GEOMETRIA - Prova scritta del 1° febbraio 2011

(Prof. M. Spera)

① Si consideri la superficie parametrica

$$\Sigma : \begin{cases} x = u\cos v \\ y = u \sin v \\ z = u \end{cases} \qquad \begin{array}{l} u > 0 \\ v \in [0, 2\pi) \end{array}$$

si verifichi che Σ è una superficie rigata (cos'è?). Si accerti altresì la sua sviluppabilità, mostrando che il piano tangente è costante lungo le generatrici. Si calcoli, in due modi, la curvatura gaussiana di Σ.

② Si consideri $\mathcal{C}$: $\begin{cases} x = t\cos t \\ y = t\sin t \\ z = t \end{cases} \qquad t > 0$

si verifichi che $\mathcal{C}$ è regolare, e dire se $\mathcal{C}$ risulta essere una geodetica di Σ.

Si sfrutti il fatto che in generale $\underline{b} \propto \dot{\underline{r}} \times \ddot{\underline{r}}$ [fac: verificarlo].

③ Stesso esercizio ②, per altra via:
calcolare $\sin\alpha$, α = angolo tra la generatrice di Σ passante per un dato punto $P \in \mathcal{C}$ e la tangente a $\mathcal{C}$ nello stesso punto (utilizzare $\cos\alpha = \frac{\langle\cdot,\cdot\rangle}{\|\ \|\ \|\ \|}$) e vedere se risulta soddisfatto il teorema di Clairaut (perché lo si può usare?)

Tempo a disposizione: 2h

Le risposte vanno adeguatamente giustificate.

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$$\Sigma : \quad x^2+y^2-z^2=0 \qquad z>0$$

semicono

par.

$$\begin{cases} x = u\cos v \\ y = u \sin v \\ z = u \end{cases} \qquad \begin{array}{l} u>0 \\ v \in [0,2\pi) \end{array} \qquad \underline{r} = (u\cos v, u\sin v, u)$$

sup. rigata : generatrici $v = v_0$ (cost)

$$\begin{cases} x = u\cos v_0 \\ y = u\sin v_0 \\ z = u \end{cases} \qquad \text{(rette..)}$$

Il piano tangente è costante lungo le generatrici (sviluppabilità)

verifica analitica:

$$\underline{r}_u = (\cos v, \sin v, 1)$$
$$\underline{r}_v = (-u\sin v, u\cos v, 0)$$

$$E = 2 \qquad F = 0 \qquad G = u^2$$

$$\underline{r}_u \times \underline{r}_v = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} =$$

$$\underline{i}\,(-u\cos v) - \underline{j}\, u\sin v + \underline{k}\,(u\cos^2 v + u\sin^2 v) =$$

$$= -u\cos v\, \underline{i} - u\sin v\, \underline{j} + u\, \underline{k}$$

$$\underline{N} = \frac{\underline{r}_u \times \underline{r}_v}{\underbrace{\| \quad \|}_{\sqrt{u^2+u^2}}} = \frac{1}{\sqrt{2}}\left[-\cos v\, \underline{i} - \sin v\, \underline{j} + \underline{k}\right]$$

che non dipende da u.

$$\sqrt{u^2+u^2} = \sqrt{2u^2} = \sqrt{2}\,|u| = \sqrt{2}\,u \quad (u>0)$$

$$\Rightarrow \boxed{K = 0}$$

$$\underline{r}_{uu} = (0, 0, 0)$$

$$\underline{r}_{uv} = (-\sin v, \cos v, 0) = \underline{r}_{vu} \qquad \underline{N} = \frac{1}{\sqrt{2}}(-\cos v, -\sin v, 1)$$

$$\underline{r}_{vv} = (-u\cos v, -u\sin v, 0)$$

$$e = \langle \underline{r}_{uu}, \underline{N} \rangle = 0$$

$$f = \langle \underline{r}_{uv}, N \rangle = +\sin v \cos v - \sin v \cos v = 0$$

$$g = \langle \underline{r}_{vv}, \underline{N} \rangle = u\cos^2 v + u \sin^2 v = u$$

$$K = \frac{eg - f^2}{EG - F^2} = 0 \quad , \text{di nuovo}$$

(2)

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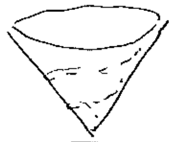
$$\mathscr{C}: \begin{cases} x = t\cos t \\ y = t\sin t \\ z = t \end{cases} \qquad \mathscr{C} \subset \Sigma$$

"elica"



$\mathscr{C}$ regolare. ($\underline{r} = \underline{r}(t)$ iniettiva e $\mathscr{C}^6$ $\dot{\underline{r}}(t) \neq \underline{0}$

$\underline{r} = (t\cos t, t\sin t, t)$
$\dot{\underline{r}} = (\cos t - t\sin t, \sin t + t\cos t, 1)$
...

Dire se $\mathscr{C}$ è una geodetica.

Sugg: sfruttare $\underline{b} \propto \dot{\underline{r}} \times \ddot{\underline{r}}$

verifica dettagliata: $\underline{t}' = k \cdot \underline{n}$ $\qquad ds = \|\dot{\underline{r}}\| dt$

$$k \cdot \underline{n} = \frac{d\underline{t}}{ds} = \frac{d\underline{t}}{dt}\frac{dt}{ds} = \frac{d}{dt}\left(\frac{\dot{\underline{r}}}{\|\dot{\underline{r}}\|}\right)\frac{1}{\|\dot{\underline{r}}\|}$$

$$= \left\{\frac{\ddot{\underline{r}}}{\|\dot{\underline{r}}\|} + \dot{\underline{r}}\frac{d}{dt}\left(\frac{1}{\|\dot{\underline{r}}\|}\right)\right\}\frac{1}{\|\dot{\underline{r}}\|} = \left\{\frac{\ddot{\underline{r}}}{\|\dot{\underline{r}}\|} + \dot{\underline{r}}\left(-\frac{1}{\|\dot{\underline{r}}\|^2}\right)\right\}\cdot\frac{1}{\|\dot{\underline{r}}\|}$$

$$= \frac{\|\dot{\underline{r}}\| \cdot \ddot{\underline{r}} - \dot{\underline{r}}}{\|\dot{\underline{r}}\|^3}$$

$$\underline{b} = \underline{t} \times \underline{n} \qquad k\,\underline{b} = \underset{/\!/\ \frac{\dot{\underline{r}}}{\|\dot{\underline{r}}\|}}{\underline{t}} \times k\,\underline{n}$$

$$= \ldots = \frac{\dot{\underline{r}} \times \ddot{\underline{r}}}{\|\dot{\underline{r}}\|^3} \qquad \left(\Rightarrow\ k = \frac{\|\dot{\underline{r}} \times \ddot{\underline{r}}\|}{\|\dot{\underline{r}}\|^3}\right)$$

Calcoliamo allora $\langle \underline{b}, \underline{N} \rangle$, o, più esattamente:

$$\langle \dot{\underline{r}} \times \ddot{\underline{r}}, \underline{r}_u \times \underline{r}_v \rangle$$

$$\ddot{\underline{r}} = (-\sin t - \sin t - t\cos t,\ \cos t + \cos t - t\sin t,\ 0)$$

$$= (-2\sin t - t\cos t,\ 2\cos t - t\sin t,\ 0)$$

$$\dot{\underline{r}} \times \ddot{\underline{r}} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos t - t\sin t & \sin t + t\cos t & 1 \\ -2\sin t - t\cos t & 2\cos t - t\sin t & 0 \end{vmatrix} = \underline{i}\,(-2\cos t + t\sin t) - \underline{j}\,(2\sin t + t\cos t)$$

$$+ \underline{k}\left\{(\cos t - t\sin t)(2\cos t - t\sin t) + (2\sin t + t\cos t)(\sin t + t\cos t)\right\}$$

$$2\cos^2 t - \cancel{2t\sin t\cos t} - \cancel{t\cos t\sin t} + t^2\sin^2 t$$
$$+ 2\sin^2 t + \cancel{t\sin t\cos t} + \cancel{2t\sin t\cos t} + t^2\cos^2 t$$
$$= 2 + t^2$$

$$\dot{\underline{r}} \times \ddot{\underline{r}} = (-2\cos t + t\sin t)\,\underline{i} - (2\sin t + t\cos t)\,\underline{j} + (2 + t^2)\,\underline{k}$$

$$\underline{N} = \underline{N}(t) = \frac{1}{\sqrt{2}}\left[-\cos t\,\underline{i} - \sin t\,\underline{j} + \underline{k}\right]$$

$$\langle\, , \rangle = \frac{1}{\sqrt{2}}\left\{+2\cos^2 t - \cancel{t\sin t\cos t} + 2\sin^2 t + \cancel{t\sin t\cos t} + 2 + t^2\right\}$$

$$= \frac{1}{\sqrt{2}}\left\{4 + t^2\right\} \neq 0$$

$$\dot{\underline{r}} \times \ddot{\underline{r}} \neq \underline{0} \quad \forall\, t > 0 \quad \Rightarrow \quad k \neq 0 \quad \forall\, t > 0$$

③ Clairaut :

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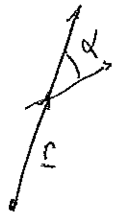


calcoliamo $\cos\alpha$
↑ azimut

$$\underline{r} = (t\cos t,\ t\sin t,\ t)$$
$$\dot{\underline{r}} = (\cos t - t\sin t,\ \sin t + t\cos t,\ 1)$$

$$\cos\alpha = \frac{\langle \underline{r}, \dot{\underline{r}} \rangle}{\|\underline{r}\|\ \|\dot{\underline{r}}\|}$$

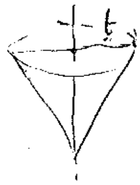
$$\langle \underline{r}, \dot{\underline{r}} \rangle = t\cos^2 t - \cancel{t^2 \sin t\cos t} + t\sin^2 t + \cancel{t^2\sin t\cos t} + t$$
$$= 2t \qquad \|\underline{r}\| = \sqrt{2}\, t \qquad (t > 0)$$

$$\|\dot{\underline{r}}\|^2 = \cos^2 t + t^2\sin^2 t - \cancel{2t\sin t\cos t} + \sin^2 t + t^2\cos^2 t + \cancel{2t\sin t\cos t} + 1$$
$$= 2 + t^2 \qquad \|\dot{\underline{r}}\| = \sqrt{t^2+2}$$

$$\cos\alpha = \frac{2t}{\sqrt{2}\, t\sqrt{t^2+2}} = \frac{\sqrt{2}}{\sqrt{t^2+2}} = \sqrt{\frac{2}{t^2+2}} \qquad (\leq 1)$$

$$\sin\alpha = +\sqrt{1-\cos^2\alpha} = \sqrt{1 - \frac{2}{t^2+2}} = \sqrt{\frac{t^2+2-2}{t^2+2}} = \frac{t}{\sqrt{t^2+2}}$$

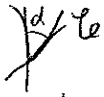
raggio del parallelo
$= t$



$$t \cdot \sin\alpha(t) = \frac{t^2}{\sqrt{t^2+2}}$$

che non è costante

Clairaut



$r_{par} \cdot \sin\alpha = \text{cost}$
se γ è geodetica