

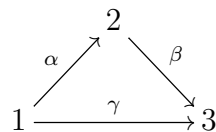
IV. Exercises Representation Theory

10. Exercise: Let Q be a finite quiver bound by an admissible ideal $I = \langle \rho_1, \rho_2, \dots, \rho_s \rangle \subseteq KQ$. Show:

- (a) $\text{Mod}(KQ/I)$ can be identified with a full subcategory of $\text{Mod}(KQ)$ in a canonical way.
- (b) The equivalences Φ and Ψ between $\text{Mod}(KQ)$ and $\text{Rep}_K(Q)$ induce, via restriction to full subcategories, mutually inverse equivalences between $\text{Mod}(KQ/I)$ and $\text{Rep}_K(Q, I)$, the full subcategory of representations $V \in \text{Rep}_K(Q)$ respecting the relations $\rho_1, \rho_2, \dots, \rho_s$ (that is, $V(\rho_i) = 0$ for all $i = 1, \dots, s$). Moreover, these equivalences send finite-dimensional objects to finite-dimensional objects.
- (c) Determine all simple and all indecomposable projective representations of the bound quiver (Q, I) (up to isomorphism). (HINT: For the projectives, recall how the corresponding vector spaces and their bases and the linear maps between them are *explicitly* given in the case $I = 0$.) 10 P.

11. Exercise: Let Q be a finite quiver with $Q_0 = \{1, \dots, n\}$. For each $i \in Q_0$. Describe the indecomposable injective K -linear representations I_1, \dots, I_n , where I_i is the injective envelope of the simple representation S_i . (HINT: Consider also the opposite quiver Q^{op} .) 10 P.

12. Exercise: Let K be a field and Q the quiver (no relations).



- (a) Determine all indecomposable projective representations and their radicals.
- (b) Determine all indecomposable injective representations and their socles.
- (c) Determine the minimal projective resolutions of the simple modules. 10 P.

13. Exercise: Let Q be the same quiver as in Exercise 12.

(a) Compute the representation $\nu(S_1)$.

(b) Compute the representation $\tau(S_1)$.

10 P.

14. Exercise: Let Q be the quiver $1 \begin{array}{c} \xleftarrow{\alpha} \\ \xrightarrow{\beta} \end{array} 2$. Let $I = \langle \beta\alpha \rangle$. Compute minimal projective resolutions of the simple representations S_1 and S_2 of the bound quiver (Q, I) .

10 P.

15. Exercise: Let Q be the same quiver as in Exercise 14.

(a) Let $I = \langle \alpha\beta, \beta\alpha \rangle$ and consider K -linear representations of the bound quiver (Q, I) . Show that the minimal projective resolutions of S_1 and S_2 , respectively, are

$$\begin{aligned} \cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_2 \rightarrow P_1 \rightarrow P_2 \rightarrow P_1 \rightarrow S_1 \rightarrow 0 \\ \cdots \rightarrow P_1 \rightarrow P_2 \rightarrow P_1 \rightarrow P_2 \rightarrow P_1 \rightarrow P_2 \rightarrow S_2 \rightarrow 0 \end{aligned}$$

(Thus, KQ/I has infinite global dimension.)

(b) Let $I = \langle \alpha\beta\alpha \rangle$. Let V be the K -linear representation $K \begin{array}{c} \xleftarrow{0} \\ \xrightarrow{1} \end{array} K$ of the bound quiver (Q, I) . Show that there is a short exact sequence $0 \rightarrow V \rightarrow P_2 \rightarrow V \rightarrow 0$. Deduce that S_1 has infinite projective dimension. Compute also the projective dimension of S_2 .

10 P.

GENERAL REMARK: In Exercises 12.–15. all computations should be done in terms of representations (given by vector spaces and linear maps between them) in an *explicit* way.