GEOMETRIA

Prof. M. Spera

Prova scritta del 24/9/2008

a.a. 2007/08

Tempo a disposizione: 2h.

Le risposte vanno adeguatamente giustificate

(1) Sia data la curva $\mathscr{C}$: $\begin{cases} x^2+y^2=1 \\ y^2+z^2=1 \end{cases}$

[Se ne determinino i punti singolari e se ne abbozzi il grafico...] Se ne calcoli la curvatura e si determini il piano osculatore in $P:(1,0,1)$ (utilizzare il calcolo implicito, usando $x=x(s)$ ecc. ma non si determini s!).

Fac. Si calcoli la torsione di $\mathscr{C}$ in P.

(2) Sia data Σ: $\begin{cases} x = u\cos v \\ y = u\sin v \\ z = \log u \end{cases}$ $\quad u>0$, $v\in[0,2\pi)$

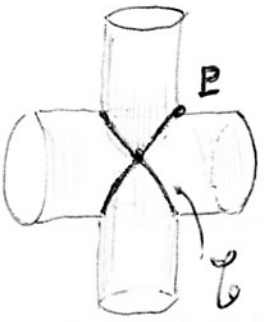
se ne determinino la prima e seconda forma fondamentale, la curvatura gaussiana e media, nonché, in due modi, le curvature principali.

(3) Dati gli spazi topologici

$X =$  e $Y =$ 

si dica, giustificando la risposta, se essi risultano omeomorfi

(1) $\mathscr{C}: \begin{cases} x^2+y^2=1 \\ y^2+z^2=1 \end{cases}$ $\quad f = x^2+y^2-1=0$, $\quad g = y^2+z^2-1=0$



pti angolati:

$\nabla f \parallel \nabla g$

$\nabla f = (2x, 2y, 0)$

$\nabla g = (0, 2y, 2z)$

$\nabla f \times \nabla g = \underline{0} \iff x = z = 0 \Rightarrow y = \pm 1$

$\Rightarrow$ pti sing: $(0, \pm 1, 0)$

Sia $P: (1, 0, 1)$ $(\in \mathscr{C})$

deriviamo le $\begin{cases} f=0 \\ g=0 \end{cases}$ e $x'^2+y'^2+z'^2=1$

$x = x(s)$ ecc. $\qquad ' = \dfrac{d}{ds}$

$\begin{cases} xx'+yy'=0 \\ yy'+zz'=0 \end{cases}$ $\qquad (**)\ x'x''+y'y''+z'z''=0$

$(*) \begin{cases} x'^2+xx''+y'^2+yy''=0 \\ y'^2+yy''+z'^2+zz''=0 \end{cases}$

troviamo: $x'=0 \quad z'=0$
in P $\qquad \Rightarrow y' = \pm 1$ (scegliamo +, ma k non cambia)

$\left.\begin{array}{ll} x''+1=0 & x''=-1 \\ y''=0 & \\ z''+1=0 & z''=-1 \end{array}\right)$

in P

$\underline{t}: (0, +1, 0)$
$\quad \| $
$\underline{r}'$

$\underline{r}'' = (-1, 0, -1)$

$k = \|\underline{r}''\| = \sqrt{2}$

$k(P) = \sqrt{2}$

Piano osculatore: $\langle \underline{r} - \underline{r}_P \,;\, \underline{r} \times \underline{r}'' \rangle = 0$

$$\begin{vmatrix} x-1 & y & z-1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x-1 & z-1 \\ -1 & -1 \end{vmatrix} = 0 \qquad \begin{vmatrix} x-1 & z-1 \\ 1 & 1 \end{vmatrix} = 0$$

$$x-1-(z-1) = 0 \qquad x - 1 - z + 1 = 0$$

$$\boxed{x - z = 0}$$

in P

$\underline{r}' : (0 \quad 1$

$\underline{r}'' \; (-1 \quad 0$

Parte facoltativa

calcoliamo $\tau(P)$: ci occorre $\underline{r}'''$

deriviamo ancora le $(*)$ e $(**)$

$$2x'x'' + x'x'' + xx''' + 2y'y'' + y'y'' + yy''' = 0$$

$$\boxed{3x'x'' + 3y'y'' + xx''' + yy''' = 0}$$

$$2y'y'' + y'y'' + yy''' + 2z'z'' + z'z'' + zz''' = 0$$

$$\boxed{3y'y'' + 3z'z'' + yy''' + zz''' = 0}$$

$$\boxed{x''^2 + x'x''' + y''^2 + y'y''' + z''^2 + z'z''' = 0}$$

In ℓ : $(1, 0, 1)$ si ha:

$$\begin{array}{l} \underline{\ell} \equiv \underline{r} : (1,0,1) \\ \underline{r}' = (0,1,0) \\ \underline{r}'' = (-1,0,-1) \end{array}$$

$x''' = 0$

$z''' = 0$

$\underline{r}''' : (0, -2, 0)$

$1 + 1 + y''' = 0 \Rightarrow y''' = -2$

$$\tau = -\frac{|\underline{r}' \;\; \underline{r}'' \;\; \underline{r}'''|}{k^2} \Rightarrow \tau(\ell) = 0$$

$(\underline{r}' \,//\, \underline{r}''')$

deducibile a priori per ragioni di simmetria

Dunque:

$k(\ell) = \sqrt{2} \qquad \tau(\ell) = 0$

È tuttavia immediato constatare che ℓ non è piana.

②

$$\begin{cases} x = u\cos v \\ y = u\sin v \\ z = \log u \end{cases} \qquad \begin{array}{l} u > 0 \\ \\ v \in [0, 2\pi) \end{array}$$

$$\underline{r} = (u\cos v,\ u\sin v,\ \log u)$$

$$\underline{r}_u = \left(\cos v,\ \sin v,\ \frac{1}{u}\right)$$

$$\underline{r}_v = (-u\sin v,\ u\cos v,\ 0)$$

$$\underline{r}_{uu} = \left(0,\ 0,\ -\frac{1}{u^2}\right)$$

$$\underline{r}_{uv} = (-\sin v,\ \cos v,\ 0)$$

$$\underline{r}_{vv} = (-u\cos v,\ -u\sin v,\ 0)$$

$$E = \|\underline{r}_u\|^2 = 1 + \frac{1}{u^2} = \frac{u^2+1}{u^2}$$

$$F = 0 \qquad G = u^2$$

$$\boxed{\begin{array}{l} E = \dfrac{u^2+1}{u^2} \\ F = 0 \\ G = u^2 \end{array}}$$

$$\underline{N} = \frac{\underline{r}_u \times \underline{r}_v}{\|\ \cdot\ \|} = \frac{\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos v & \sin v & \frac{1}{u} \\ -u\sin v & u\cos v & 0 \end{vmatrix}}{\|\ \cdot\ \|} =$$

$$= \frac{\underline{i}\left[-\cos v\right] - \underline{j}\sin v + \underline{k}\,u}{\sqrt{1+u^2}}$$

$$e = \langle \underline{r}_{uu}, \underline{N} \rangle = -\frac{1}{u^2}\frac{u}{\sqrt{1+u^2}} = -\frac{1}{u\sqrt{1+u^2}}$$

$$f = \langle \underline{r}_{uv}, \underline{N} \rangle = \dots = 0$$

$$g = \langle \underline{r}_{vv}, \underline{\mathbf{N}} \rangle = +\frac{u}{\sqrt{u^2+1}}$$

$$K = \frac{eg - f^2}{EG - F^2} = \frac{-\frac{1}{1+u^2}}{u^2+1} = -\frac{1}{(1+u^2)^2}$$

$$H = \frac{1}{2}\frac{eG - \cancel{2Ff} + Eg}{EG - \cancel{F^2}} = \frac{1}{2}\frac{eG+Eg}{EG}$$

$$= \frac{1}{2}\left(\underbrace{\frac{e}{E}}_{k_1} + \underbrace{\frac{g}{G}}_{k_2}\right) \text{, dove}$$

$$\frac{e}{E} = -\frac{1}{u\sqrt{1+u^2}}\frac{u^2}{1+u^2} = -\frac{u}{(1+u^2)^{3/2}}$$

$$\frac{g}{G} = \frac{u}{\sqrt{1+u^2}}\cdot\frac{1}{u^2} = \frac{1}{u\sqrt{1+u^2}}$$

$$H = \frac{1}{2}\left[-\frac{u}{(1+u^2)^{3/2}} + \frac{1}{u}\frac{1}{(1+u^2)^{1/2}}\right] = \frac{1}{2}\frac{-u^2+1+u^2}{u(1+u^2)^{3/2}}$$

$$= \frac{1}{2}\cdot\frac{1}{u(1+u^2)^{3/2}}$$

controllo: curv. del meridiano

$$z = \log u$$

$$z' = \frac{1}{u}$$

$$z'' = -\frac{1}{u^2}$$

$$k = \frac{z''}{(1+z'^2)^{3/2}}$$

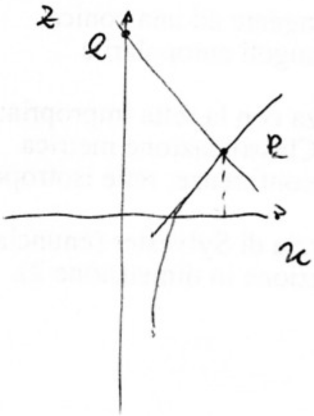
$$\frac{-\frac{1}{u^2}}{\left(1+\frac{1}{u^2}\right)^{3/2}} = -\frac{u^3}{u^2(1+u^2)^{3/2}}$$

$$= -\frac{u}{(1+u^2)^{3/2}}$$

★ variante

k_1 : cm. del meridiano (v. calcolo pag. precedente)

$$k_2 = \frac{1}{\text{gran normale}}$$



tangente in P a $z = \log x$

$$Z - \log x = \frac{1}{x}\left(U - x\right)$$

var. ausiliarie

normale $\tilde{n}$: $$Z - \log x = -x\left(U - x\right)$$

$Q =$ asse $z \cap \tilde{n}$ $\quad Z - \log x = x^2$

$U = 0$ $\quad Z_Q = x^2 + \log x$

$$\overline{QP} = \sqrt{\underbrace{\left(x^2 + \cancel{\log x} - \cancel{\log x}\right)^2}_{x^4} + x^2} = \sqrt{x^2(1+x^2)}$$

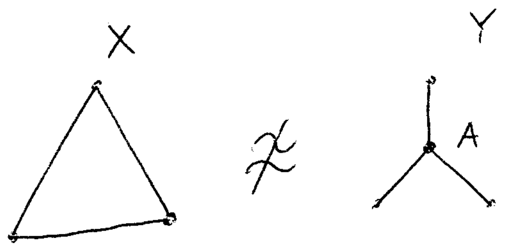
$$= x\sqrt{1+x^2}$$

$P : (x, \log x)$

$Q : (0, x^2 + \log x)$

$$\Rightarrow k_2 = \frac{1}{x\sqrt{1+x^2}} \quad \checkmark$$

3)



(anche se sono entrambi compatti e connessi)

Sia p.a. $f: Y \to X$ un omeomorfismo

$f\big|_{Y \setminus \{A\}} : Y \setminus \{A\} \to X \setminus \{f(A)\}$

è un omeomorfismo, ma ciò è assurdo:

$Y \setminus \{A\}$ è sconnesso, $X \setminus \{f(A)\}$ è

comunque connesso.