Geometria – Prova scritta del 9/7/2008

(Prof. M. Spera)

1. Data la parabola $\mathcal{P}: y^2 = 2px$ $(p>0)$

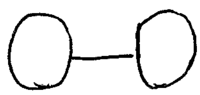
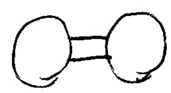
se ne determini l'evoluta $\mathcal{E}_{\mathcal{P}}$, abbozzandone il grafico.

[Si può procedere in due modi ...]

2. Data la parabola $\mathcal{P}$ dell'ex. 1, sia Σ la superficie di rivoluzione ottenuta facendo ruotare $\mathcal{P}$ attorno all'asse x.

Se ne calcolino le curvature principali e la curvatura gaussiana in un punto generico.

3. Dati gli spazi topologici

X =  Y = 

si dica se essi sono omeomorfi, giustificando dettagliatamente la risposta.

Tempo a disposizione: 2h. Le risposte vanno adeguatamente giustificate.

① $y^2 = 2px$ $\quad \mathcal{P}$

1°: usiamo il metodo dell'inviluppo [delle normali]

troviamo la normale in $P: (x, y) \in \mathcal{P}$

$$2px - y^2 = 0 \qquad f(x, y) = 0$$

tangente: $f_x^0 (x - x_0) + f_y^0 (y - y_0) = 0$

$$f_x = 2p$$
$$f_y = -2y$$

$$2p(X - x) - 2y(Y - y) = 0$$

$$\cancel{2}p\left(X - \frac{y^2}{2p}\right) - \cancel{2}y(Y - y) = 0 \qquad \text{tangenti}$$

normali: $(*)$ $y\left(X - \frac{y^2}{2p}\right) + p(Y - y) = 0$

$\frac{\partial}{\partial y}() = 0$ $\qquad \left(X - \frac{y^2}{2p}\right) + y \cdot \left(\frac{-\cancel{2}y}{\cancel{2}p}\right) - p = 0$

$$X - \frac{y^2}{2p} - \frac{y^2}{p} - p = 0$$

$$X - \frac{3}{2p} y^2 - p = 0$$

$$X - p = 3\frac{y^2}{2p}$$

$$X - p = 3x \qquad (*) \quad X - p = \frac{3}{2}\frac{y^2}{p}$$

$$\Rightarrow \quad X - x = p + 2x$$

sostituendo in (*) si ha:

$$y\,(p+2x) + p\,(Y-y) = 0$$

$$Y - y = -\frac{y\,(p+2x)}{p}$$

$$Y = y\left(1 - \frac{p+2x}{p}\right) = y\,\frac{(-2x)}{p}$$

$$**) \quad Y = y\,\frac{(-2)}{p}\,\frac{y^2}{2p} = -\frac{y^3}{p^2}$$

(*) e (**) forniscono

$$(X-p)^3 = \frac{27}{8}\frac{y^6}{p^3}$$

$$Y^2 = \frac{y^6}{p^4} \quad \Rightarrow \quad y^6 = p^4 Y^2$$

$$\Rightarrow \quad (X-p)^3 = \frac{27}{8}\,p\,Y^2$$

$$\boxed{27\,p\,Y^2 = 8(X-p)^3}$$ vedi oltre

[parabola semicubica]

2° : "standard"

$$t = y \qquad P = \left(\frac{y^2}{2p} , y \right)$$

$$\|\dot P\|^2 = \frac{y^2}{p^2} + 1 = \frac{y^2 + p^2}{p^2} \qquad \dot P = \left(\frac{y}{p} , 1 \right)$$

$$\ddot P = \left(\frac{1}{p} , 0 \right)$$

$$i \dot P = \left(-1 , \frac{y}{p} \right)$$

$$\langle i\dot P , \ddot P \rangle = - \frac{1}{p}$$

$$- \frac{y^2 + p^2}{p}$$

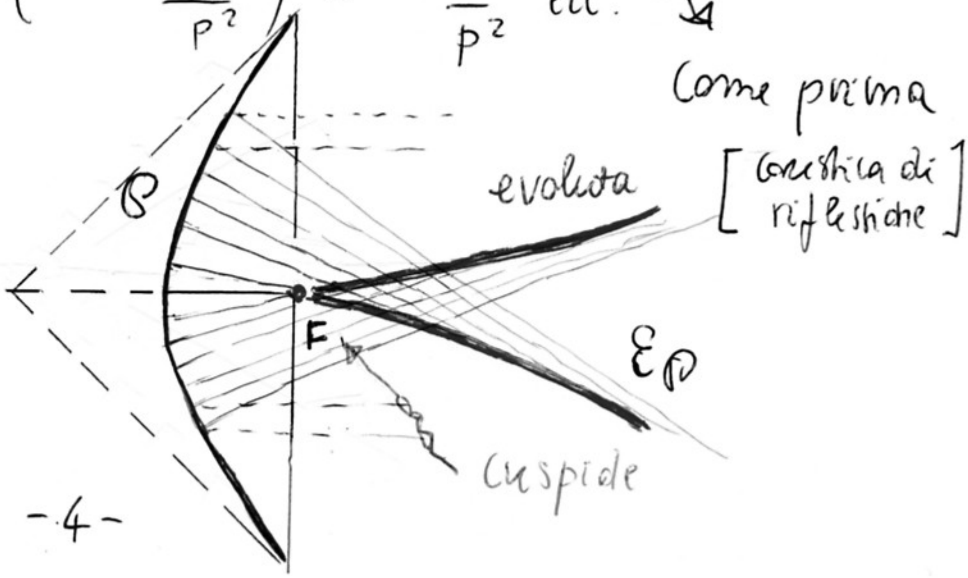
$$Q - P = \frac{\|\dot P\|^2}{\langle i\dot P , \ddot P \rangle} \, i\dot P = \frac{\frac{y^2+p^2}{p^2}}{-\frac{1}{p}} \left(-1 , \frac{y}{p} \right)$$

$$X - x = \frac{y^2 + p^2}{p} = \frac{y^2}{p} + p \implies X - p = \frac{y^2}{p} + x$$

$$Y - y = - \frac{y^2 + p^2}{p^2} \, y$$

$$Y = y \left(1 - \frac{y^2 + p^2}{p^2} \right) = - \frac{y^3}{p^2} \text{ ecc.}$$

Come prima



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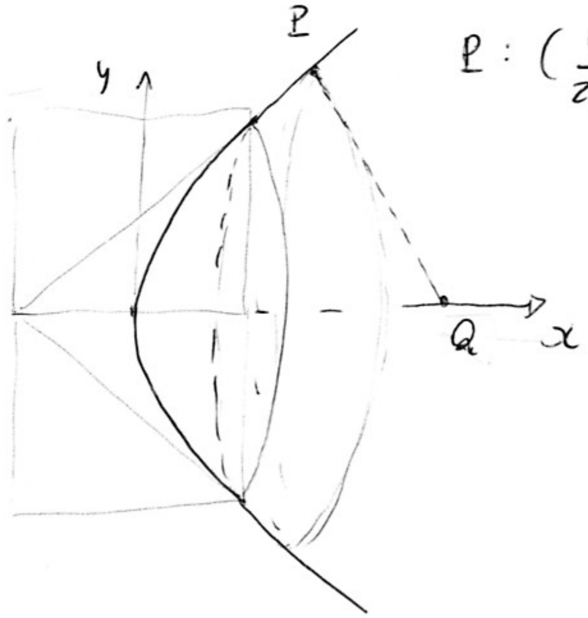


$P : \left(\frac{y^2}{2p} , y \right)$

$y^2 = 2px \quad \rightsquigarrow$ par. di rotazione Σ

$$z^2 + y^2 = 2px$$

calcoliamone la curvatura in un pto generico: essa è costante lungo i paralleli, pertanto basterà determinarla in $P \in \Sigma \cap \{z = 0\}$

Si ha $K = k_1 \, k_2$

k_1: curv. meridiano

k_2: $\frac{1}{N}$ $\quad N =$ gran normale

Calcoliamo k_1, sfruttiamo i calcoli dell'es. 1

$$k_1 = \frac{\langle i\dot{P}, \ddot{P} \rangle}{\|\dot{P}\|^3} \qquad \left(= \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \right)$$

$t = y$

$$\kappa_1 = - \frac{\frac{y^2+p^2}{p}}{\left(\frac{y^2+p^2}{p^2}\right)^{3/2}}$$

$P: \left(\frac{y^2}{2p}, y\right)$

$\dot{P} = \left(\frac{y}{p}, 1\right)$

$\ddot{P} = \left(\frac{1}{p}, 0\right)$

$i\dot{P} \ (-1, \frac{y}{p})$

$\langle i\dot{P}, \ddot{P} \rangle = -\frac{1}{p}$

$\|\dot{P}\|^3 = \left(\frac{y^2+p^2}{p^2}\right)^{3/2}$

$$= - \frac{-\frac{1}{p}}{\left(\frac{y^2+p^2}{p^2}\right)^{3/2}}$$

$$= - \frac{1}{p\left(\frac{y^2+p^2}{p^2}\right)^{\frac{3}{2}}} = - \frac{1}{\frac{(y^2+p^2)^{3/2}}{p^2}} = - \frac{p^2}{(y^2+p^2)^{3/2}}$$

Calcoliamo la grannormale:

riprendiamo la ★ di 1) $\swarrow$ ⚠ cambio di not.

$$y\left(X - \frac{y^2}{2p}\right) + p(Y-y) = 0$$

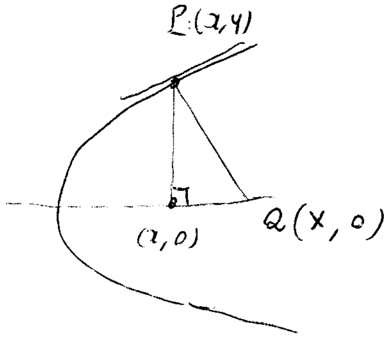
troviamo Q: poniamo $Y=0$

$$\Rightarrow \cancel{y}\left(X - \frac{y^2}{2p}\right) = p\cancel{y} \qquad y \neq 0$$

$$X - \frac{y^2}{2p} = p \Rightarrow X = p + \frac{y^2}{2p} \left(= \frac{2p^2+y^2}{2p}\right)$$

Calcoliamo $\overline{PQ} =$

$$\left[\left(p + \frac{y^2}{2p} - \frac{y^2}{2p} \right)^2 + y^2 \right]^{\frac{1}{2}}$$



$$= \left(p^2 + y^2 \right)^{\frac{1}{2}}$$

$$\Rightarrow \quad k_2 = - \frac{1}{\sqrt{p^2+y^2}} \qquad k_1 = \frac{-p^2}{\left(p^2+y^2\right)^{3/2}}$$

⚠

$$\Rightarrow \quad K = k_1 k_2 = \frac{p^2}{(p^2+y^2)^2} > 0$$

[quadrica a punti ellittici]

in generale $\quad K = \dfrac{p^2}{(p^2+\rho^2)^2}$

su un parallelo di raggio ρ

Variante (standard)

Sup. di rot di $y^2 = 2px$ attorno all'asse x

$$\rho^2 = 2px$$

$$\begin{cases} x = \dfrac{\rho^2}{2p} \\ y = \rho\cos\varphi \\ z = \rho\sin\varphi \end{cases} \qquad \rho^2 = y^2+z^2$$

$$\underline{r} = \left(\frac{\rho^2}{2p}, \rho\cos\varphi, \rho\sin\varphi \right)$$

$$\underline{r}_\rho = \left(\frac{\rho}{p}, \cos\varphi, \sin\varphi \right)$$

$$\underline{r}_\varphi = \left(0, -\rho\sin\varphi, \rho\cos\varphi \right)$$

$$E = \|r_\rho\|^2 = 1 + \frac{\rho^2}{p^2} = \frac{p^2+\rho^2}{p^2}$$

$$F = 0$$

$$G = \|\underline{r}_\varphi\|^2 = \rho^2$$

$$\underline{r}_\rho \times \underline{r}_\varphi = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\rho}{p} & \cos\varphi & \sin\varphi \\ 0 & -\rho\sin\varphi & \rho\cos\varphi \end{vmatrix} = \underline{i}\,\rho - \underline{j}\,\frac{\rho^2}{p}\cos\varphi - \underline{k}\,\frac{\rho^2}{p}\sin\varphi$$

$$\| \quad \|^2 = \rho^2 + \frac{\rho^4}{p^2} = \rho^2\,\frac{p^2+\rho^2}{p^2}$$

anche:

$$\|\underline{r}_\rho \times \underline{r}_\varphi\| = \sqrt{EG} = \sqrt{\frac{p^2+\rho^2}{p^2}\,\rho^2} = \sqrt{p^2+\rho^2}\,\frac{\rho}{p}$$

$$\underline{r}_{\rho\rho} = \left(\frac{1}{p} \,,\, 0 \,,\, 0 \right)$$

$$\underline{r}_{\rho\varphi} = \left(0 \,,\, -\sin\varphi \,,\, \cos\varphi \right)$$

$$\underline{r}_{\varphi\varphi} = \left(0 \,,\, -\rho\cos\varphi \,,\, -\rho\sin\varphi \right)$$

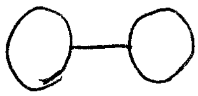
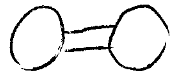
$$e = \frac{\langle \underline{r}_{\rho\rho} \,,\, \underline{r}_\rho \times \underline{r}_\varphi \rangle}{\|\quad\|} = \frac{\cancel{\rho}}{\cancel{p}} \cdot \frac{\cancel{p}}{\cancel{\rho}\sqrt{p^2+\rho^2}} = \frac{1}{\sqrt{p^2+\rho^2}}$$

$$f = 0$$

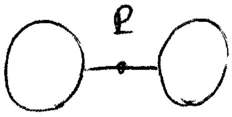
$$g = \frac{\rho^{\cancel{3}\,2}}{\cancel{p}} \cdot \frac{\cancel{p}}{\cancel{\rho}\sqrt{p^2+\rho^2}} = \frac{\rho^2}{\sqrt{p^2+\rho^2}}$$

$$K = \frac{e\,g}{EG} = \frac{\cancel{\rho^2}}{p^2+\rho^2}\;\frac{1}{\cancel{\rho^2}\,\frac{p^2+\rho^2}{p^2}}$$

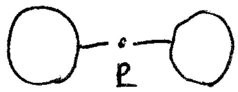
$$= \frac{p^2}{(p^2+\rho^2)^2} \quad \checkmark$$

③ $X =$  e $Y =$ 

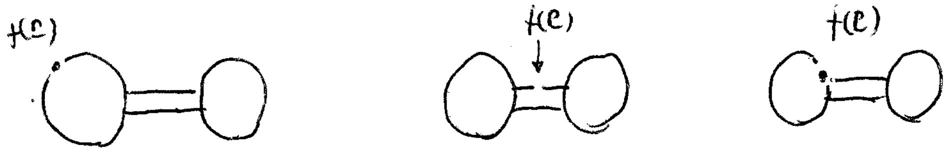
non sono omeomorfi: infatti, detto f un eventuale omeomorfismo, si avrebbe che



$$f\big|_{X\setminus\{e\}} : X\setminus\{e\} \longrightarrow Y\setminus\{f(e)\}$$

è ancora un omeomorfismo. Ma $X\setminus\{e\}$ è sconnesso 

, mentre $Y\setminus\{f(e)\}$ è connesso:



ecc.

Nota X e Y sono <u>loc.</u> connessi per archi

Dunque connessione = conn. per archi