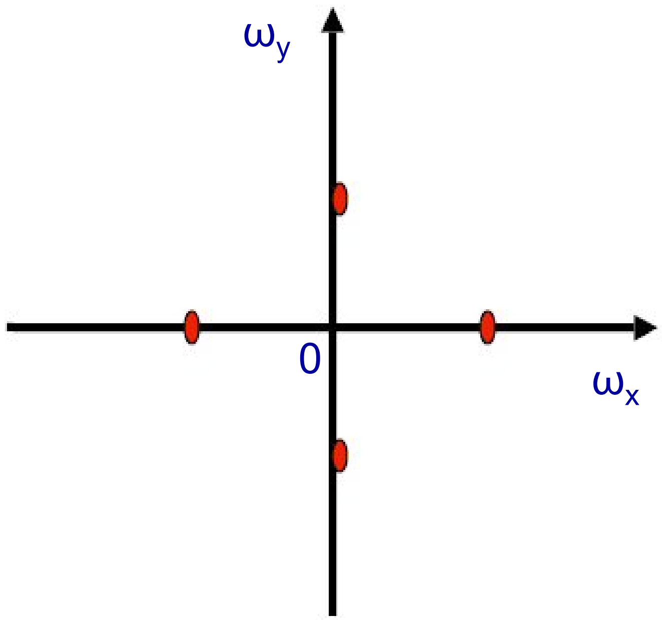
Vertical grating



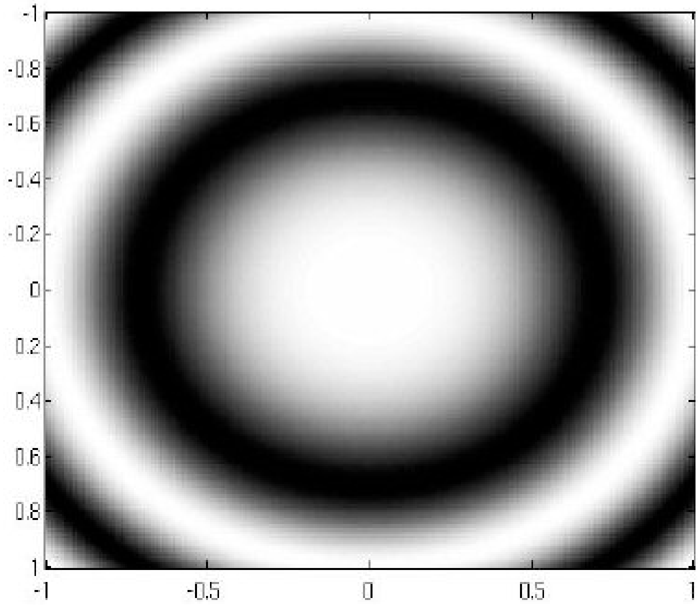


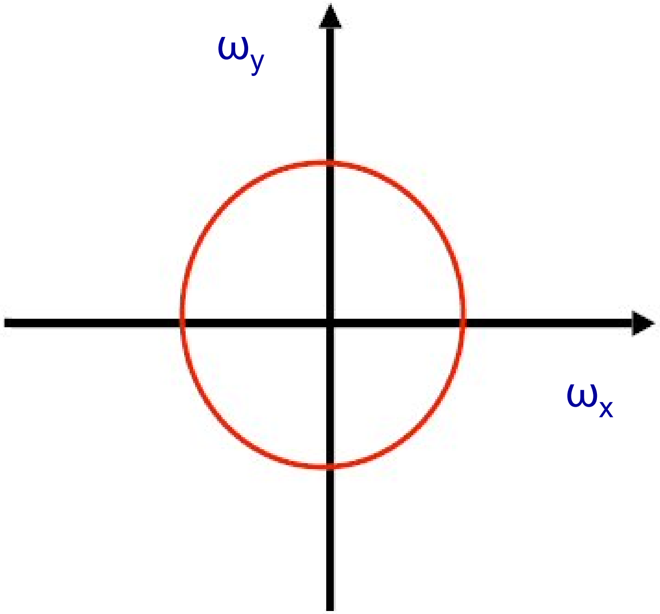
Double grating





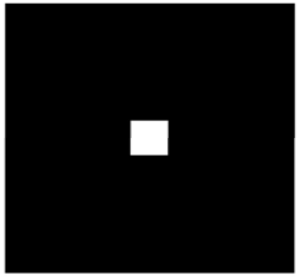
Smooth rings

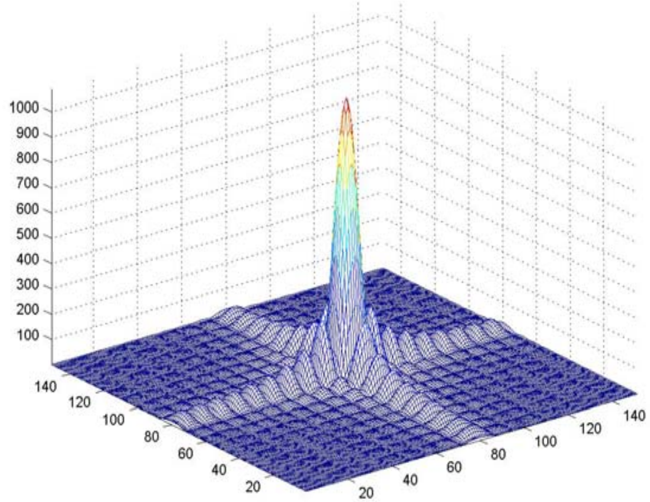


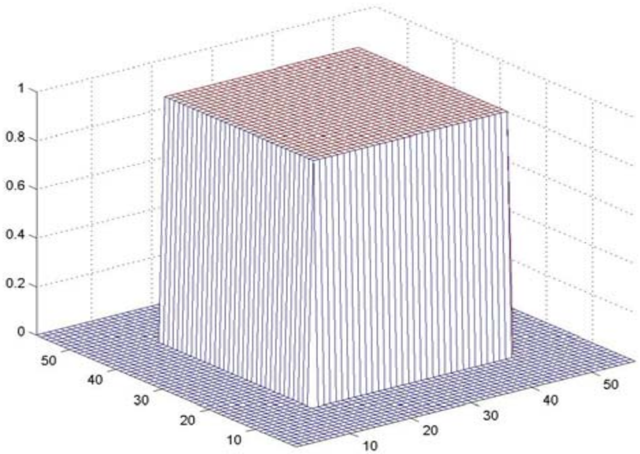


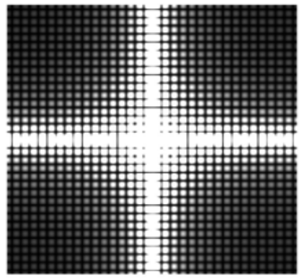
2D box

2D sinc

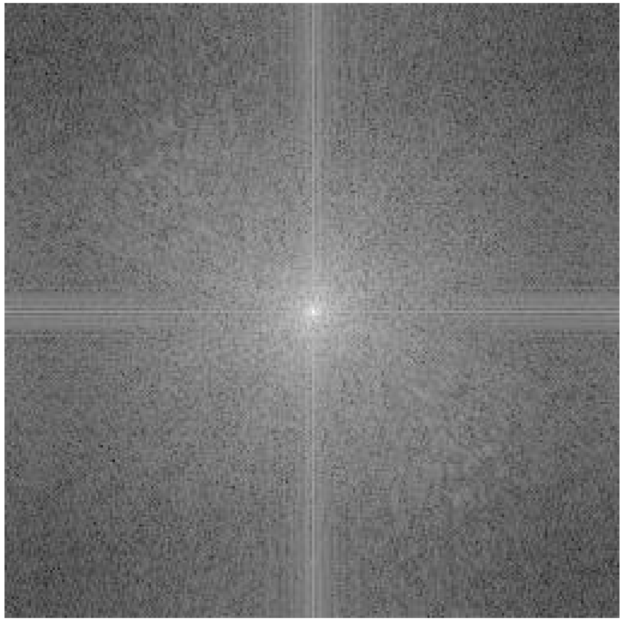






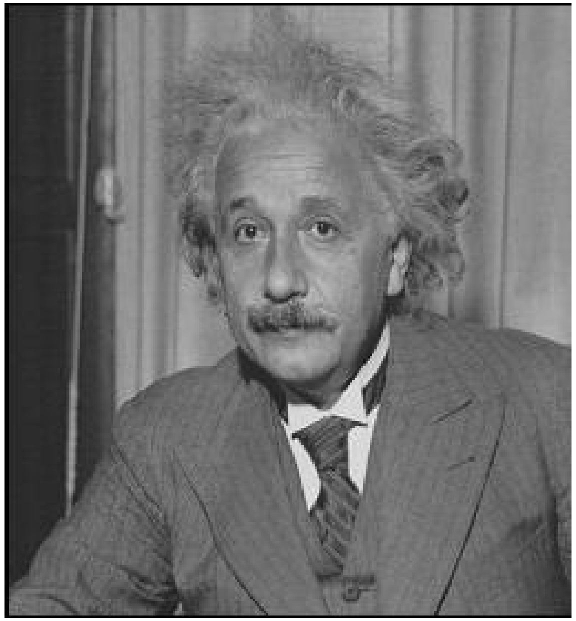


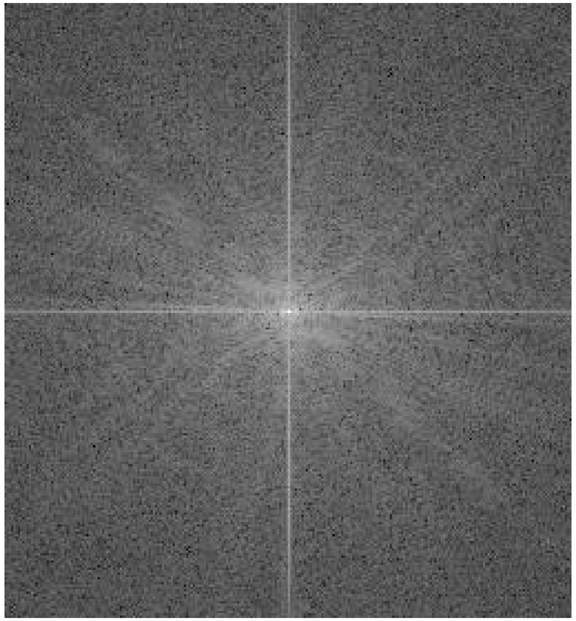
Margherita Hack



log amplitude of the spectrum

Einstein





log amplitude of the spectrum

What we are going to analyze

- 2D Fourier Transform of continuous signals (2D-CTFT)

1D $$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t} dt, \; f(t) = \int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} dt$$

- 2D Fourier Transform of discrete space signals (2D-DTFT)

1D $$F(\Omega) = \sum_{k=-\infty}^{\infty} f[k] e^{-j\Omega k}, \; f[k] = \frac{1}{2\pi} \int_{2\pi} F(\Omega) e^{j\Omega k} dt$$

- 2D Discrete Fourier Transform (2D-DFT)

1D $$F_r = \sum_{k=0}^{N_0-1} f[k] e^{-jr\Omega_0 k}, \; f_{N_0}[k] = \frac{1}{N_0} \sum_{r=0}^{N_0-1} F_r e^{jr\Omega_0 k}, \Omega_0 = \frac{2\pi}{N_0}$$

2D Continuous Fourier Transform

- Continuous case (*x* and *y* are real) – 2D-CTFT (notation 1)

$$\hat{f}\left(\omega_x,\omega_y\right)=\int_{-\infty}^{+\infty} f\left(x,y\right)e^{-j\left(\omega_x x+\omega_y y\right)}dxdy$$

$$f\left(x,y\right)=\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\hat{f}\left(\omega_x,\omega_y\right)e^{j\left(\omega_x x+\omega_y y\right)}d\omega_x d\omega_y$$

$$\iint f\left(x,y\right)g^*\left(x,y\right)dxdy=\frac{1}{4\pi^2}\iint\hat{f}\left(\omega_x,\omega_y\right)\hat{g}^*\left(\omega_x,\omega_y\right)d\omega_x d\omega_y$$ Parseval formula

$$f=g\rightarrow\iint\left|f\left(x,y\right)\right|^2 dxdy=\frac{1}{4\pi^2}\iint\left|\hat{f}\left(\omega_x,\omega_y\right)\right|^2 d\omega_x d\omega_y$$ Plancherel equality

2D Continuous Fourier Transform

- Continuous case (x and y are real) – 2D-CTFT

$$\omega_x = 2\pi u$$

$$\omega_y = 2\pi v$$

$$\hat{f}(u,v) = \int_{-\infty}^{+\infty} f(x,y) e^{-j2\pi(ux+vy)} dxdy$$

$$f(x,y) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \hat{f}(u,v) e^{j2\pi(ux+vy)} (2\pi)^2 \, dudv =$$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \hat{f}(u,v) e^{j2\pi(ux+vy)} (2\pi)^2 \, dudv$$

2D Continuous Fourier Transform

- 2D Continuous Fourier Transform (notation 2)

$$\hat{f}(u,v) = \int_{-\infty}^{+\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

$$f(x,y) = \int_{-\infty}^{+\infty} \hat{f}(u,v)e^{j2\pi(ux+vy)}dudv =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left| f(x,y)\right|^2 dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|\hat{f}(u,v)\right|^2 dudv$$

Plancherel's equality

2D Discrete Fourier Transform

The independent variable (t,x,y) is discrete

$$F_r = \sum_{k=0}^{N_0-1} f[k]e^{-jr\Omega_0 k}$$

$$f_{N_0}[k] = \frac{1}{N_0}\sum_{r=0}^{N_0-1} F_r e^{jr\Omega_0 k}$$

$$\Omega_0 = \frac{2\pi}{N_0}$$

$$F[u,v] = \sum_{i=0}^{N_0-1}\sum_{k=0}^{N_0-1} f[i,k]e^{-j\Omega_0(ui+vk)}$$

$$f_{N_0}[i,k] = \frac{1}{N_0{}^2}\sum_{u=0}^{N_0-1}\sum_{v=0}^{N_0-1} F[u,v]e^{j\Omega_0(ui+vk)}$$

$$\Omega_0 = \frac{2\pi}{N_0}$$

Delta

- Sampling property of the 2D-delta function (Dirac's delta)

$$\int_{-\infty}^{\infty} \delta(x - x_0, y - y_0) f(x, y) dxdy = f(x_0, y_0)$$

- Transform of the delta function

$$F\left\{\delta(x, y)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x, y) e^{-j2\pi(ux+vy)} dxdy = 1$$

$$F\left\{\delta(x - x_0, y - y_0)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - x_0, y - y_0) e^{-j2\pi(ux+vy)} dxdy = e^{-j2\pi(ux_0+vy_0)}$$

shifting property

Constant functions

- Inverse transform of the impulse function

$$F^{-1}\{\delta(u,v)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(u,v) e^{j2\pi(ux+vy)} du dv = e^{j2\pi(0x+v0)} = 1$$

- Fourier Transform of the constant (=1 for all x and y)

$$k(x,y) = 1 \qquad \forall x, y$$

$$F\{k\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j2\pi(ux+vy)} dx dy = \delta(u,v)$$

Trigonometric functions

- Cosine function oscillating along the x axis
  - Constant along the y axis

$$s(x, y) = \cos(2\pi fx)$$

$$F\left\{\cos(2\pi fx)\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos(2\pi fx) e^{-j2\pi(ux+vy)} dxdy =$$

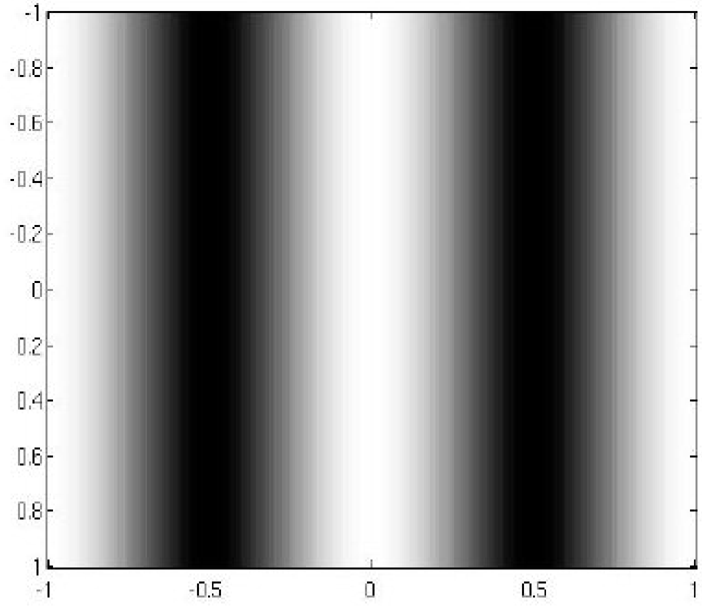
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{e^{j2\pi(fx)} + e^{-j2\pi(fx)}}{2}\right] e^{-j2\pi(ux+vy)} dxdy$$

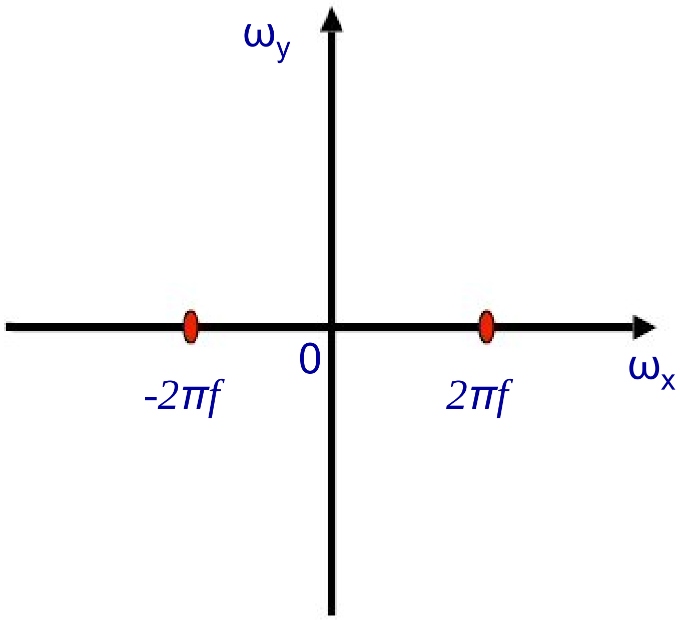
$$= \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[e^{-j2\pi(u-f)x} + e^{-j2\pi(u+f)x}\right] e^{-j2\pi vy} dxdy =$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{-j2\pi vy} dy \int_{-\infty}^{\infty} \left[e^{-j2\pi(u-f)x} + e^{-j2\pi(u+f)x}\right] dx = \frac{1}{2} 1 \int_{-\infty}^{\infty} \left[e^{-j2\pi(u-f)x} + e^{-j2\pi(u+f)x}\right] dx =$$

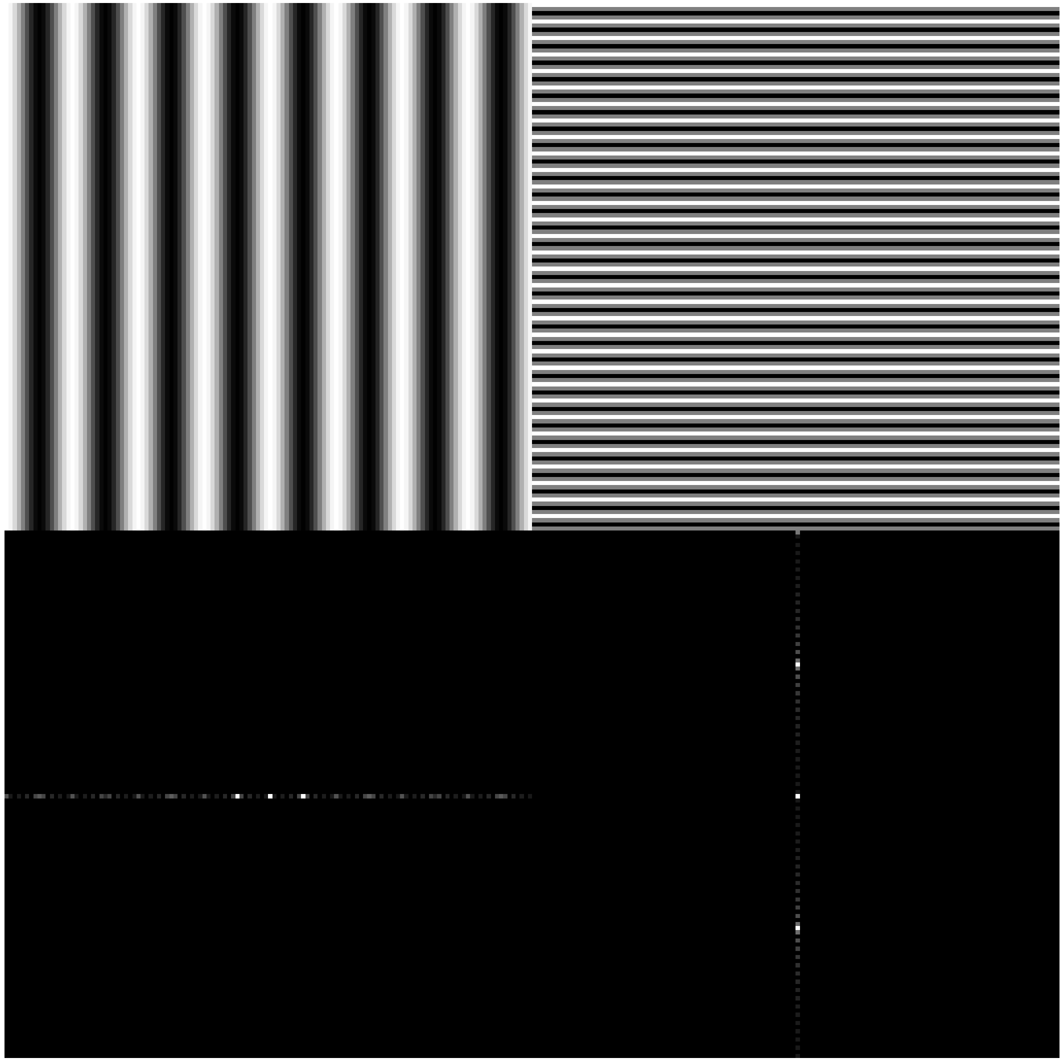
$$\frac{1}{2}\left[\delta(u-f) + \delta(u+f)\right]$$

Vertical grating

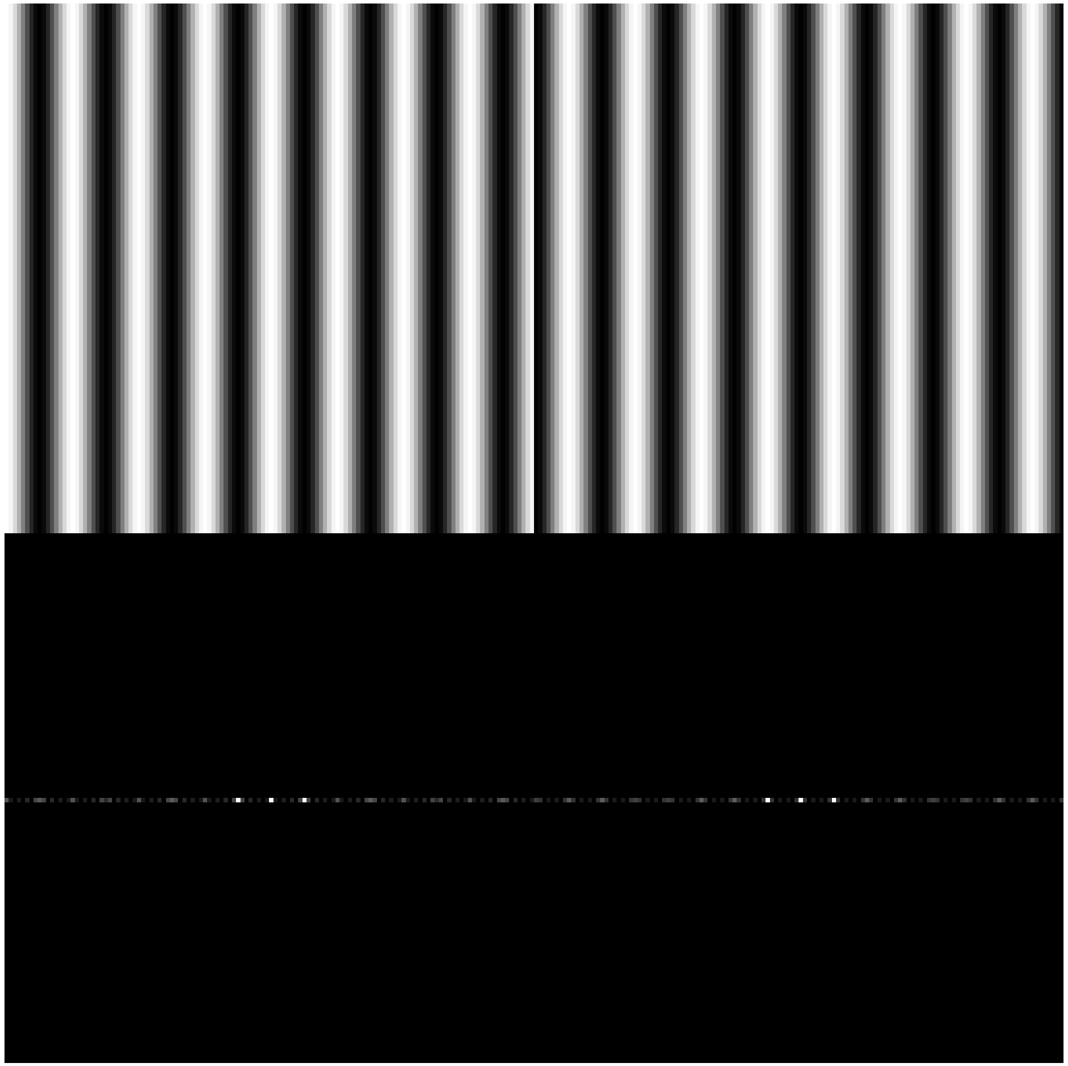




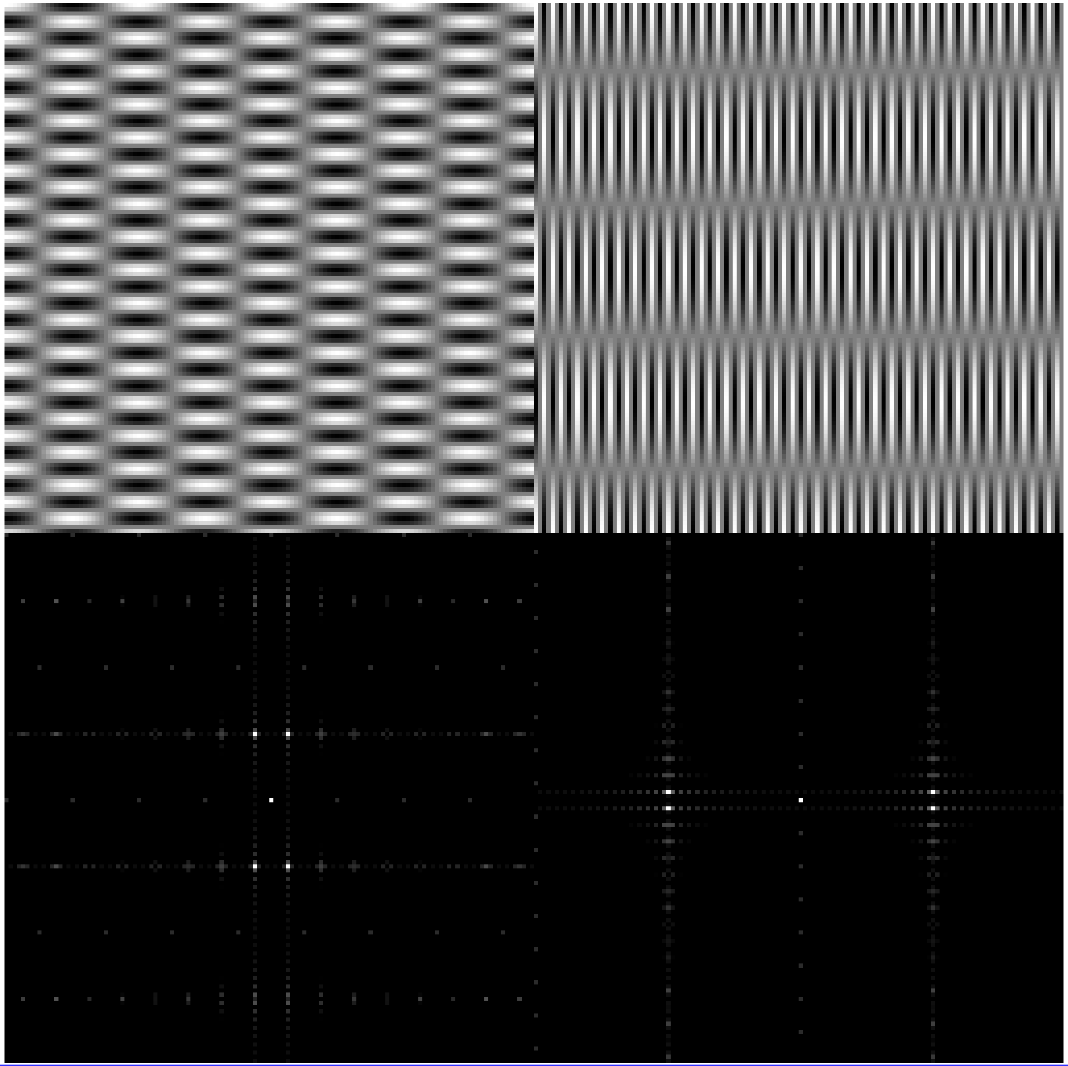
Ex. 1



Ex. 2

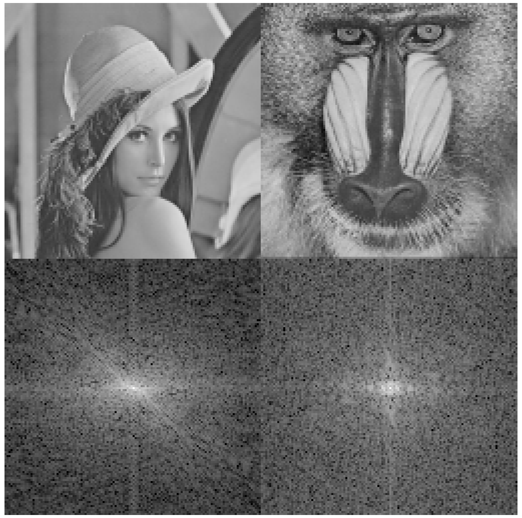


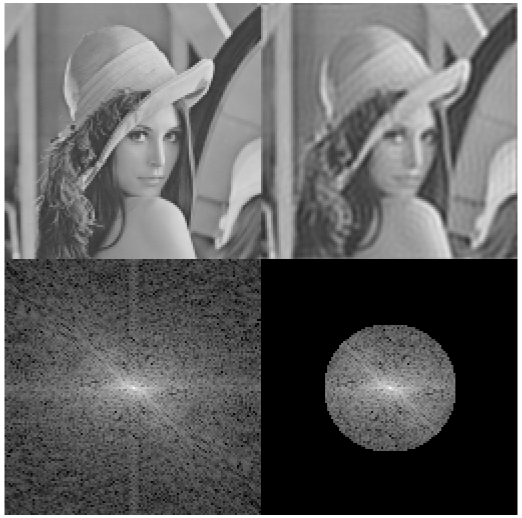
Ex. 3

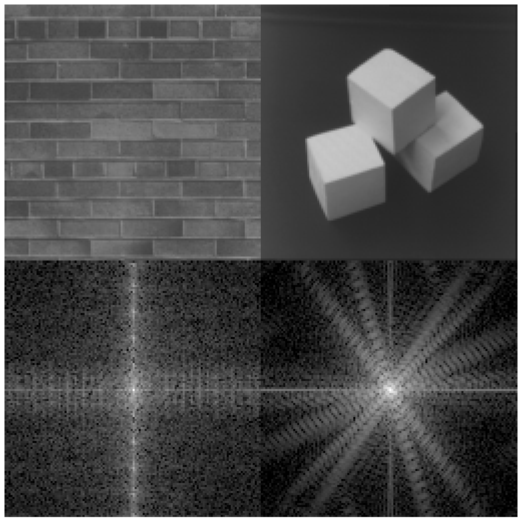


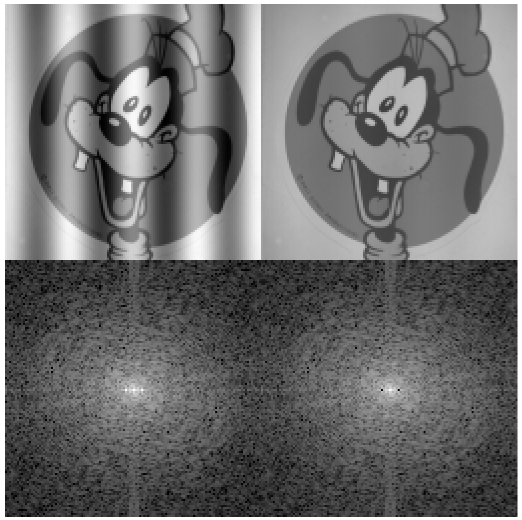
Magnitudes

Examples









Properties

- Linearity $$af(x,y)+bg(x,y) \Leftrightarrow aF(u,v)+bG(u,v)$$
- Shifting $$f(x-x_0, y-x_0) \Leftrightarrow e^{-j2\pi(ux_0+vy_0)}F(u,v)$$
- Modulation $$e^{j2\pi(u_0x+v_0y)}f(x,y) \Leftrightarrow F(u-u_0, v-v_0)$$
- Convolution $$f(x,y)*g(x,y) \Leftrightarrow F(u,v)G(u,v)$$
- Multiplication $$f(x,y)g(x,y) \Leftrightarrow F(u,v)*G(u,v)$$
- Separability $$f(x,y)=f(x)f(y) \Leftrightarrow F(u,v)=F(u)F(v)$$

Separability

1. Separability of the 2D Fourier transform
    – 2D Fourier Transforms can be implemented as a sequence of 1D Fourier Transform operations performed *independently* along the two axis

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi ux}e^{-j2\pi vy}dxdy = \int_{-\infty}^{\infty} e^{-j2\pi vy}dy\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi ux}dx =$$

$$= \int_{-\infty}^{\infty} F(u,y)e^{-j2\pi vy}dy = F\left(u,v\right)$$

2D DFT ⇒ 1D DFT along the rows → 1D DFT along the cols

Separability

- Separable functions can be written as $f\left(x,y\right)=f\left(x\right)g\left(y\right)$

2. The FT of a separable function is the product of the FTs of the two functions

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-j2\pi(ux+vy)}dxdy=$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(x)g\left(y\right)e^{-j2\pi ux}e^{-j2\pi vy}dxdy=\int_{-\infty}^{\infty}g\left(y\right)e^{-j2\pi vy}dy\int_{-\infty}^{\infty}h(x)e^{-j2\pi ux}dx=$$

$$=H\left(u\right)G\left(v\right)$$

$$f\left(x,y\right)=h\left(x\right)g\left(y\right)\Rightarrow F\left(u,v\right)=H\left(u\right)G\left(v\right)$$

2D Fourier Transform of a Discrete function

- Fourier Transform of a 2D *a-periodic* signal defined over a 2D discrete grid
  - The grid can be thought of as a 2D brush used for sampling the continuous signal with a given spatial resolution (T_x,T_y)

1D

$$F(\Omega) = \sum_{k=-\infty}^{\infty} f[k] e^{-j\Omega k}, \; f[k] = \frac{1}{2\pi} \int_{2\pi} F(\Omega) e^{j\Omega k} dt$$

2D

$$F(\Omega_x, \Omega_y) = \sum_{k_1=-\infty}^{+\infty} \sum_{k_2=-\infty}^{+\infty} f[k_1, k_2] e^{-j\left(k_1\Omega_x + k_2\Omega_y\right)}$$

$$f[k] = \frac{1}{4\pi^2} \int_{2\pi} \int_{2\pi} F(\Omega_x, \Omega_y) e^{j\left(k_1\Omega_x + k_2\Omega_y\right)} d\Omega_x \Omega_y$$

Ω_x,Ω_y: normalized frequency

Unitary frequency notations

$$\begin{cases} \Omega_x = 2\pi u \\ \Omega_y = 2\pi v \end{cases}$$

$$F(u,v) = \sum_{k_1=-\infty}^{+\infty} \sum_{k_2=-\infty}^{+\infty} f[k_1,k_2] e^{-j2\pi(k_1 u + k_2 v)}$$

$$f[k_1,k_2] = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} F(u,v) e^{-j2\pi(k_1 u + k_2 v)} du dv$$

- The integration interval for the inverse transform has width=1 instead of 2π
  - It is quite common to choose

$$\frac{-1}{2} \leq u,v < \frac{1}{2}$$

Properties

- Periodicity: 2D Fourier Transform of a *discrete* a-periodic signal is *periodic*
  - *The period is 1 for the unitary frequency notations and 2π for normalized frequency notations*.
  - Proof (referring to the firsts case)

$$F(u+k, v+l) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] e^{-j2\pi\left((u+k)m+(v+l)n\right)}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] e^{-j2\pi\left(um+vn\right)} \overbrace{e^{-j2\pi km}}^{1} \overbrace{e^{-j2\pi ln}}^{1}$$

Arbitrary integers

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n] e^{-j2\pi(um+vn)}$$

$$= F(u,v)$$

Properties

- Linearity
- shifting
- modulation
- convolution
- multiplication
- separability
- energy conservation properties also exist for the 2D Fourier Transform of discrete signals.
- NOTE: in what follows, (k_1,k_2) is replaced by (m,n)

2D DTFT Properties

- Linearity $$af[m,n]+bg[m,n] \Leftrightarrow aF(u,v)+bG(u,v)$$
- Shifting $$f[m-m_0,n-n_0] \Leftrightarrow e^{-j2\pi(um_0+vn_0)}F(u,v)$$
- Modulation $$e^{j2\pi(u_0m+v_0n)}f[m,n] \Leftrightarrow F(u-u_0,v-v_0)$$
- Convolution $$f[m,n]*g[m,n] \Leftrightarrow F(u,v)G(u,v)$$
- Multiplication $$f[m,n]g[m,n] \Leftrightarrow F(u,v)*G(u,v)$$
- Separable functions $$f[m,n]=f[m]f[n] \Leftrightarrow F(u,v)=F(u)F(v)$$
- Energy conservation $$\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left|f[m,n]\right|^2 = \int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\left|F(u,v)\right|^2 dudv$$

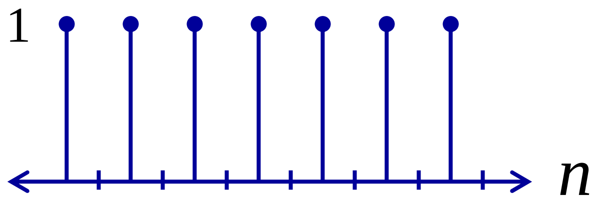
Impulse Train

- Define a *comb* function (impulse train) as follows

$$comb_{M,N}[m,n] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \delta[m-kM, n-lN]$$

where *M* and *N* are integers

$comb_2[n]$



Appendix

2D-DTFT: delta

- Define *Kronecker delta function*

$$\delta[m,n]=\begin{cases}1, & \text{for } m=0 \text{ and } n=0\\ 0, & \text{otherwise}\end{cases}$$

- DT Fourier Transform of the Kronecker delta function

$$F(u,v)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\left[\delta[m,n]e^{-j2\pi(um+vn)}\right]=e^{-j2\pi(u0+v0)}=1$$

2D DT Fourier Transform: constant

- Fourier Transform of 1

$$f[k,l]=1, \forall k,l$$

$$F[u,v]=\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\left[1e^{-j2\pi(uk+vl)}\right]=$$

$$=\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\delta(u-k,v-l)$$

periodic with period 1
along u and v

To prove: Take the inverse Fourier Transform of the Dirac delta function and use the fact that the Fourier Transform has to be periodic with period 1.