Optimal dynamic advertising model

We denote by $G(t)$ the stock of goodwill of the product/service at time t. Following the model of Nerlove and Arrow (1962), $G(t)$ summarizes the effects of present and past advertising on demand; goodwill needs an advertising effort to increase, but is subject to spontaneous decay. The goodwill value is the joint result of an advertising process, which is decided by the manufacturer, and of a known exogenous and deterministic interference.

The manufacturer's action is the advertising intensity $u(t) \geq 0$ (the activation level of an advertising medium) and we assume that the goodwill level evolves over time according to the differential equation

$$\dot{G}(t) = \gamma u(t) - \xi - \delta G(t), \tag{1}$$

where

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- $\delta > 0$ the goodwill-depreciation rate;
- $\gamma u > 0$ represents the effect of activating the advertising medium at level $u \in [0; u_{\max}]$;
- $\xi > 0$ is the constant exogenous interference.

We assume that we know the goodwill level

$$G(0) = \alpha > 0 \tag{2}$$

at the initial time. Since the term $\gamma u(t) - \xi$ can be negative (as well as positive), this may lead to negative values of $G(t)$. These negative values are admissible, but the corresponding demand is 0. Following Luca Grosset and Bruno Viscolani (2009), we assume that the product-demand rate is a piecewise linear function of its goodwill,, i.e.,

$D(G) = \beta \max(G, 0) =: \beta.[G]^+$, where the constant $\beta > 0$.

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The manufacturer's discounted profit over the infinite horizon $[0, +\infty)$ is

$$\Pi(u) := \int_0^{+\infty} e^{-\rho t}[\pi D(G(t)) - \frac{\kappa}{2}u^2(t)]dt, \tag{3}$$

where $\rho > 0$ is the discount parameter, $\pi D(G(t))$ is the manufacturer's marginal profit and $\frac{\kappa}{2}u^2(t)$ denotes the advertising costs at the time t.

The considered optimal control problem is to maximize the criterion (3) under the dynamics (1) and the initial condition (2). Further we follow the approach of Luca Grosset and Bruno Viscolani (2009) for describing some properties of the solution of this optimal control problem.

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Remark.

The set of the reachable goodwill values is compact, because the motion equation is (1) and the set $[0, u_{\max}]$ is compact. From here, it follows that the optimal control problem (to maximize the criterion (3) under the dynamics (1) and the initial condition (2)) has a solution (cf., also, an existence theorem of Seierstad and Sydsaeter from 1987).

Notations.

Given a function $u : [0, \tau) \to [0, u_{\max}]$ or a function $u : [0, \infty) \to [0, u_{\max}]$, we denote by $u_\tau : [0, \tau) \to [0, u_{\max}]$ the function defined as $\begin{cases} u(t), & t \in [0, \tau), \\ 0, & t \in [\tau, +\infty). \end{cases}$ We denote by $G(t, u)$ the value at time t of the solution of (1) corresponding to the control function u with the initial condition (2).

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Lemma 1. (Luca Grosset and Bruno Viscolani (2009))

If u is an admissible control such that for some $\tau > 0$ and $\theta > 0$

$$\left(\begin{array}{rl} G(t,u) > 0, & \text{for each } t \in [0,\tau), \\ G(t,u) < 0, & \text{for each } t \in [\tau,\tau+\theta), \\ G(\tau,u) & = G(\tau+\theta,u) = 0, \end{array} \right.$$

then u cannot be optimal for the problem (3), (1) and (2).

Proof.

Let us assume that u is optimal control for the problem (3), (1) and (2). Then $\Pi(u) = A + B + C$, where

$$A = \int_0^\tau e^{-\rho t}[\pi\beta G(t,u) - \frac{\kappa}{2}u^2(t)]dt,$$

Proof of Lemma 1 (continuation).

$$B = -\frac{\kappa}{2}\int_{\tau}^{\tau+\theta} e^{-\rho t}u^2(t)dt,$$

$$C = \int_{\tau+\theta}^{\infty} e^{-\rho t}\pi\beta[G(t,u)]^+ - \frac{\kappa}{2}u^2(t)]dt.$$

If we assume that $B = 0$, then we obtain that $u(t) = 0$ for almost every $t \in [\tau, \tau+\theta)$. Since $G(\tau, u) = 0$, the Cauchy formula implies that

$$G(\tau+\theta, u) = G(\tau, u)e^{-\delta(\tau+\theta)} - \xi\int_{\tau}^{\tau+\theta} e^{-\delta(\tau+\theta-s)}ds < 0,$$

and this contradicts the assumptions of Lemma 1. Hence $B < 0$.

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Proof of Lemma 1 (continuation).

Let us assume that $C < -B$. Then $\Pi(u_\tau) = A > A + B + C = \Pi(u)$, and this contradicts the optimality of u. Also, $C < \infty$ because the control function is bounded and the solution of (1) is

$$G(t,u) = e^{-\delta t} G_0 - \frac{\xi}{\delta}(1 - e^{-\delta t}) + \gamma \int_0^t e^{-\delta(t-s)} u(s) ds \leq$$

$$\leq G_0 + 0 + \frac{\gamma}{\delta} u_{\max} \left(1 - e^{-\delta t}\right) \leq G_0 + \frac{\gamma}{\delta} u_{\max},$$

and hence $G(\cdot, u)$ is bounded. Let us define the control $u^*(t) = \begin{cases} u(t), & \text{if } t \in [0,\tau), \\ u(t+\theta), & \text{if } t \in [\tau, \infty). \end{cases}$ The corresponding state variable $G(t,u^*) = \begin{cases} G(t,u), & \text{if } t \in [0,\tau), \\ G(t+\theta, u), & \text{if } t \in [\tau,\infty). \end{cases}$

Proof of Lemma 1 (continuation).

Thus u^* is an admissible control such that

$$\Pi(u) - \Pi(u^*) = -\frac{\kappa}{2}\int_{\tau}^{\tau+\theta} e^{-\rho t} u^2(t)dt$$

$$+ \int_{\tau+\theta}^{\infty} e^{-\rho t}[\pi\beta[G(t,u)]^+ - \frac{\kappa}{2}u^2(t)]dt$$

$$- \int_{\tau}^{\infty} e^{-\rho t}[\pi\beta[G(t+\theta,u)]^+ - \frac{\kappa}{2}u^2(t+\theta)]$$

$$= B + C(1 - e^{\rho\theta}) < 0$$

(because $B < 0$, $C \geq -B > 0$ and $1 - e^{\rho\theta} < 0$), and hence, the control u is not optimal.

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Remark.

This Lemma has an interesting interpretation from a practical point of view. If the goodwill (and therefore the demand) reaches the value zero at time $\tau > 0$, then it will be negative (the demand will remain 0) for all $t > \tau$.

Notation.

If $w : [0, +\infty) \to [0, u_{\max}$ is an admissible control, the we denote by (τ, w) (instead of $\left(\tau, w_{|_{[0,\tau]}}\right)$) the time-control pair, restricting w to the interval $[0, \tau]$.

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Lemma 2. (Luca Grosset and Bruno Viscolani (2009))

The time-control pair (τ, u) is an optimal solution of the variable final-time problem of maximizing

$$\Psi(T, v) = \int_0^T e^{-\rho t}[\pi\beta G(t, u) - \frac{\kappa}{2} v^2(t)]dt \qquad (4)$$

subject to

$$\dot{G}(t, v) = -\delta G(t, v) + \gamma v(t) - \xi, \qquad (5)$$

$$G(0, v) = \alpha > 0, \;\; G(t, v) \geq 0 \qquad (6)$$

if and only if the control u_τ , obtained from u by extending u to the interval $[0, \infty)$ with a 0 value, is an optimal solution for the optimal control problem (to maximize the criterion (3) under the dynamics (1) and the initial condition (2)) and $G(\tau, u_\tau) = 0$.

Proof of Lemma 2.

$\Rightarrow$ Let the time-control pair u be an optimal solution for the variable final-time problem (4)-(6). First of all, if $G(\tau, u) > 0$, then for sufficiently small $\varepsilon > 0$ we have that $G(\tau + \varepsilon, u_\tau) > 0$. But this implies that $\Psi(\tau, u) < \Psi(\tau + \varepsilon, u_\tau)$ and this contradicts the optimality of u. Therefore we must have $G(\tau, u) = 0$. We have proved that $G(\tau, u_\tau) = G(\tau, u) = 0$. We notice that the control u_τ is an admissible solution for problem (1)-(3) and that

$$\Psi(\tau, u) = \int_0^\tau e^{-\rho t}[\pi\beta G(t, u) - \frac{\kappa}{2}u^2(t)]dt = \Pi(u_\tau). \qquad (7)$$

Proof of Lemma 2 (continuation).

Let us assume that u_τ is not optimal for problem (1)-(3), and let w be an optimal control of problem (1)-(3), which exists because of Remark 1, then we have that $\Pi(u_\tau) < \Pi(w)$, and therefore

$$\Psi(\tau, u) = \Pi(u_\tau) < \Pi(w). \tag{8}$$

Now, either of the following cases may occur:
(a) there exists a time $\theta > 0$ such that $G(\theta, w) = 0$;
(b) $G(t, w) > 0$ for all $t \in [0, \infty)$.

Case (a)

Case (a) implies that (τ, u) is not optimal for problem (4)-(6) because, from the inequality (8) and Lemma 1, we obtain that $\Psi(\tau, u) < \Pi(w) = \Psi(\theta, w)$, which is a contradiction.

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Proof of Lemma 2 (continuation): Case (b)

We have that

$\lim_{T\to\infty} = \lim_{T\to\infty} \int_0^T e^{-\rho t}[\pi\beta G(t,u) - \frac{\kappa}{2}u^2(t)]dt = \Pi(w)$. From here and the inequality (8), we obtain that there exists some $T > 0$ such that $\Psi(\tau, u) < \Psi(T, w)$ which contradicts the optimality of u for the problem (4)-(6).

$\Leftarrow$ Let u be an optimal solution for problem (1)-(3) with $G(\tau, u) = 0$ and let us assume that (τ, u) is not optimal for the variable final time problem (4)-(6). Therefore, there exists a control w and a time $\theta \in (0, \infty]$ such that

$$\Psi(\tau, u) < \Psi(\theta, w).$$

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Proof of Lemma 2 (continuation).

Also, we have that

$$\Psi(\theta, w) = \int_0^{\tau} e^{-\rho t}[\pi\beta G(t, w) - \frac{\kappa}{2} w^2(t)]dt \leq$$

$$\int_0^{\tau} e^{-\rho t}[\pi\beta G(t, w) - \frac{\kappa}{2} w^2(t)]dt + \int_{\tau}^{\infty} e^{-\rho t}\pi\beta[G(t, w)]^+ dt = \Pi(u_\tau).$$

Moreover, by Lemma 1, we have that $\Psi(\tau, u) = \Pi(\tau, u_\tau)$ (because, after τ the state variable is always negative, and hence the optimal control must also be equal to 0 after τ. Thus we have obtained that

$$\Pi(u) = \Psi(\tau, u) < \Psi(\theta, w) \leq \Pi(w_\tau),$$

and so u cannot be optimal for problem (1)-(3).

A differential game

We consider a differential game with two players (each player is a firm or her product or brand). Let G_j be the brand's goodwill of the product/service of the j-th player at time t, $t \in [0, +\infty)$. We assume that the demand rate for a brand is proportional to its goodwill stock, i.e. $D_j(G_j) = \beta \max(0, G_j)$, $j = 1, 2$, where $\beta > 0$ is a parameter measuring goodwill's efficiency. Denote by $u_j(t) \geq 0$ the advertising effort of player $j \in \{1, 2\}$ at time $t \in [0, +\infty)$. For each index $j \in \{1, 2\}$ the j-th player maximize the following criterion

$$J_j(u_1, u_2) := \int_0^\infty e^{-\rho t}\{\pi_j \beta [G_j(t)]^+ - \frac{\kappa}{2} u_j^2(t)\} dt \qquad (9)$$

subject to

$$\dot{G}_j(t) = \gamma_j u_j(t) - \xi_i u_i(t) - \delta G_j(t), \; G_j(0) = \alpha_j > 0. \qquad (10)$$

Optimal dynamic advertising model

A differential game

Here $i \in \{1, 2\} \setminus \{j\}$, $\gamma_j > 0$ is the advertising effort efficiency (or a scaling parameter transforming advertising money into goodwill), ξ_i is the interference factor of competitor i-th's advertising efforts on own j-th's goodwill evolution, δ is a decay parameter and $J_j(u_1, u_2)$ is discounted profit of the j-th player corresponding to the advertising efforts u_1 and u_2) of the both players.

Lemma 3 (Bruno Viscolani, Georges Zaccour (2009)).

We set

$$u_j^* = \frac{\beta}{\delta + \rho} \frac{\gamma_j \pi_j}{\kappa_j}, j = 1, 2.$$

Let $(\bar{u}_j(t), G_j(t, \bar{u}_j, u_k^*))$, $t \in [0, +\infty)$ be a bounded optimal solution of j-th firm's problem, where $k = \{1, 2\} \setminus \{j\}$. If the goodwill path $G_j(t, \bar{u}_j, u_k^*) > 0$ is strictly positive for all $t \in [0, +\infty)$, then the optimal advertising effort is

$$\bar{u}_j(t) = u_j^* \text{ for each } t \in [0, +\infty).$$

Let us consider the following infinite horizon problem:

$$I(x_0, u) = \int_0^{\infty} e^{-rt} L(x(t), u(t))\, dt \;\rightarrow\; \max \tag{11}$$

subject to

$$\dot{x}(t) = f(x(t), u(t), t) \;\text{ and }\; x(0) = x_0. \tag{12}$$

The state $x(t)$ belongs to an open subset $\mathcal{X}$ of R^n, and the control $u(t)$ is a bounded measurable function taking its values from a compact subset $\mathcal{U}$ of R^m. The functions $f : \mathcal{X} \times U \times R_+ \rightarrow R^n$ and $L : \mathcal{X} \times U \times R_+ \rightarrow R$ are continuous and continuously differentiable with respect to the state variable. The corresponding differentials are denoted by f_x and L_x.

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A trajectory-control pair $(x(t), u(t)), 0 \leq t < \infty$, is *admissible* if $x(\cdot)$ is a solution of (12) with control $u(\cdot)$ on $[0, \infty)$ and if the integral (11) converges. Let $(x(t), u(t)), 0 \leq t < \infty$, be an admissible trajectory-control pair.

A trajectory-control pair $(x^*(t), u^*(t)), 0 \leq t < \infty$, is *optimal* if it is admissible and optimal in the set of admissible trajectory-control pairs, i.e. for any admissible trajectory-control pair $(x(t), u(t)), 0 \leq t < \infty$, the value of the integral (11) is not greater than its value corresponding to $(x(\cdot), u(\cdot))$.

Let us define the following Hamiltonian $\hat{H}^c : \mathcal{X} \times \mathcal{U} \times R^n \to R$:

$$\hat{H}^c(x, u, p, t) := L(x, u) + \sum_{j=1}^{n} p_j f_j(x, u, t)$$

and the maximized Hamiltonian

$$\hat{H}^{*c}(x, p, t) = \max_{u \in U} \hat{H}^c(x, u, p, t).$$

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Theorem 1 (Atle Seierstadt, Knut Sydsaeter (1977)) .

Let the following assumptions hold true: Let $(x^*(t), u^*(t))$, $0 \leq t < \infty$, be an admissible trajectory-control pair and $\mu(t)$, $0 \leq t < \infty$, be an absolutely continuous function such that the following assumptions hold true:

A1. The function $\hat{H}^{*c}(x, m, t)$ is well defined for each $(x, p) \in \mathcal{X} \times R^n$ and concave with respect to x for each fixed point $(p) \in R^n$;

A2. For almost all $t \in [0, +\infty)$

$$\frac{d}{dt}\mu_i(t) = r\mu_i(t) - \frac{\partial \hat{H}^c}{\partial x_i}(x^*(t), u^*(t), \mu(t), t).$$

A3. For almost all $t \in [0, +\infty)$

$$\hat{H}^c\left(x^*(t), u^*(t), \mu(t), t\right) = \hat{H}^{c*}\left(x^*(t), \mu(t), t\right)$$

and the maximum is achieved at the unique point $u^*(t)$.

Theorem 1 (continuation).

If the components of the state variable $x(t)$ and the co-state variable $\mu(t)$, $t \in [0, \infty)$, take nonnegative values, and

$$\lim_{T \to \infty} e^{-rT} \sum_{j=1}^{n} \mu_j(T) \cdot x_j^*(T) = 0,$$

then the trajectory-control pair $(x^*(t), u^*(t)), 0 \leq t < \infty$, is optimal.

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Proof of Lemma 3.

We define the Hamiltonian as follows:

$$H(G_j, u_j, \mu) = \pi_j \beta G_j - \frac{\kappa_j}{2} u_j^2 + \mu(\gamma_j u_j - \xi_i u_i(t) - \delta G_j))$$

Then, clearly, the condition A1 holds true. Let $u^*(t)$ maximize the function $H(G_j^*(t), u_j^*(t), \mu(t))$. Then the gradient $\frac{\partial}{\partial G} H(G_j^*(t), u_j^*(t), \mu(t)) = 0$ implies that $u^*(t) = \frac{\mu(t)\gamma_j}{\kappa_j}$. Since the Hamiltonian is a concave function with respect to u_j, then $u^* = \frac{\mu(t)\gamma_j}{\kappa_j}$ maximize the Hamiltonian, i.e. the condition A3 is fulfilled.

The adjoint equation is $\dot{m}u(t) = -\pi_j\beta + (\rho + \delta)\mu(t)$. If we set $\mu(t) := \frac{\pi_j \beta}{\rho + \delta}$, then this μ is a solution of the adjoint equation.

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But then $u^*(t) = \frac{\beta}{\rho+\delta}\frac{\pi_j\gamma_j}{\kappa_j}$ and the condition A2 holds true. Clearly, G and μ take non negative values. Moreover,

$$0 = \lim_{T\to\infty} e^{-\rho T}\left(\mu(T)G_j(T)\right) =$$

$$e^{-\rho T}\frac{\pi_j\beta}{\rho+\delta}\left(e^{-\delta T}\alpha_j + \frac{\gamma_j u_j^*}{\delta}(1-e^{-\delta T}) - \xi_i\int_0^T e^{-\delta(T-t)}u_i(t)dt\right)$$

because u_i is a bounded function. Applying Theorem 1, we complete the proof of Lemma 3.

Lemma 4 (Bruno Viscolani, Georges Zaccour (2009)).

We set
$$u_j^* = \frac{\beta}{\delta+\rho}\frac{\gamma_j \pi_j}{\kappa_j}, j = 1, 2.$$

Let $(\bar{u}_j(t), G_j(t, \bar{u}_j, u_k^*))$, $t \in [0, +\infty)$ be a bounded optimal solution of j-th firm's problem, where $k = \{1, 2\} \setminus \{j\}$. If the goodwill path $G_j(t, \bar{u}_j, u_k^*) > 0$ for all $t \in [0, \tau)$ and $G_j(\tau, \bar{u}_j, u_k^*) = 0$, then the optimal advertising effort is

$$u_j^\tau = u_j^* \left[1 - e^{-(\delta+\rho)(t-\tau)}\right]^+ \text{ with } \frac{\beta}{\delta+\rho}\frac{\gamma_j \pi_j}{\kappa_j},$$

where τ is a solution of the following exit time equation:

$$\xi_i e^{\delta T} \int_0^T e^{-\delta(T-s)} u_i(s) ds = \tag{13}$$

$$\frac{\gamma_j u_j^*}{\delta}\left(e^{\delta\tau} - 1\right) - \frac{\gamma_j u_j^* \left(e^{\delta\tau} - e^{-(\delta+\rho)\tau}\right)}{2\delta+\rho} + \alpha_j.$$

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Proof of Lemma 4.

In view of Lemma 2, any optimal control u_j for the original problem under the present assumptions is related to that of the variable-final-time problem (4)-(6). From Lemma 1, we obtain that $G_j(t, u_j) \leq 0$ for $t \geq \tau$, and hence $u_j^{\tau}(t) = 0$ for all $t > \tau$. Let us consider optimal-control problem (4)-(6) without the state condition $G_j(t, u^{\tau}) \geq 0$. We call this the relaxed variable-final-time problem. The corresponding Hamiltonian is

$$H(G_j, u_j, \lambda, t) = e^{-\rho t}[\pi\beta G_j - \frac{\kappa_j}{2} u_j^2] + \lambda(\gamma_j u_j - \xi_\xi u_i(t) - \delta G_j)$$

which is a strictly concave function with respect to u_j and its first partial derivative with respect to u_j is

$$\frac{\partial}{\partial x} H(G_j, u_j, \lambda, t) = -\kappa_j u_j e^{-\rho t} + \gamma_j \lambda.$$

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Hence there exists a single maximum point

$$u_j^T(t) = \frac{e^{\rho t}\gamma_j}{\kappa_j}[\lambda(t)]^+.$$

The adjoint equation is:

$\dot{\lambda}(t) = -\pi_j \beta e^{-\rho t} + \delta\lambda(t)$ with transversality condition $\lambda(T) = 0$.

The solution of this equation is

$$\lambda(t) = \frac{\pi_j \beta}{\delta + \rho}\left(1 - e^{(\delta+\rho)(t-T)}\right) e^{-\rho t}.$$

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Hence the possible optimal control is

$$u_j^{\tau}(t) = \frac{\pi_j \beta \gamma_j}{\kappa_j(\rho+\delta)}\left(1 - e^{(\delta+\rho)(t-T)}\right).$$

Since, $0 = H(G_j(T), u_j(T), \lambda(T), T) =$

$$e^{-\rho T}[\pi_j \beta G_j(T) - \frac{\kappa_j}{2} u_j^2(T)] + \lambda(T)(\gamma_j u_j(T) - \xi_i u_i(T) - \delta G_j(T))$$

$$\text{i.e. } 0 = G_j(T) \text{ (because } \lambda(T) = 0 \text{ and } u_j^{\tau}(T) = 0).$$

Since $G_j(0) = \alpha_j$, we obtain that $0 = G_j(T)$ is just the exist equation for τ.

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Definition.

Assume that the i-th player chooses the control u_i^* and the j-th player chooses the control u_j^*. It is said that the j-th player is a strong player if his best response to u_i^* is u_j^*. It is said that the j-th player is a weak player if his best response is the exit control u_j^T.

Therem 2 (Bruno Viscolani, Georges Zaccour (2009)).

- If both players are strong, then (u_1^*, u_2^*) is the unique open-loop Nash equilibrium.
- If the i-th players is strong and the j-th players is weak, then (u_i^*, u_j^T) is the unique open-loop Nash equilibrium.

Open problems

★ What is the open-loop Nash equilibrium if both players are weak?
★ How to characterize strong and weak players?

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