2D Discrete Fourier Transform (DFT)

Outline

- Circular and linear convolutions
- 2D DFT
- 2D DCT
- Properties
- Other formulations
- Examples

In words

- Given 2 sequences of length N and M, let y[k] be their linear convolution

$$y[k] = f[k] * h[k] = \sum_{n=-\infty}^{+\infty} f[n]h[k-n]$$

- y[k] is also equal to the circular convolution of the two suitably zero padded sequences making them consist of the same number of samples

$$c[k] = f[k] \otimes h[k] = \sum_{n=0}^{N_0-1} f[n]h[k-n]$$

$$N_0 = N_f + N_h - 1: \text{ length of the zero-padded seq}$$

- In this way, the linear convolution between two sequences having a different length (filtering) can be computed by the DFT (which rests on the circular convolution)
  - The procedure is the following
    - Pad f[n] with N_h-1 zeros and h[n] with N_f-1 zeros
    - Find Y[r] as the product of F[r] and H[r] (which are the DFTs of the corresponding zero-padded signals)
    - Find the inverse DFT of Y[r]
- **Allows to perform linear filtering using DFT**

2D Discrete Fourier Transform

- Fourier transform of a 2D signal defined over a discrete finite 2D grid of size MxN

or equivalently

- Fourier transform of a 2D set of samples forming a bidimensional sequence
- As in the 1D case, 2D-DFT, though a self-consistent transform, can be considered as a mean of calculating the transform of a 2D sampled signal defined over a discrete grid.
- The signal is periodized along both dimensions and the 2D-DFT can be regarded as a sampled version of the 2D DTFT

2D Discrete Fourier Transform (2D DFT)

- 2D Fourier (discrete time) Transform (DTFT) [Gonzalez]

$$F(u,v)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} f[m,n]e^{-j2\pi(um+vn)}$$

a-periodic signal
periodic transform

- 2D Discrete Fourier Transform (DFT)

$$F[k,l]=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]e^{-j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}$$

periodized signal
periodic and **sampled** transform

2D DFT can be regarded as a sampled version of 2D DTFT.

2D DFT: Periodicity

- A [M,N] point DFT is periodic with period [M,N]
  - Proof

$$F[k,l]=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f[m,n]e^{-j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}$$

$$F[k+M,l+N]=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f[m,n]e^{-j2\pi\left(\frac{k+M}{M}m+\frac{l+N}{N}n\right)}$$

$$=\frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f[m,n]e^{-j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}\cancelto{1}{e^{-j2\pi\left(\frac{M}{M}m+\frac{N}{N}n\right)}}$$

$$=F[k,l]$$

(In what follows: spatial coordinates=k,l, frequency coordinates: u,v)

2D DFT: Periodicity

- Periodicity

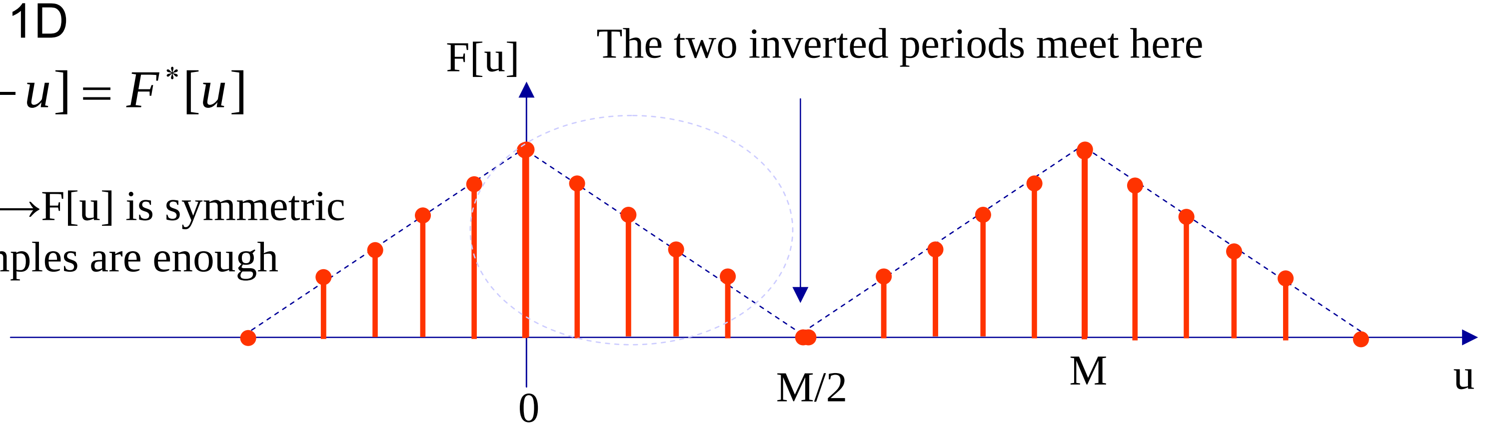
$$F[u,v] = F[u+mM,v] = F[u,v+nN] = F[u+mM,v+nN]$$

$$f[k,l] = f[k+mM,l] = f[k,l+nN] = f[k+mM,l+nN]$$

- This has important consequences on the implementation and energy compaction property
  - 1D

$$F[N-u] = F^*[u]$$

f[k] real→F[u] is symmetric
M/2 samples are enough



Periodicity: 1D

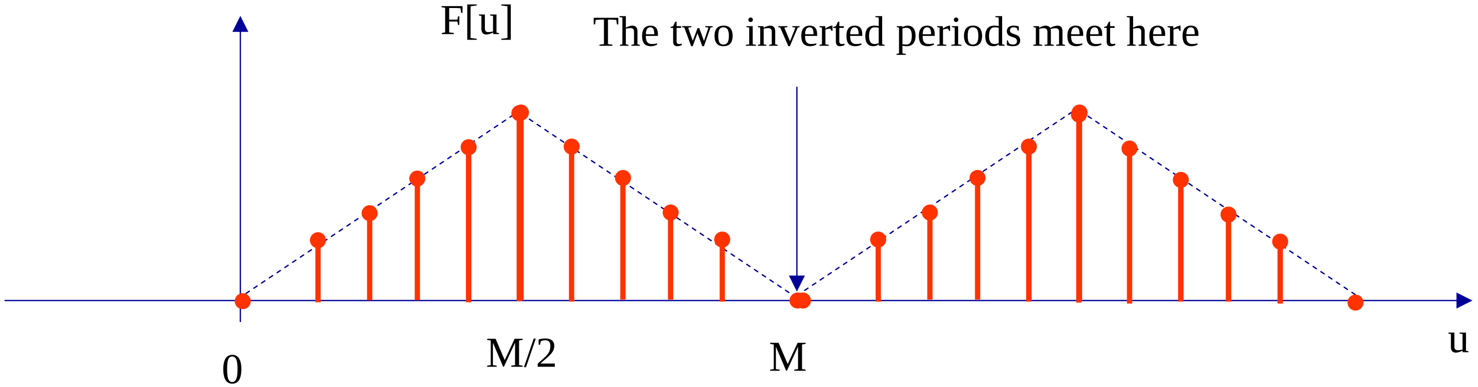
$$f[k] \leftrightarrow F[u]$$

$$f[k]e^{j2\pi\frac{u_0 k}{M}} \leftrightarrow F[u-u_0]$$

$$u_0 = \frac{M}{2} \rightarrow e^{j2\pi\frac{u_0 k}{M}} = e^{j2\pi\frac{Mk}{2M}} = e^{j\pi k} = (-1)^k$$

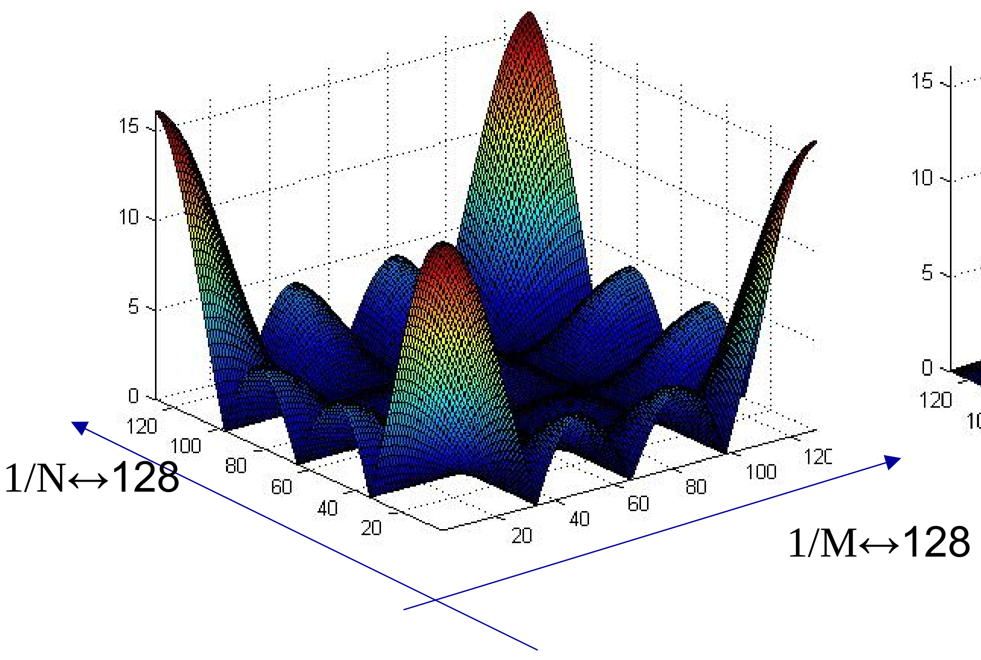
$$(-1)^k f[k] \leftrightarrow F[u-\frac{M}{2}]$$

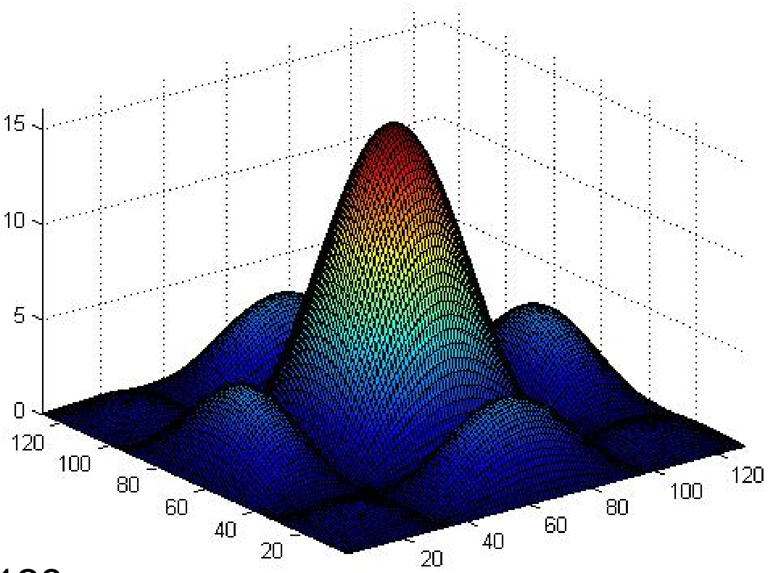
changing the sign of every other sample of the DFT puts F[0] at the center of the interval [0,M]



It is more practical to have one complete period positioned in [0, M-1]

Periodicity

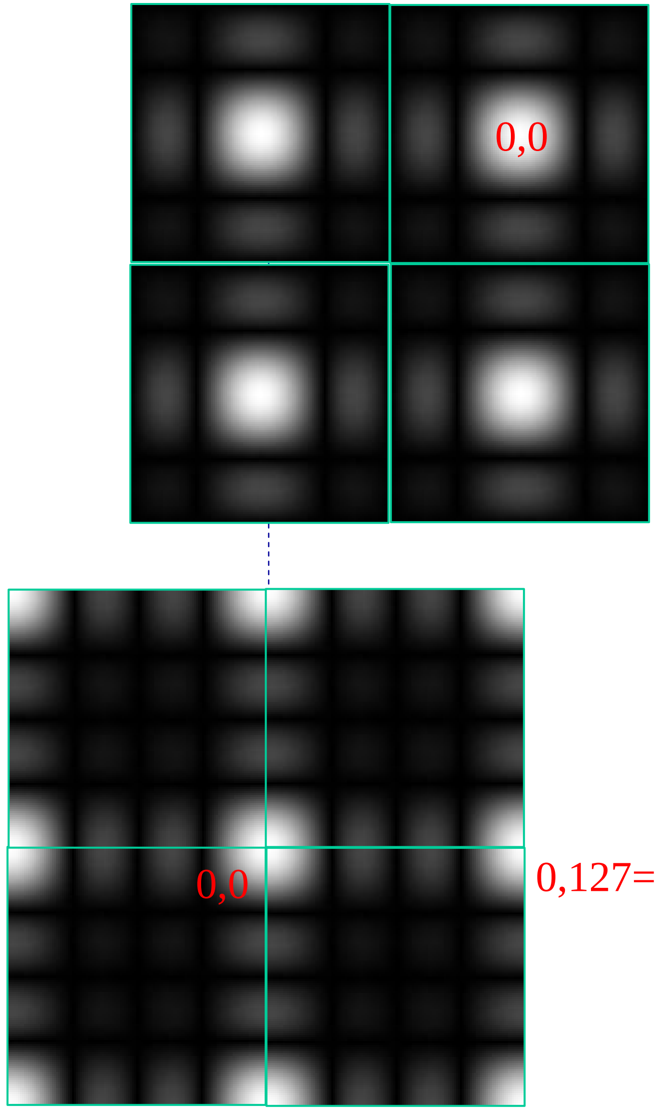




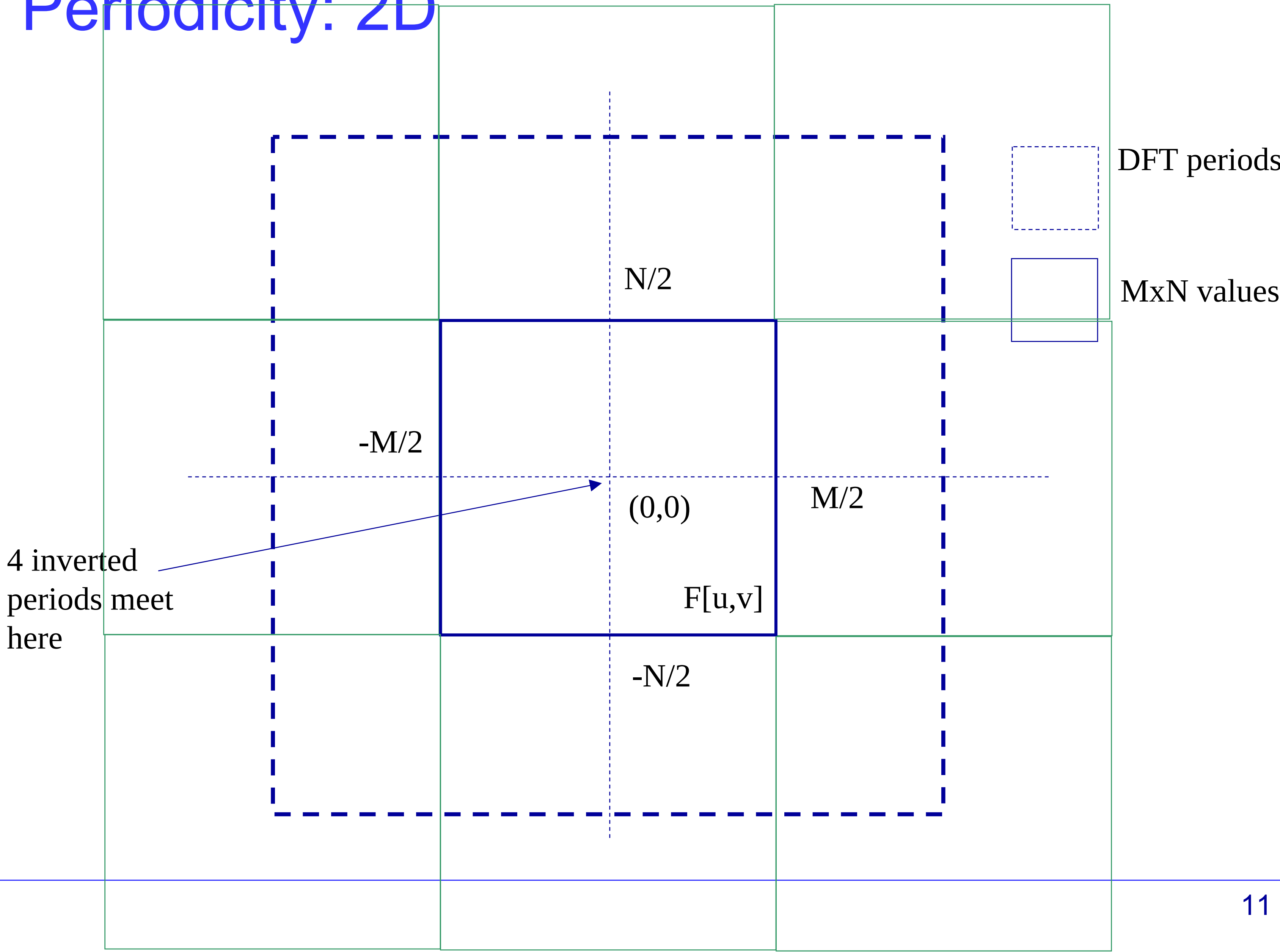
I 4 semiperiodi si incontrano ai vertici

I 4 semiperiodi si incontrano al centro

Periodicity



Periodicity: 2D



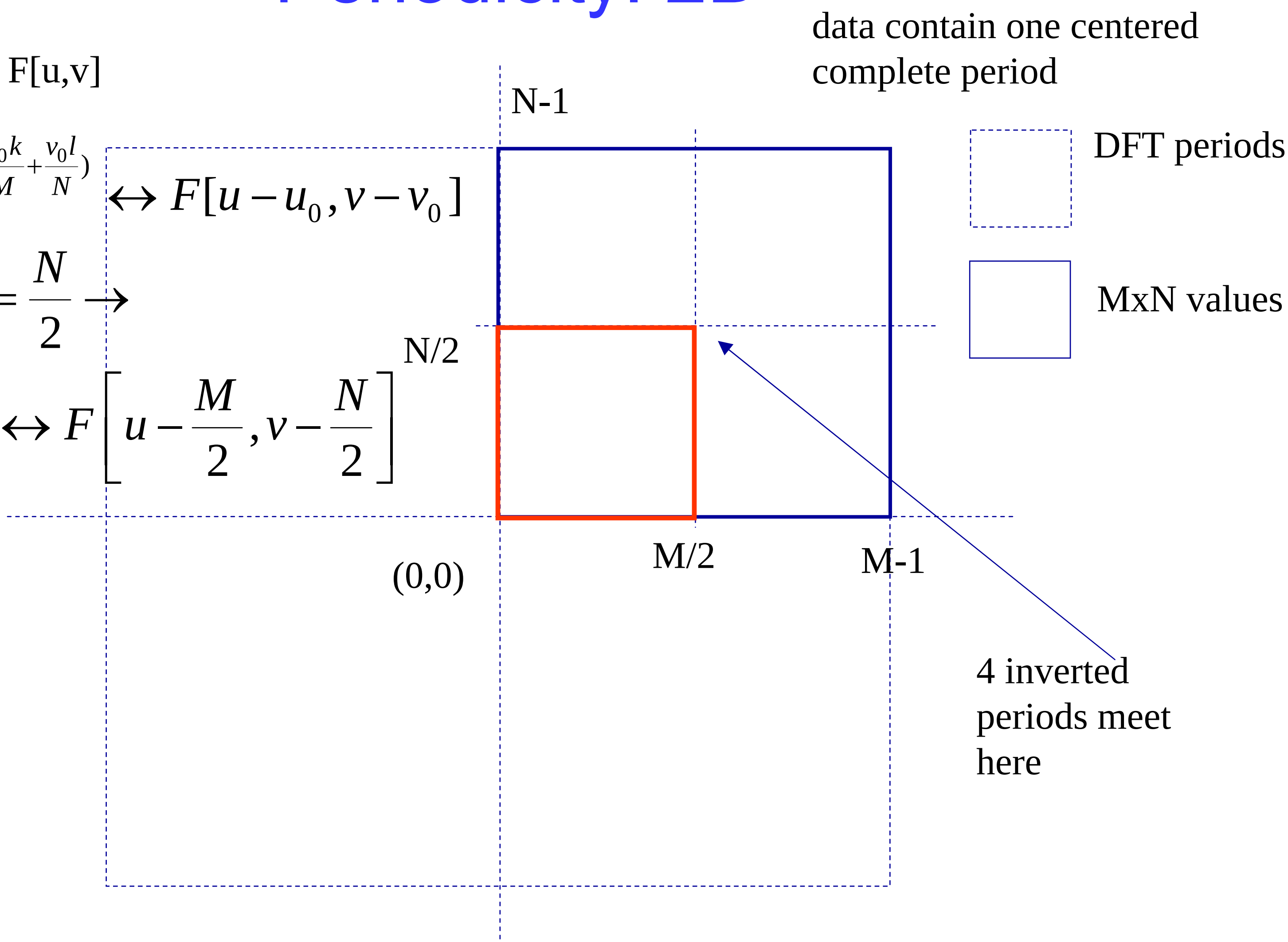
Periodicity: 2D

F[u,v]

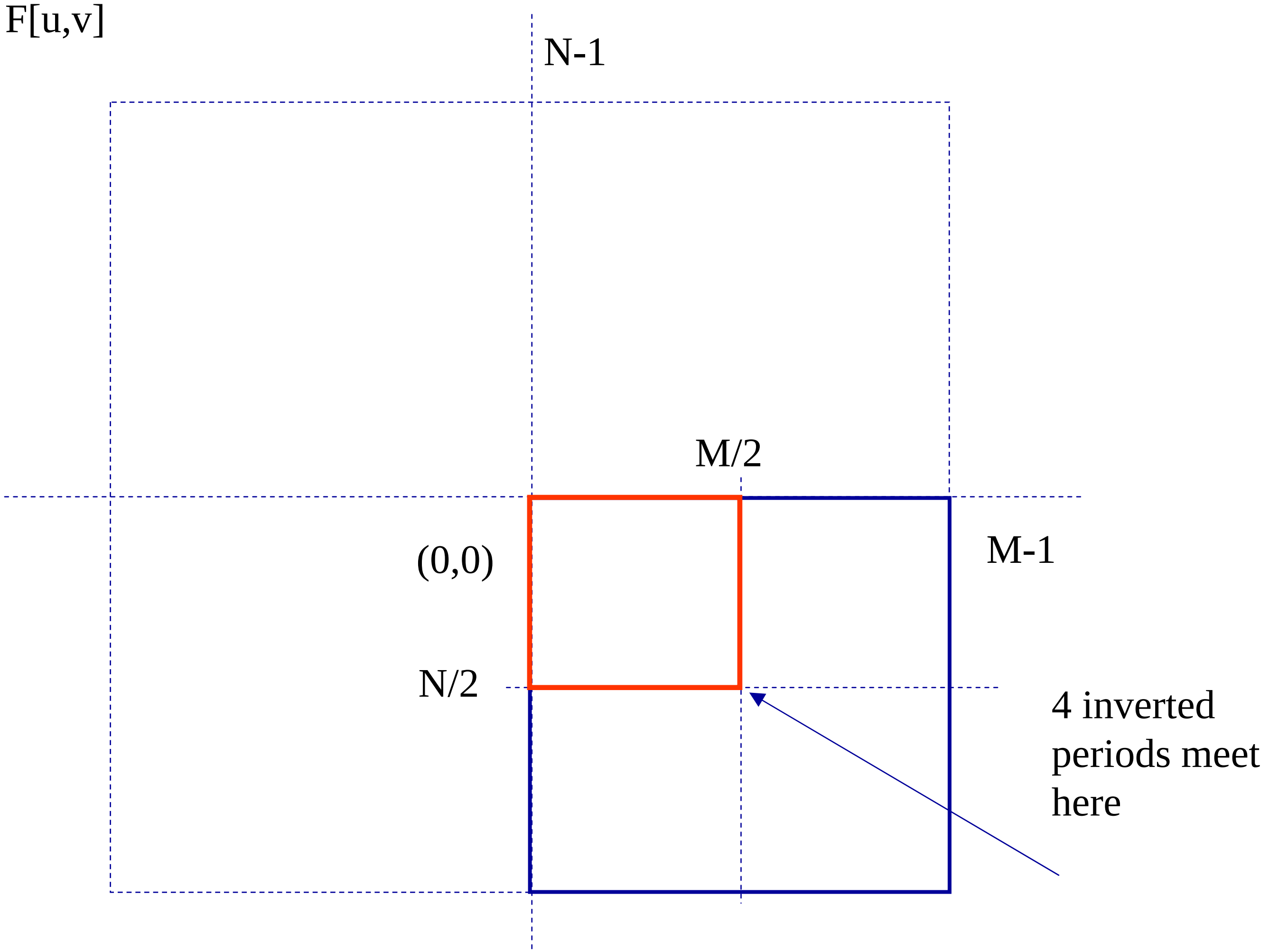
$$f[k,l]e^{j2\pi\left(\frac{u_0 k}{M}+\frac{v_0 l}{N}\right)} \leftrightarrow F[u-u_0, v-v_0]$$

$$u_0 = \frac{M}{2}, v_0 = \frac{N}{2} \rightarrow$$

$$(-1)^{k+l} f[k] \leftrightarrow F\left[u-\frac{M}{2}, v-\frac{N}{2}\right]$$



Periodicity: 2D



Periodicity in spatial domain

- [M,N] point inverse DFT is periodic with period [M,N]

$$f[m,n] = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

$$f[m+M, n+N] = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}(m+M) + \frac{l}{N}(n+N)\right)}$$

$$= \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)} \cancelto{1}{e^{j2\pi\left(\frac{k}{M}M + \frac{l}{N}N\right)}}$$

$$= f[m,n]$$

Angle and phase spectra

$$F[u,v] = |F[u,v]| e^{j\Phi[u,v]}$$

$$|F[u,v]| = \left[\mathrm{Re}\{F[u,v]\}^2 + \mathrm{Im}\{F[u,v]\}^2 \right]^{1/2}$$ modulus (amplitude spectrum)

$$\Phi[u,v] = \arctan\left[\frac{\mathrm{Im}\{F[u,v]\}}{\mathrm{Re}\{F[u,v]\}} \right]$$ phase

$$P[u,v] = |F[u,v]|^2$$ power spectrum

For a real function

$$F[-u,-v] = F^*[u,v]$$ conjugate symmetric with respect to the origin

$$|F[-u,-v]| = |F[u,v]|$$

$$\Phi[-u,-v] = -\Phi[u,v]$$

Translation and rotation

$$f[k,l]e^{j2\pi\left(\frac{m}{M}k+\frac{n}{N}l\right)} \leftrightarrow F\left[u-m,v-l\right]$$

$$f\left[k-m,l-n\right] \leftrightarrow F\left[u,v\right]^{-j2\pi\left(\frac{m}{M}k+\frac{n}{N}l\right)}$$

$$\begin{cases} k = r\cos\vartheta \\ l = r\sin\vartheta \end{cases} \qquad \begin{cases} u = \omega\cos\varphi \\ l = \omega\sin\varphi \end{cases}$$

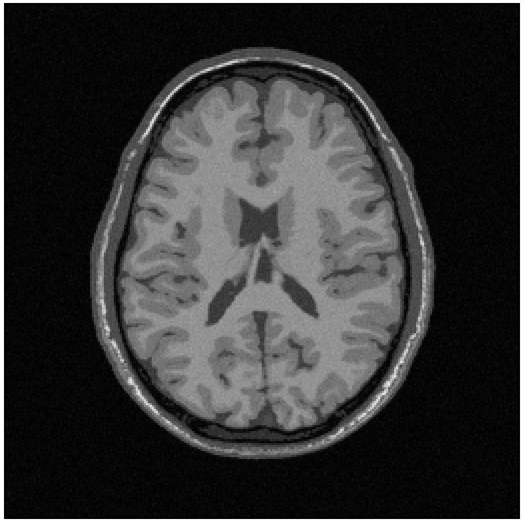
$$f\left[r,\vartheta+\vartheta_0\right] \leftrightarrow F\left[\omega,\varphi+\vartheta_0\right]$$

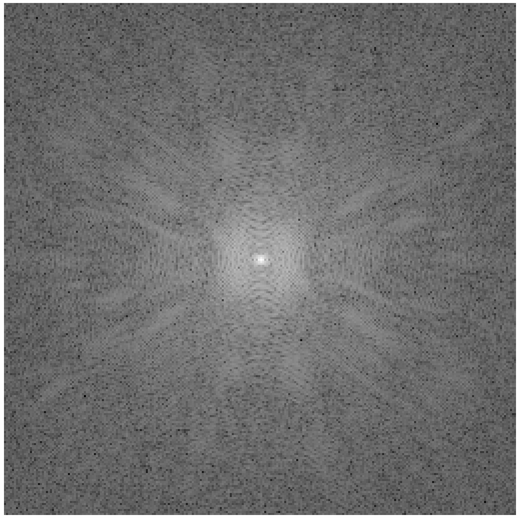
Rotations in spatial domain correspond equal rotations in Fourier domain

mean value

$$F[0,0] = \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f[n,m]$$

DC coefficient





Separability

- The discrete two-dimensional Fourier transform of an image array is defined in series form as

$$F[k,l] = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

- inverse transform

$$f[m,n] = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

  - Because the transform kernels are separable and symmetric, the two dimensional transforms can be computed as sequential row and column one-dimensional transforms.
  - The basis functions of the transform are complex exponentials that may be decomposed into sine and cosine components.

2D DFT: summary

TABLE 4.1
Summary of some important properties of the 2-D Fourier transform.

Property	Expression(s)
Fourier transform	$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$
Inverse Fourier transform	$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$
Polar representation	$F(u,v) = \lvert F(u,v)\rvert e^{-j\phi(u,v)}$
Spectrum	$\lvert F(u,v)\rvert = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$, $R = \text{Real}(F)$ and $I = \text{Imag}(F)$
Phase angle	$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$
Power spectrum	$P(u,v) = \lvert F(u,v)\rvert^2$
Average value	$\bar{f}(x,y) = F(0,0) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$
Translation	$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$ $f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}$ When $x_0 = u_0 = M/2$ and $y_0 = v_0 = N/2$, then $f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$ $f(x-M/2, y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$

2D DFT: summary

Conjugate symmetry	$F(u,v) = F^*(-u,-v)$ $\lvert F(u,v)\rvert = \lvert F(-u,-v)\rvert$
Differentiation	$\dfrac{\partial^n f(x,y)}{\partial x^n} \Leftrightarrow (ju)^n F(u,v)$ $(-jx)^n f(x,y) \Leftrightarrow \dfrac{\partial^n F(u,v)}{\partial u^n}$
Laplacian	$\nabla^2 f(x,y) \Leftrightarrow -(u^2+v^2)F(u,v)$
Distributivity	$\Im[f_1(x,y)+f_2(x,y)] = \Im[f_1(x,y)] + \Im[f_2(x,y)]$ $\Im[f_1(x,y)\cdot f_2(x,y)] \neq \Im[f_1(x,y)]\cdot\Im[f_2(x,y)]$
Scaling	$af(x,y) \Leftrightarrow aF(u,v),\ f(ax,by) \Leftrightarrow \dfrac{1}{\lvert ab\rvert} F(u/a, v/b)$
Rotation	$x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$ $f(r, \theta+\theta_0) \Leftrightarrow F(\omega, \varphi+\theta_0)$
Periodicity	$F(u,v) = F(u+M,v) = F(u,v+N) = F(u+M,v+N)$ $f(x,y) = f(x+M,y) = f(x,y+N) = f(x+M,y+N)$
Separability	See Eqs. (4.6-14) and (4.6-15). Separability implies that we can compute the 2-D transform of an image by first computing 1-D transforms along each row of the image, and then computing a 1-D transform along each column of this intermediate result. The reverse, columns and then rows, yields the same result.

2D DFT: summary

Property	Expression(s)
Computation of the inverse Fourier transform using a forward transform algorithm	$\frac{1}{MN} f^*(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F^*(u, v) e^{-j2\pi(ux/M + vy/N)}$ This equation indicates that inputting the function $F^*(u, v)$ into an algorithm designed to compute the forward transform (right side of the preceding equation) yields $f^*(x, y)/MN$. Taking the complex conjugate and multiplying this result by MN gives the desired inverse.
Convolution†	$f(x, y) * h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) h(x - m, y - n)$
Correlation†	$f(x, y) \circ h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m, n) h(x + m, y + n)$
Convolution theorem†	$f(x, y) * h(x, y) \Leftrightarrow F(u, v) H(u, v)$; $f(x, y) h(x, y) \Leftrightarrow F(u, v) * H(u, v)$
Correlation theorem†	$f(x, y) \circ h(x, y) \Leftrightarrow F^*(u, v) H(u, v)$; $f^*(x, y) h(x, y) \Leftrightarrow F(u, v) \circ H(u, v)$

2D DFT: summary

Some useful FT pairs:

Impulse	$\delta(x, y) \Leftrightarrow 1$
Gaussian	$A\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2(x^2+y^2)} \Leftrightarrow Ae^{-(u^2+v^2)/2\sigma^2}$
Rectangle	$\mathrm{rect}[a, b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
Cosine	$\cos(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow \frac{1}{2}\left[\delta(u + u_0, v + v_0) + \delta(u - u_0, v - v_0)\right]$
Sine	$\sin(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow j\frac{1}{2}\left[\delta(u + u_0, v + v_0) - \delta(u - u_0, v - v_0)\right]$

† Assumes that functions have been extended by zero padding.

other formulations

2D Discrete Fourier Transform

- 2D Discrete Fourier Transform (DFT)

$$F[k,l] = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

where $l = 0,1,\ldots,N-1$

$k = 0,1,\ldots,M-1$

- Inverse DFT

$$f[m,n] = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

2D Discrete Fourier Transform

- It is also possible to define DFT as follows

$$F[k,l]=\frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]e^{-j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}$$

where $k=0,1,\ldots,M-1$
$l=0,1,\ldots,N-1$

- Inverse DFT

$$f[m,n]=\frac{1}{\sqrt{MN}}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} F[k,l]e^{j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}$$

2D Discrete Fourier Transform

- Or, as follows

$$F[k,l] = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]e^{-j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}$$

where $k = 0,1,\ldots,M-1$ and $l = 0,1,\ldots,N-1$

- Inverse DFT

$$f[m,n] = \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} F[k,l]e^{j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}$$

2D DFT

- The discrete two-dimensional Fourier transform of an image array is defined in series form as

$$\mathcal{F}(u,v) = \frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1} F(j,k)\exp\left\{\frac{-2\pi i}{N}(uj+vk)\right\}$$

- inverse transform

$$F(j,k) = \frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} \mathcal{F}(u,v)\exp\left\{\frac{2\pi i}{N}(uj+vk)\right\}$$

2D DCT

Discrete Cosine Transform

2D DCT

- based on most common form for 1D DCT

$$C(u) = \alpha(u) \sum_{x=0}^{N-1} f(x) \cos\left[\frac{\pi(2x+1)u}{2N}\right],$$

u,x=0,1,…, N-1

$$f(x) = \sum_{u=0}^{N-1} \alpha(u) C(u) \cos\left[\frac{\pi(2x+1)u}{2N}\right],$$

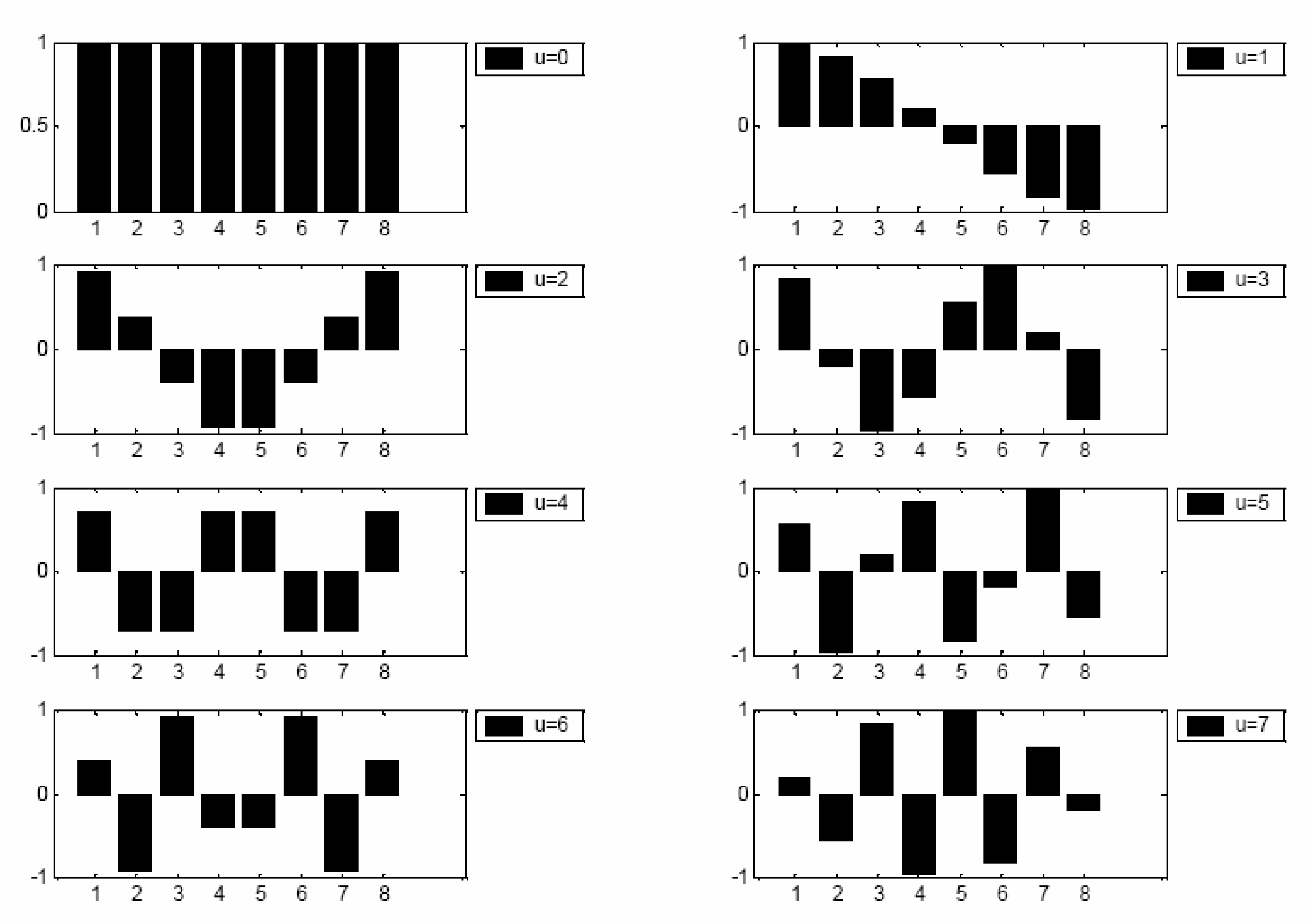
$$\alpha(u) = \begin{cases} \sqrt{\frac{1}{N}} & for \quad u = 0 \\ \sqrt{\frac{2}{N}} & for \quad u \neq 0. \end{cases}$$

$$C(u=0) = \sqrt{\frac{1}{N}} \sum_{x=0}^{N-1} f(x).$$

“mean” value

1D basis functions

Figure 1



Cosine basis functions are orthogonal

2D DCT

- Corresponding 2D formulation

direct
$$C(u,v)=\alpha(u)\alpha(v)\sum_{x=0}^{N-1}\sum_{y=0}^{N-1}f(x,y)\cos\left[\frac{\pi(2x+1)u}{2N}\right]\cos\left[\frac{\pi(2y+1)v}{2N}\right],$$

u,v=0,1,…., N-1

$$\alpha(u)=\begin{cases}\sqrt{\frac{1}{N}} & for \quad u=0\\ \sqrt{\frac{2}{N}} & for \quad u\neq 0.\end{cases}$$

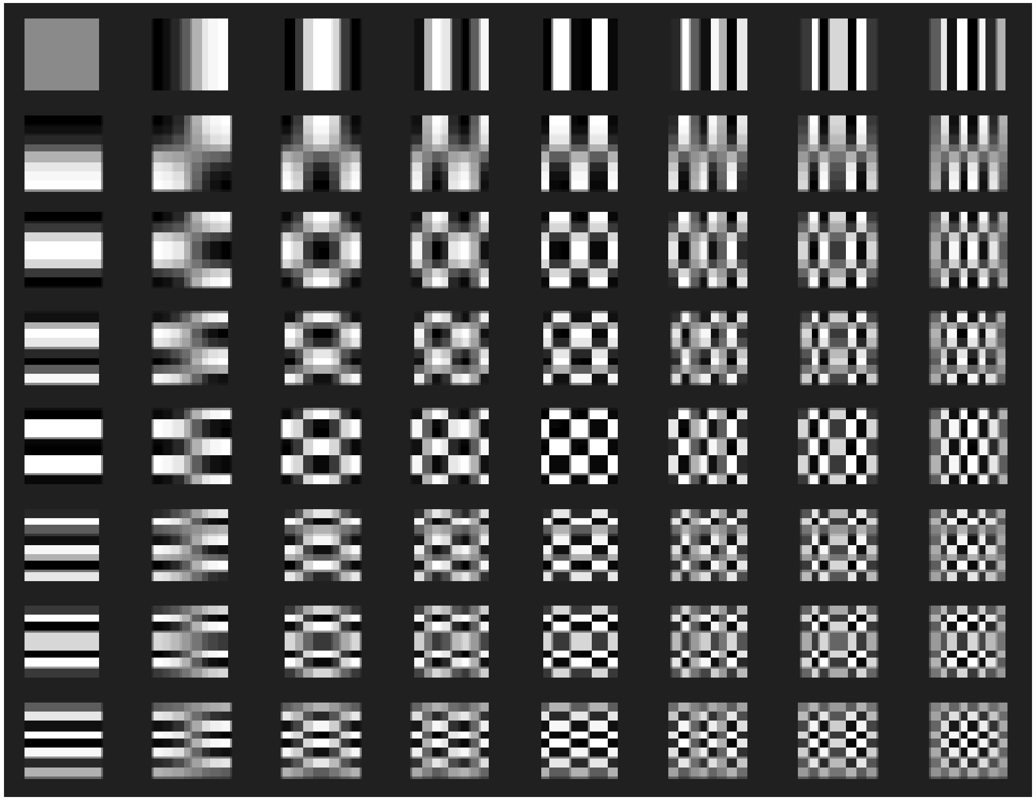
inverse
$$f(x,y)=\sum_{u=0}^{N-1}\sum_{v=0}^{N-1}\alpha(u)\alpha(v)C(u,v)\cos\left[\frac{\pi(2x+1)u}{2N}\right]\cos\left[\frac{\pi(2y+1)v}{2N}\right],$$

2D basis functions

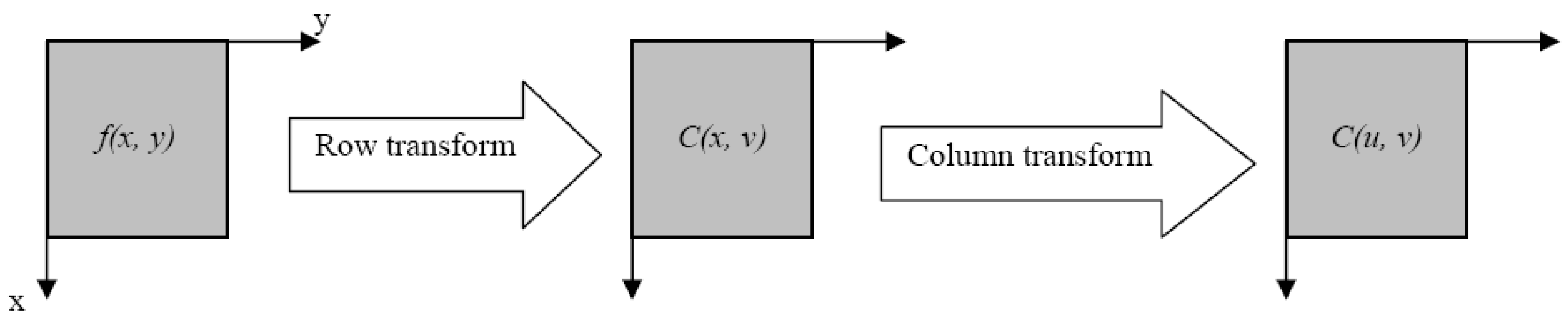
- The 2-D basis functions can be generated by multiplying the horizontally oriented 1-D basis functions (shown in Figure 1) with vertically oriented set of the same functions.
- The basis functions for N = 8 are shown in Figure 2.
  - The basis functions exhibit a progressive increase in frequency both in the vertical and horizontal direction.
  - The top left basis function assumes a constant value and is referred to as the *DC coefficient*.

2D DCT basis functions

Figure 2



Separability



The inverse of a multi-dimensional DCT is just a separable product of the inverse(s) of the corresponding one-dimensional DCT , e.g. the one-dimensional inverses applied along one dimension at a time

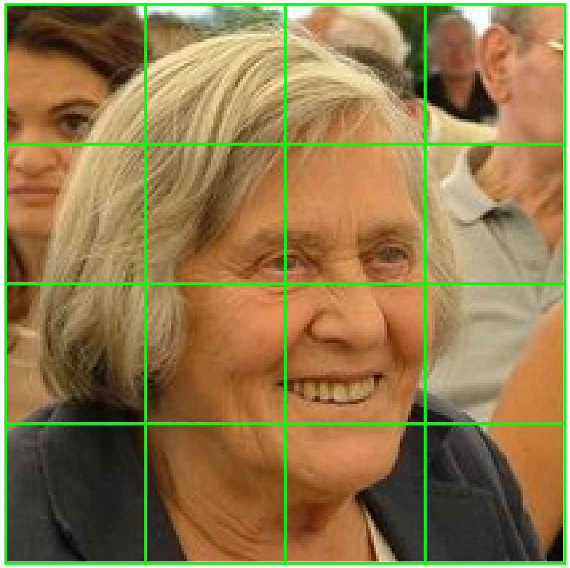
Block-based implementation

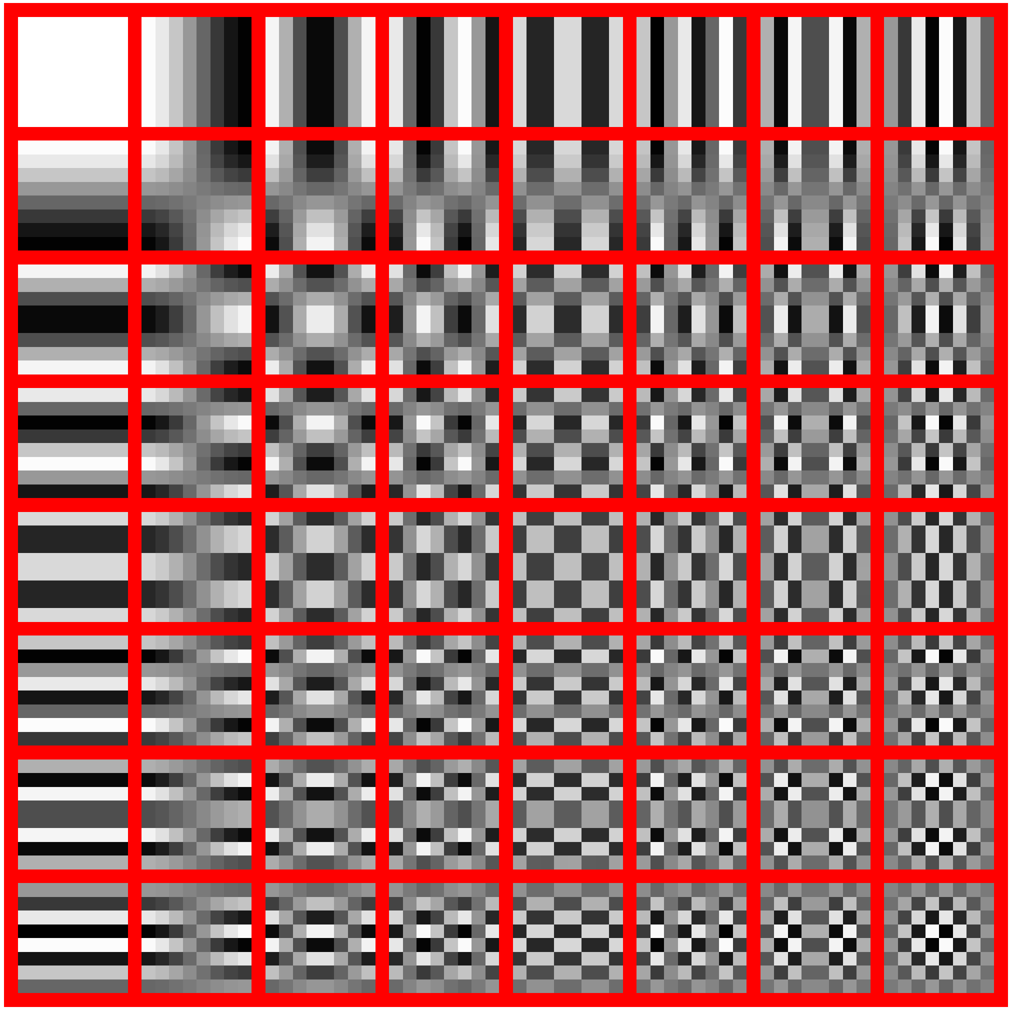
Basis function

Block-based transform

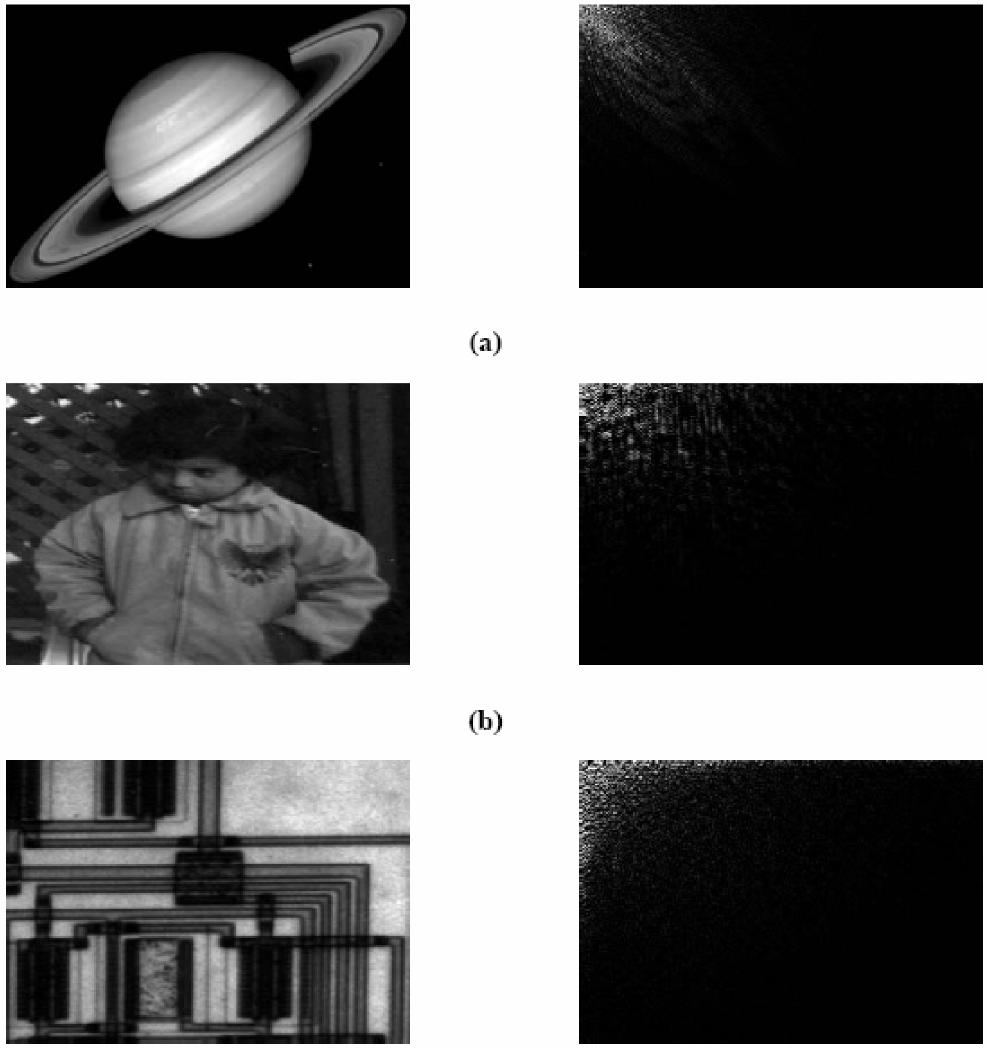
Block size
N=M=8

The source data (8x8) is transformed to a linear combination of these 64 frequency squares.



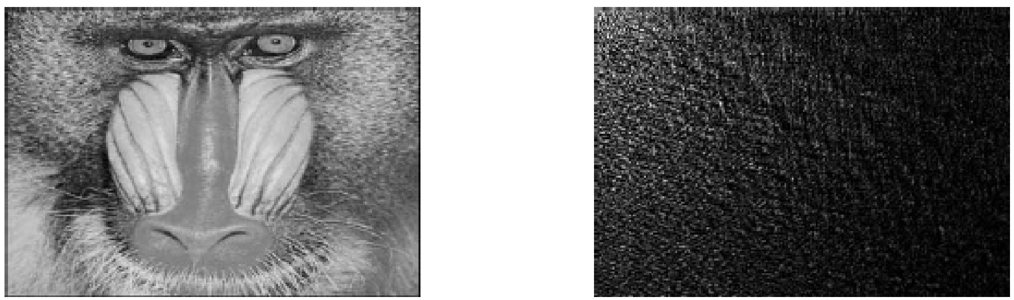


Energy compaction

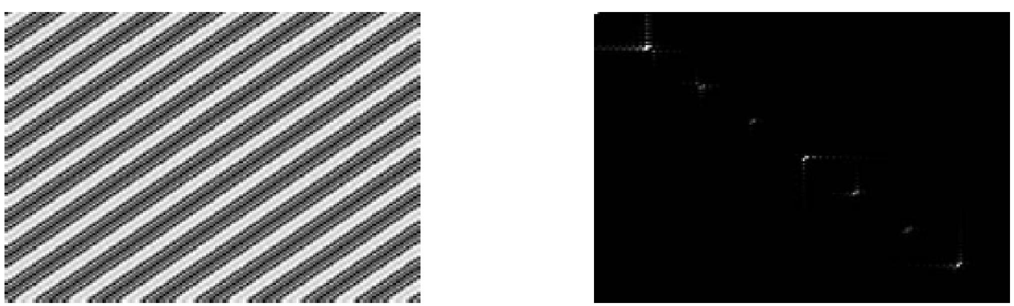


Energy compaction

(d)



(e)



Appendix

- Eulero's formula

$$A(j,k\,;u,v) = \exp\left\{\frac{-2\pi i}{N}(uj+vk)\right\} = \cos\left\{\frac{2\pi}{N}(uj+vk)\right\} - i\sin\left\{\frac{2\pi}{N}(uj+vk)\right\}$$

$$B(j,k\,;u,v) = \exp\left\{\frac{2\pi i}{N}(uj+vk)\right\} = \cos\left\{\frac{2\pi}{N}(uj+vk)\right\} + i\sin\left\{\frac{2\pi}{N}(uj+vk)\right\}$$