Varianti e approfondimenti
Lezione IX bis

Siano $a = a(t)$, $b = b(t)$, $t \in I$

f. differenziabili, con $a^2 + b^2 \equiv 1$

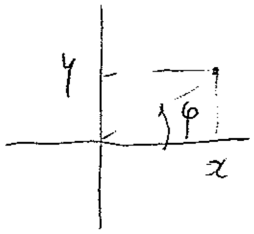
e $a(t_0) = \cos\varphi_0$, $b(t_0) = \sin\varphi_0$.

t_0

Allora
$$a(t) = \cos\varphi(t) \qquad b(t) = \sin\varphi(t)$$
con
$$\varphi(t) = \varphi_0 + \int_{t_0}^{t} (a\dot{b} - b\dot{a})\,dt$$

Dim: Segue subito dall'espressione della forma angolare
$$d\varphi(x,y) = \frac{x\,dy - y\,dx}{x^2+y^2} \qquad (x,y) \neq (0,0)$$

$$\varphi - \varphi_0 = \int_{t_0}^{t} \frac{x\dot{y} - y\dot{x}}{x^2+y^2}\,dt \ \ldots$$



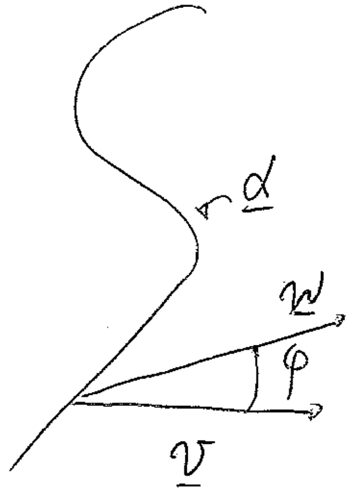
In modo più "pedestre": sia
$$f(t) = (a - \cos\varphi)^2 + (b - \sin\varphi)^2$$

mostriamo che $f(t) \equiv 0$. In t_0 vale 0;

derivando si ha
$$\dot{f}(t) = 2(a - \cos\varphi)(\dot{a} + \sin\varphi\,\dot{\varphi}) + 2(b - \sin\varphi)(\dot{b} - \cos\varphi\,\dot{\varphi})$$

$= \ldots = 0$ [usando $a\dot{a} + b\dot{b} = 0$, che viene da $a^2 + b^2 = 1$] e $\dot{\varphi} = a\dot{b} - \dot{a}b$

★ Proposizione



$\|\underline{v}\| \equiv 1$

$\|\underline{w}\| = 1$

$$\boxed{\left[\frac{\nabla \underline{w}}{dt}\right] - \left[\frac{\nabla}{dt}\underline{v}\right] = \frac{d\varphi}{dt}} \quad (\star)$$

[in particolare, se $\frac{\nabla}{dt}\underline{v} \equiv \underline{0}$, i.e. $\underline{v}$ è parallelo lungo α , $\left[\frac{\nabla}{dt}\underline{w}\right] = \frac{d\varphi}{dt}$, e se

$\underline{w} = \underline{\alpha}'$, è $k_g = \frac{d\varphi}{dt}$

↖ versore tangente

↑ curvatura geodetica

★ tasso di variazione angolare di $\underline{\alpha}'$ rispetto ad un vettore parallelo lungo $\underline{\alpha}$. [per una geodetica è $k_g = 0$]

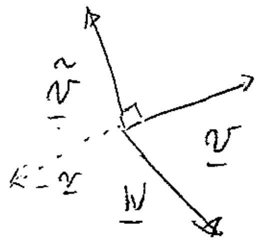
$$\boxed{\langle \underline{\dot{v}}, \underline{\tilde{v}} \rangle = \left[\frac{\nabla \underline{v}}{dt}\right]} \quad \text{ricordare}$$

$\underline{N} \times \underline{v}$

Dim. Poniamo

$$\tilde{v} = N \times v \qquad \tilde{w} = N \times w$$

Sia $w = \cos\varphi\, v + \sin\varphi\, \tilde{v}$



$$\tilde{w} = N \times w = \cos\varphi\,(N \times v) + \sin\varphi\,(N \times \tilde{v})$$
$$= \cos\varphi\, \tilde{v} - \sin\varphi\, v$$

$$\Rightarrow \dot{\tilde{w}} = -\sin\varphi\, \dot{\varphi}\, \tilde{v} + \cos\varphi\, \dot{\tilde{v}} - \cos\varphi\, \dot{\varphi}\, v - \sin\varphi\, \dot{v}$$

Ma

$$\langle \dot{\tilde{w}}\, w \rangle = -\langle \dot{w}, \tilde{w} \rangle = -\left[\frac{\nabla w}{dt}\right],$$

da un lato, e, dall'altro, utilizzando le formule precedenti e le

$$\boxed{\begin{aligned} &\langle v, \tilde{v} \rangle = 0 \\ &\langle v, \dot{v} \rangle = 0 \\ &\langle \dot{v}, \tilde{v} \rangle = -\langle v, \dot{\tilde{v}} \rangle \\ &\langle \dot{w}, \tilde{w} \rangle = -\langle w, \dot{\tilde{w}} \rangle \end{aligned}}$$

$$\imath = -\sin^2\varphi\, \dot{\varphi} + \cos^2\varphi\, \langle v\, \dot{\tilde{v}} \rangle - \cos^2\varphi\, \dot{\varphi} - \sin^2\varphi\, \langle \tilde{v}, \dot{v} \rangle$$

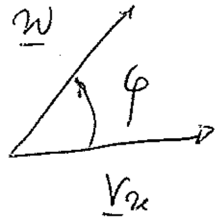
$$= -\dot{\varphi} - \langle \tilde{v}, \dot{v} \rangle = -\dot{\varphi} - \left[\frac{\nabla v}{dt}\right]$$

da cui l'asserto (cambiando i segni)

Dimostriamo ora la seguente formula generale

$\underline{w} = \underline{w}(t)$ lungo $\underline{\alpha}$, $\|\underline{w}\| \equiv 1$

$$\left[\frac{\nabla \underline{w}}{dt}\right] = \frac{1}{2\sqrt{EG}} \left\{ G_u \frac{dv}{dt} - E_v \frac{du}{dt} \right\} + \frac{d\varphi}{dt}$$



Sia $\underline{e}_1 = \dfrac{r_u}{\sqrt{E}}$, $\underline{e}_2 = \dfrac{r_v}{\sqrt{G}}$ $\qquad \underline{e}_1 \times \underline{e}_2 = \underline{N}$

$$\left[\frac{\nabla \underline{w}}{dt}\right] = \left[\frac{\nabla \underline{e}_1}{dt}\right] + \frac{d\varphi}{dt}$$

Ma $\left[\dfrac{\nabla e_1}{dt}\right] = \langle \dot{\underline{e}}_1, \underline{e}_2 \rangle = \left\langle \dfrac{\partial e_1}{\partial u} \dot{u} + \dfrac{\partial e_1}{\partial v} \dot{v}, \underline{e}_2 \right\rangle$

$$= \left\langle \frac{\partial e_1}{\partial u}, \underline{e}_2 \right\rangle \dot{u} + \left\langle \frac{\partial e_1}{\partial v}, \underline{e}_2 \right\rangle \dot{v}$$

ma $\left\langle \dfrac{\partial}{\partial u}\left(\dfrac{r_u}{\sqrt{E}}\right), \dfrac{\underline{r}_v}{\sqrt{G}} \right\rangle = \Big\langle \dfrac{r_{uu}}{\sqrt{E}} - \underbrace{\tfrac{1}{2} E^{-\frac{3}{2}} E_u \underline{r}_u, \underline{r}_v \Big\rangle}_{\langle\ ,\ \rangle = 0}$

$$= \frac{\overbrace{\langle \underline{r}_{uu}, \underline{r}_v \rangle}^{= -\frac{1}{2} E_v}}{\sqrt{E}} = -\frac{1}{2} \frac{1}{\sqrt{EG}} E_v$$

Analogamente

$$\left\langle \frac{\partial \underline{e}_1}{\partial v}, \underline{e}_2 \right\rangle = -\frac{1}{2} \frac{E_v}{\sqrt{EG}}$$

Sostituendo, troviamo, in definitiva

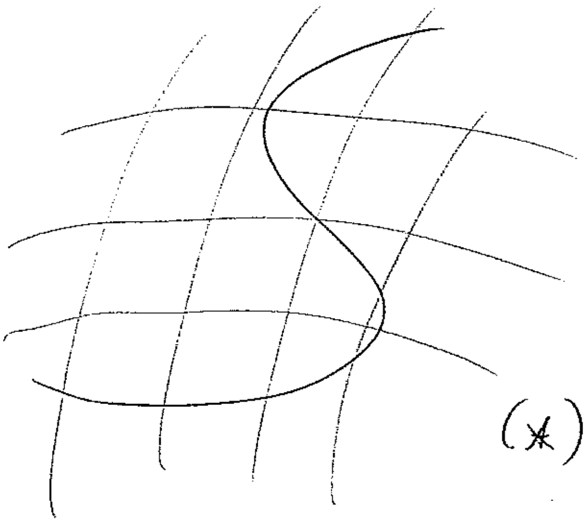
$$\left[\frac{\nabla \underline{w}}{dt} \right] = \frac{1}{2\sqrt{EG}} \left\{ G_u \dot{v} - E_v \dot{u} \right\} + \dot{\varphi}$$

Corollario: $\underline{w}$ è parallelo $\Leftrightarrow$

$$\dot{\varphi} = -\frac{1}{2\sqrt{EG}} \left\{ G_u \dot{v} - E_v \dot{u} \right\}$$

$\Rightarrow$ integrando si ottiene $\varphi = \varphi(t)$ ...

** Formula di Liouville



$\underline{\alpha}$ regolare

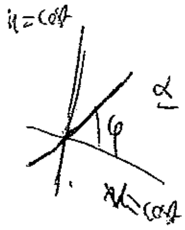
parametr. ortogonale $(F=0)$

$\underline{\alpha} = \underline{\alpha}(s)$, s lungh. d'arco

$$(*) \quad \underline{\alpha}' = \underline{r_u}\, u' + \underline{r_v}\, v' = \left(\frac{r_u}{\sqrt{E}}\right)(\sqrt{E}\,u') + \left(\frac{r_v}{\sqrt{G}}\right)(\sqrt{G}\,v')$$

$$\boxed{k_g = \underbrace{(k_g)_1}_{v=\text{cost}} \cos\varphi + \underbrace{(k_g)_2}_{u=\text{cost}} \sin\varphi + \frac{d\varphi}{ds}}$$

curve coordinate



$' = \dfrac{d}{ds}$

Dim.
$$k_g = \frac{1}{2\sqrt{EG}} \left\{ G_u\, v' - E_v\, u' \right\} + \varphi'$$

$v = \text{cost} \Rightarrow v' = 0 \qquad (k_g)_1 = -\dfrac{1}{2E\sqrt{G}} E_v$

$\dfrac{du}{ds} = \dfrac{1}{\sqrt{E}}$ è pure $(*)$ $\qquad (k_g)_2 = \dfrac{1}{2G\sqrt{E}} G_u v'$

$$\sqrt{E}\,u' = \langle \underline{\alpha}', \underbrace{\tfrac{r_u}{\sqrt{E}}}_{e_1} \rangle = \cos\varphi \qquad \sqrt{G}\,v' = \langle \underline{\alpha}', \underbrace{\tfrac{r_v}{\sqrt{G}}}_{e_2} \rangle = \sin\varphi$$

$\dfrac{dv}{ds} = \dfrac{1}{\sqrt{G}}$

$\boxed{g_{ij}\, u_i'\, u_j' \equiv 1}$ $\Rightarrow$ sostituendo, si conclude $\square$

☆ Teorema di Gauss-Bonnet locale

(forma generale)

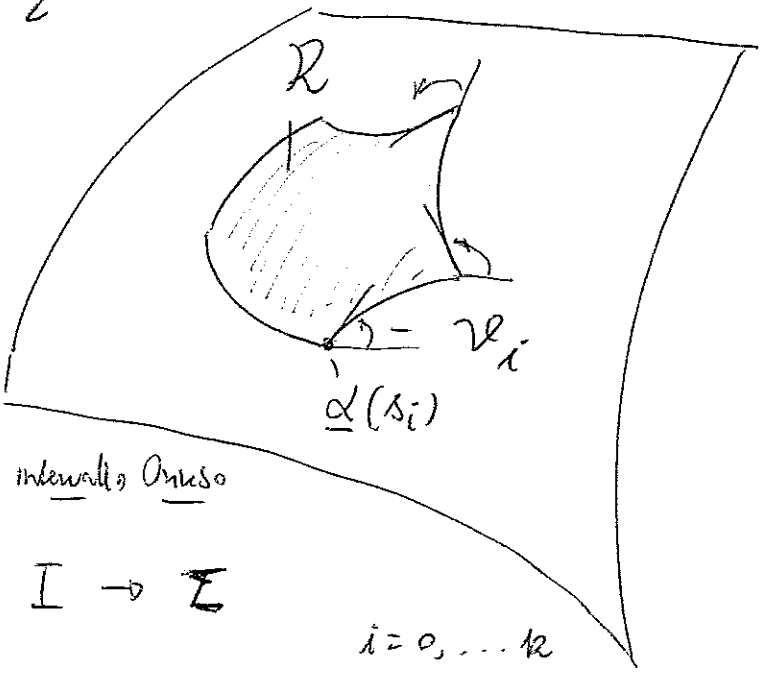
Σ : $\underline{r} : U \to \mathbb{R}^3$ superficie regolare,

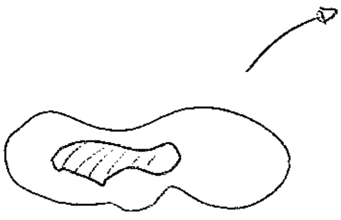
$\underline{r} : U \to \Sigma$ parametrizzazione ortogonale

(abuso di notazione) $(F = 0)$ ← ipotesi di comodo

($U \approx$ disco aperto , $\underline{r}$ compatibile con ...

l'orientazione di Σ





$(I \to U \to \mathbb{R}^3 \ldots)$

intervallo chiuso

$\underline{\alpha} : I \to \Sigma$

$i = 0, \ldots k$

$\underline{\alpha} = \underline{\alpha}(s)$ $\quad s$: lunghezza d'arco

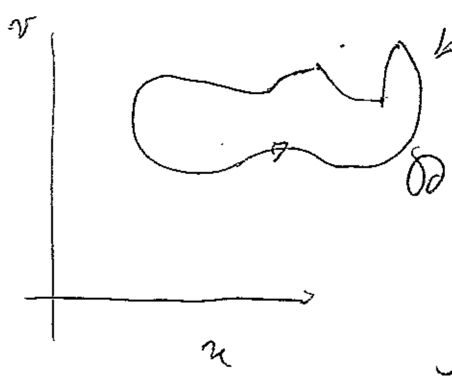
si ha:

$$\boxed{\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(s)\, ds + \iint_R K\, d\sigma + \sum_{i=0}^{k} \vartheta_i = 2\pi}$$

k_g: curvatura geodetica

si ha:

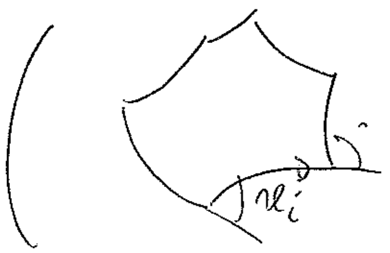
$$\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} k_g(s)\,ds = \iint_{\underline{r}^{-1}(R) = D} \left[\left(\frac{E_v}{2\sqrt{EG}}\right)_v + \left(\frac{G_u}{2\sqrt{EG}}\right)_u\right] du\,dv + \sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} \frac{d\varphi_i}{ds}\,ds$$



Ma (Gauss, ***)

$$\iint_D (\quad) = -\iint_D K\sqrt{EG}\,du\,dv = -\iint_R K\,d\sigma$$

e, per il teorema della rotazione delle tangenti **



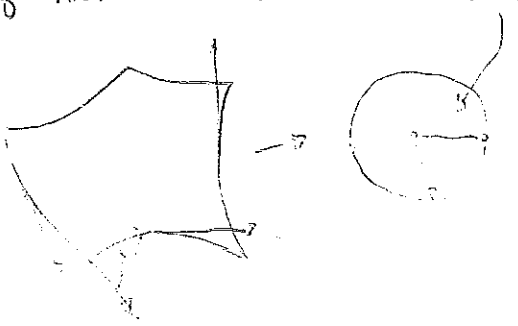
$$\sum_{i=0}^{k}\int_{s_i}^{s_{i+1}} \frac{d\varphi_i}{ds}\,ds = \sum_{i=0}^{k}\left(\varphi(s_{i+1}) - \varphi(s_i)\right)$$

$$= +2\pi - \sum_{i=0}^{k}\vartheta_i$$

angoli esterni

la curva è or. positivamente

* argomento euristico: il vettore ruota di $+2\pi$, tenendo conto dei salti...

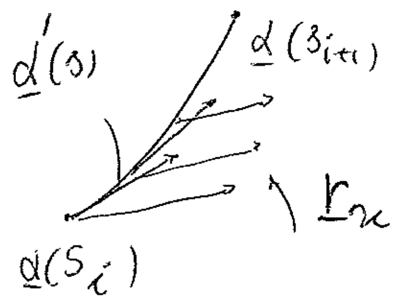


Sia $\underline{\alpha}$ descritta da $\begin{cases} u = u(s) \\ v = v(s) \end{cases}$

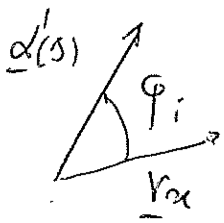
(si lavora, come sempre, sul piano dei parametri u, v)

Da:

$$\boxed{k_g(s) = \frac{1}{2\sqrt{EG}} \left\{ G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right\} + \frac{d\varphi_i}{ds}}$$



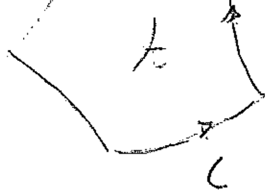
$$\left(\|\underline{\alpha}'(s)\| = 1 \right)$$



integriamo: $\displaystyle\int_{s_i}^{s_{i+1}} (\quad) = \int_{s_i}^{s_{i+1}} (\quad)$

sommiamo: $\displaystyle\sum_{i=0}^{n} \int_{s_i}^{s_{i+1}} \dots$

Dal teorema di Green



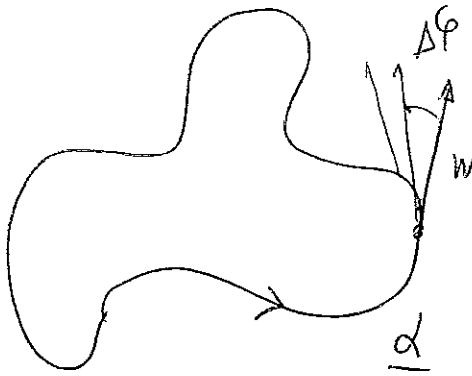
$$\int_C \left(P \frac{du}{ds} + Q \frac{dv}{ds} \right) ds = \iint_A \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} \right) du\, dv$$

e, pertanto:

$$\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(s)\,ds + \iint_R K\,d\sigma + \sum_{i=0}^{k} \vartheta_i = 2\pi$$

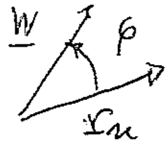
□

Commento: rideduciamo la formula di Levi-Civita



$W = W(s)$ trasporto parallelo

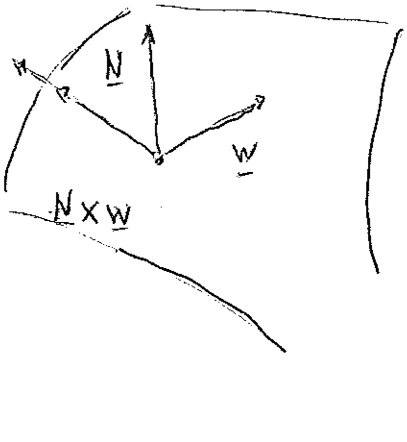
$$\frac{\nabla W}{ds} \equiv 0$$



$$0 = \int_0^l \left[\frac{\nabla W}{ds}\right] ds = \int_0^l \frac{1}{2\sqrt{EG}} \left\{ G_u \frac{dv}{ds} - E_v \frac{du}{ds} \right\} ds + \int_0^l \frac{d\varphi}{ds}\, ds$$

$\left[\frac{\nabla W}{dt}\right] = \lambda$, dove

$$\frac{\nabla W}{dt} = \lambda\,(N \times W)$$



$\left[\frac{\nabla W}{dt}\right]$ = valore algebrico di $\frac{\nabla W}{dt}$

ovviamente, è, a meno del segno, il modulo di $\frac{\nabla W}{dt} = P\,\frac{dW}{dt}$

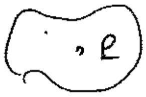
$\Rightarrow$

$$0 = -\iint_R K\,d\sigma + \varphi(l) - \varphi(0)$$

$$\boxed{\Delta\varphi = \iint_R K\,d\sigma}$$

Dal teorema del valor medio per gli integrali si trae :

$$\Delta\varphi = K(\tilde{P}) \underbrace{A(R)}_{\text{area di } R}$$



$$\Rightarrow \quad K(P) = \lim_{R \to P} \frac{\Delta\varphi}{A(R)}$$

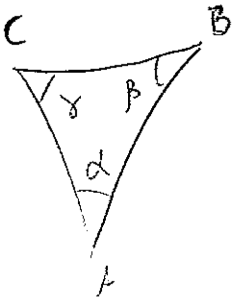
ottenendo una valida interpretazione geometrica della curvatura in termini di trasporto parallelo (Levi-Civita), suscettibile di amplissime generalizzazioni

Corollario : per un triangolo geodetico T

si ottiene nuovamente la formula di Gauss

$$\iint_T K\, d\sigma + \Sigma \vartheta_i = 2\pi$$

$$\text{ma} \quad \vartheta_i = \pi - \alpha_i$$



$$\iint_T K\, d\sigma + 3\pi - (\alpha + \beta + \gamma) = 2\pi$$

$$\Rightarrow \alpha + \beta + \gamma - \pi = \iint_T K\, d\sigma$$