Dec. 12, 2012

IV. Exercises Representation Theory

10. Exercise: Let Q be a finite quiver bound by an admissible ideal $I = \langle \rho_1, \rho_2, \ldots, \rho_s \rangle \subseteq KQ$. Show:

(a) $\mathrm{Mod}(KQ/I)$ can be identified with a full subcategory of $\mathrm{Mod}(KQ)$ in a canonical way.

(b) The equivalences Φ and Ψ between $\mathrm{Mod}(KQ)$ and $\mathrm{Rep}_K(Q)$ induce, via restriction to full subcategories, mutually inverse equivalences between $\mathrm{Mod}(KQ/I)$ and $\mathrm{Rep}_K(Q, I)$, the full subcategory of representations $V \in \mathrm{Rep}_K(Q)$ respecting the relations $\rho_1, \rho_2, \ldots, \rho_s$ (that is, $V(\rho_i) = 0$ for all $i = 1, \ldots, s$). Moreover, these equivalences send finite-dimensional objects to finite-dimensional objects.

(c) Determine all simple and all indecomposable projective representations of the bound quiver (Q, I) (up to isomorphism). (Hint: For the projectives, recall how the corresponding vector spaces and their bases and the linear maps between them are *explicitly* given in the case $I = 0$.) 10 P.

11. Exercise: Let Q be a finite quiver with $Q_0 = \{1, \ldots, n\}$. For each $i \in Q_0$. Describe the indecomposable injective K-linear representations $I_1, \ldots, I_n$, where I_i is the injective envelope of the simple representation S_i. (Hint: Consider also the opposite quiver Q^{op}.) 10 P.

12. Exercise: Let K be a field and Q the quiver

$$\begin{tikzcd} & 2 \arrow[dr, "\beta"] & \\ 1 \arrow[ur, "\alpha"] \arrow[rr, "\gamma"] & & 3 \end{tikzcd}$$

(no relations).

(a) Determine all indecomposable projective representations and their radicals.

(b) Determine all indecomposable injective representations and their socles.

(c) Determine the minimal projective resolutions of the simple modules. 10 P.

13. Exercise: Let Q be the same quiver as in Exercise 12.

(a) Compute the representation $\nu(S_1)$.

(b) Compute the representation $\tau(S_1)$. 10 P.

14. Exercise: Let Q be the quiver $1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$. Let $I = \langle \beta\alpha \rangle$. Compute minimal projective resolutions of the simple representations S_1 and S_2 of the bound quiver (Q, I). 10 P.

15. Exercise: Let Q be the same quiver as in Exercise 14.

(a) Let $I = \langle \alpha\beta,\ \beta\alpha \rangle$ and consider K-linear representations of the bound quiver (Q, I). Show that the minimal projective resolutions of S_1 and S_2, respectively, are

$$\cdots \to P_2 \to P_1 \to P_2 \to P_1 \to P_2 \to P_1 \to S_1 \to 0$$
$$\cdots \to P_1 \to P_2 \to P_1 \to P_2 \to P_1 \to P_2 \to S_2 \to 0$$

(Thus, KQ/I has infinite global dimension.)

(b) Let $I = \langle \alpha\beta\alpha \rangle$. Let V be the K-linear representation $K \underset{1}{\overset{0}{\rightleftarrows}} K$ of the bound quiver (Q, I). Show that there is a short exact sequence $0 \to V \to P_2 \to V \to 0$. Deduce that S_1 has infinite projective dimension. Compute also the projective dimension of S_2. 10 P.

General remark: In Exercises 12.–15. all computations should be done in terms of representations (given by vector spaces and linear maps between them) in an *explicit* way.

Submission of solutions: In the lecture on Jan 8, 2013.