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Master's Program in Applied Mathematics

First written test of Functional Analysis

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Solve the following problems. Justify your conclusions. Time: 120 min.

Pb 1. Let $\psi \in C^1([0,+\infty))$ be bounded together with its first derivative ψ', $\psi(0) = 0$ and $\psi(x) \geq 1$ for all $x \geq 1$. Consider $f_n : [0,+\infty) \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} 0 & \text{if } x < n, \\ \psi(x-n) & \text{if } x \geq n. \end{cases}$$

Is the sequence (f_n) bounded and/or equicontinuous? Is it relatively compact in $C([0,+\infty))$?

Pb 2. Let $(a_n) \subset [0,1]$ and $(b_n) \subset \mathbb{R}$. Then the sequence $(a_n b_n)$ is bounded if and only if for every sequence $(f_n) \subset C([0,1])$ with $\|f_n\| \to 0$ as $n \to \infty$ it follows $b_n \int_0^{a_n} f_n(x)dx \to 0$.
(Hint: You may want to ask S. Banach and H. Steinhaus for advice..)

Pb 3. Let $U : C([0,1]) \to C([0,1])$ be defined by $U(f) = f^3$ for $f \in C([0,1])$. Prove that U is continuous but not uniformly continuous if $C([0,1])$ is equipped with the supremum norm. Prove also that U is not continuous if $C([0,1])$ is endowed with the L^1-norm.

Pb 4. Consider $T : \ell^3(\mathbb{N}) \to L^3(0,1)$ defined by

$$T(x)(y) := \sum_{n=1}^{\infty} \sqrt[3]{n(n+1)}\, x_n \chi_{[\frac{1}{n+1},\frac{1}{n}]}(y), \quad y \in [0,1], \;\; x = (x_n) \in \ell^3(\mathbb{N}).$$

Prove that T is an isometry.

Pb 5. Prove that $\{f \in C([0,1] : f(0) = 0)\}$ is a dense convex subset of $L^2([0,1])$.

Pb 6. Study the convergence in $L^p(0,+\infty)$, $p > 1$, of the sequence of functions

$$f_n(x) = \frac{nx+1}{nx^2+n+1}.$$