Review

EECS 20

Lecture 38 (April 27, 2001)

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Transducive System

transduciveSystem : Values → Values

Reactive System

reactiveSystem : [Time → Values] → [Time → Values]

Discrete time: Time = $Nats_0$ = { 0, 1, 2, … }

Continuous time: Time = $Reals_+$ = { x ∈ Reals | x ≥ 0 }

A reactive system

F : [Time → Values] → [Time → Values]

is memory-free

iff

there exists a transducive system

f : Values → Values

such that

∀ x ∈ [Time → Values] , ∀ y ∈ Time,

(F (x)) (y) = f (x (y)) .

A reactive system

$F : [\ \text{Time} \to \text{Values}\ ] \to [\ \text{Time} \to \text{Values}\ ]$

is causal

iff

$\forall\ x, y \in [\ \text{Time} \to \text{Values}\ ]\ ,\ \forall\ z \in \text{Time},$

if $(\ \forall\ t \in \text{Time},\ t \leq z \ \Rightarrow\ x(t) = y(t)\ )$

then $(\ F(x)\ )(z) = (\ F(y)\ )(z)$.

The Delay System

$\text{Delay}_c : [\ \text{Time} \to \text{Values}\ ] \to [\ \text{Time} \to \text{Values}\ ]$

such that $\forall\ x \in [\ \text{Time} \to \text{Values}\ ]\ ,\ \forall\ y \in \text{Time},$

$$(\ \text{Delay}(x)\ )(y) = \begin{cases} c & \text{if } y < 1 \\ x(y-1) & \text{if } y \geq 1 \end{cases}$$

Discrete-time delay over finite set of values :

finite memory

Continuous-time delay, or infinite set of values:

infinite memory

Legal Transducive Block Diagrams

-all components are transducive systems

-no cycles

e.g., combinational circuits

Legal Reactive Block Diagrams

-all components are memory-free or delay systems

-every cycle contains at least one delay

e.g., sequential circuits

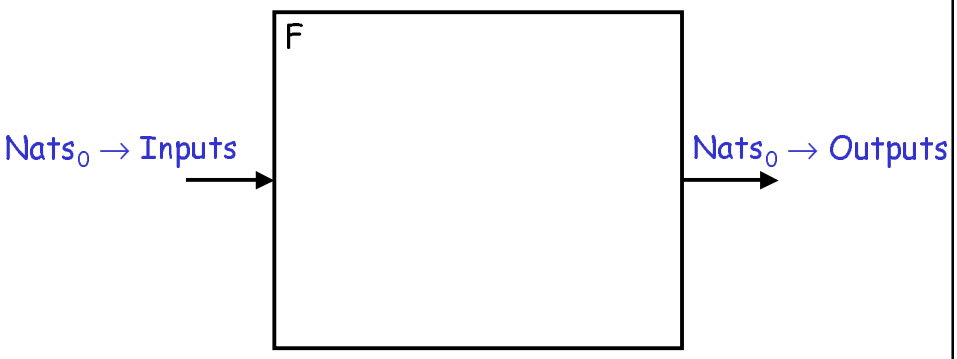
Discrete-time reactive systems with

finite memory

are naturally implemented as

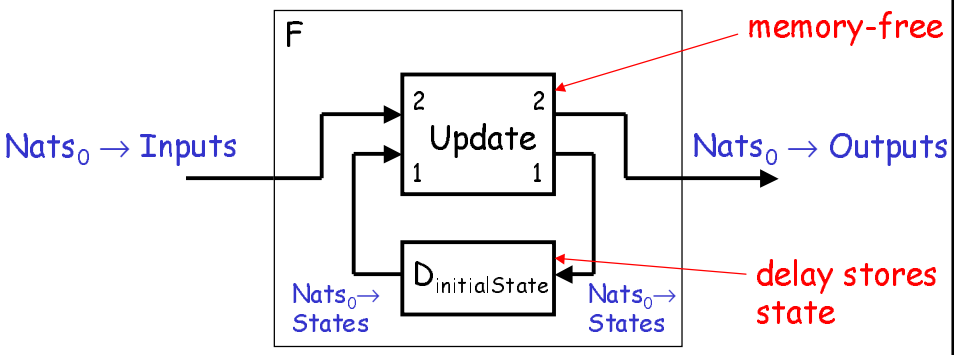
finite state machines.

A Discrete-Time Reactive System



$F : [\ Nats_0 \rightarrow Inputs\ ] \rightarrow [\ Nats_0 \rightarrow Outputs\ ]$

State Machine Implementation



update : States × Inputs → States × Outputs

initialState ∈ States

Deterministic State Machine

Inputs (set of possible input values)

Outputs (set of possible output values)

States (set of states)

initialState ∈ States

update : States × Inputs → States × Outputs

Product of State Machines

Any block diagram of N state machines with the state spaces

States1, States2, … StatesN

can be implemented by a single state machine with the state space

States1 × States2 × … × StatesN .

This is called a "product machine".

Deterministic Reactive System:

for every input signal, there is exactly one output signal.

Function:

DetSys : [Time → Inputs] → [Time → Outputs]

Nondeterministic Reactive System:

for every input signal, there is one or more output signals.

Binary relation:

$$NondetSys \subseteq [\, Time \to Inputs \,] \times [\, Time \to Outputs \,]$$

such that $\forall\, x \in [\, Time \to Inputs \,],$
$\exists\, y \in [\, Time \to Outputs \,],\ (x,y) \in NondetSys$

Every pair $(x,y) \in NondetSys$ is called a behavior.

S1 is a more detailed description of S2;

S2 is an abstraction or property of S1.

System S1 refines system S2

iff

1. Time [S1] = Time [S2] ,
2. Inputs [S1] = Inputs [S2] ,
3. Outputs [S1] = Outputs [S2] ,
4. Behaviors [S1] $\subseteq$ Behaviors [S2] .

Systems S1 and S2 are equivalent

iff

1. Time [S1] = Time [S2] ,
2. Inputs [S1] = Inputs [S2] ,
3. Outputs [S1] = Outputs [S2] ,
4. Behaviors [S1] = Behaviors [S2] .

Nondeterministic State Machine

Inputs

Outputs

States

possibleInitialStates $\subseteq$ States

possibleUpdates :

$$\text{States} \times \text{Inputs} \rightarrow P(\text{States} \times \text{Outputs}) \setminus \emptyset$$

receptiveness (i.e., machine must be prepared to accept every input)

State Machines

Deterministic

⇓ ⇑

Output-deterministic

⇓ ⇑

Nondeterministic

A state machine is deterministic

iff

1. there is only one initial state, and
2. for every state and every input,
   there is only one successor state.

A state machine is output-deterministic

iff

1. there is only one initial state, and
2. for every state and every input-output pair,
   there is only one successor state.

For deterministic M2 :

M1 is simulated by M2 iff M1 is equivalent to M2.

For output-deterministic M2 :

M1 is simulated by M2 iff M1 refines M2.

For nondeterministic M2 :

M1 is simulated by M2 implies M1 refines M2.

relation between finitely many states

condition on infinitely many behaviors

A binary relation $S \subseteq$ States [M1] $\times$ States [M2] is a simulation of M1 by M2

iff

1. $\forall$ p $\in$ possibleInitialStates [M1] ,
   $\exists$ q $\in$ possibleInitialStates [M2], (p, q) $\in$ S and
2. $\forall$ p $\in$ States [M1] , $\forall$ q $\in$ States [M2] ,
   if (p, q) $\in$ S ,
   then $\forall$ x $\in$ Inputs , $\forall$ y $\in$ Outputs , $\forall$ p' $\in$ States [M1] ,
   if (p', y) $\in$ possibleUpdates [M1] (p, x)
   then $\exists$ q' $\in$ States [M2] ,
   (q', y) $\in$ possibleUpdates [M2] (q, x) and
   (p', q') $\in$ S .

To check if M1 refines M2,
check if M1 is simulated by det(M2):

M1 refines M2

iff

M1 refines det(M2)

iff

M1 is simulated by det(M2).

output-deterministic

If M2 is an output-deterministic state machine, then a simulation S of M1 by M2 can be found as follows:

1. If $p \in$ possibleInitialStates [M1] and
   possibleInitialStates [M2] = { q },
   then $(p,q) \in S$.
2. If $(p,q) \in S$ and
   $(p',y) \in$ possibleUpdates [M1] (p,x) and
   possibleUpdates [M2] (q,x) = { q' },
   then $(p',q') \in S$.

Output-Determinization

Given: nondeterministic state machine M

Find: output-deterministic state machine det(M) that is equivalent to M

Inputs [det(M)] = Inputs [M]

Outputs [det(M)] = Outputs [M]

The Subset Construction

Let initialState [det(M)] = possibleInitialStates [M] ;

Let States [det(M)] = { initialState [det(M)] } ;

Repeat as long as new transitions can be added to det(M) :

Choose $P \in$ States [det(M)] and $(x,y) \in$ Inputs $\times$ Outputs ;

Let $Q = \{ q \in$ States [M] $\mid \exists\, p \in P,\ (q,y) \in$ possibleUpdates [M] $(p,x) \}$;

If $Q \neq \emptyset$ then

Let States [det(M)] = States [det(M)] $\cup \{Q\}$;

Let update [det(M)] $(P,x) = (Q,y)$.

Minimization Algorithm

Input : nondeterministic state machine M

Output : minimize (M), the state machine with the fewest states that is bisimilar to M

(the result is unique up to renaming of states)

A binary relation $B \subseteq States[M1] \times States[M2]$ is a bisimulation between M1 and M2

iff

A1. $\forall p \in possibleInitialStates[M1]$,

$\exists q \in possibleInitialStates[M2]$, $(p, q) \in B$, and

A2. $\forall p \in States[M1]$, $\forall q \in States[M2]$,

if $(p, q) \in B$,

then $\forall x \in Inputs$, $\forall y \in Outputs$, $\forall p' \in States[M1]$,

if $(p', y) \in possibleUpdates[M1](p, x)$

then $\exists q' \in States[M2]$,

$(q', y) \in possibleUpdates[M2](q, x)$ and

$(p', q') \in B$, and

and

B1. $\forall\, q \in \text{possibleInitialStates}\,[M2]$,

$\exists\, p \in \text{possibleInitialStates}\,[M1],\ (p, q) \in B$, and

B2. $\forall\, p \in \text{States}\,[M1]$, $\forall\, q \in \text{States}\,[M2]$,

if $(p, q) \in B$,

then $\forall\, x \in \text{Inputs}$, $\forall\, y \in \text{Outputs}$, $\forall\, q' \in \text{States}\,[M2]$,

if $(q', y) \in \text{possibleUpdates}\,[M2]\,(q, x)$

then $\exists\, p' \in \text{States}\,[M1]$,

$(p', y) \in \text{possibleUpdates}\,[M1]\,(q, x)$ and

$(p', q') \in B$.

For nondeterministic state machines M1 and M2,

M1 is equivalent to M2

$\not\Downarrow \Uparrow$

M1 simulates M2 and M2 simulates M1

$\not\Updownarrow \Uparrow$

M1 and M2 are bisimilar.

For output-deterministic state machines M1 and M2,

M1 is equivalent to M2

$\Downarrow \Uparrow$

M1 and M2 are bisimilar.

Minimization Algorithm

1. Let Q be set of all reachable states of M.
2. Maintain a set P of state sets:

   Initially let $P = \{ Q \}$.

   Repeat until no longer possible: split P.
3. When done, every state set in P represents a single state of the smallest state machine bisimilar to M.

Split P

If there exist

two state sets $R \in P$ and $R' \in P$
two states $r1 \in R$ and $r2 \in R$
an input $x \in Inputs$
an output $y \in Outputs$

such that

$\exists\, r' \in R',\ (r', y) \in possibleUpdates(r1, x)$ and

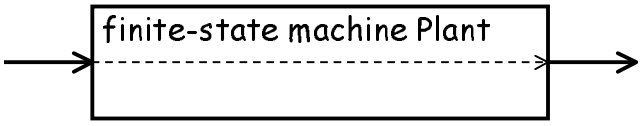
$\forall\, r' \in R',\ (r', y) \notin possibleUpdates(r2, x)$

then

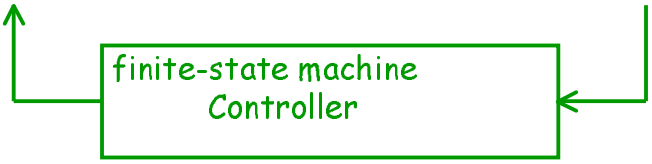
let $R1 = \{ r \in R \mid \exists\, r' \in R',\ (r', y) \in possibleUpdates(r, x) \}$;
let $R2 = R \setminus R1$;
let $P = (P \setminus \{R\}) \cup \{R1, R2\}$.

The Finite-State Safety Control Problem

Given

1. 

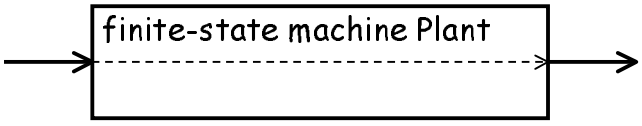
2. set Error of states of Plant

Find 

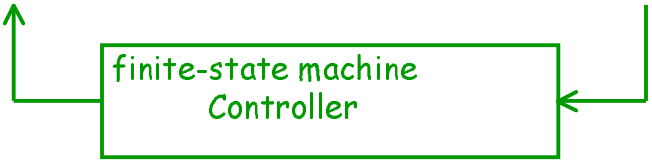
such that the composite system never enters a state in Error

The Finite-State Progress Control Problem

Given

1. 

2. set Target of states of Plant

Find 

such that the composite system is guaranteed to enter a state in Target

Compute the safety-uncontrollable states of Plant

1. Every state in Error is safety-uncontrollable.
2. For all states s,

   if for all inputs i
   there exist a safety-uncontrollable state s'
   and an output o
   such that $(s',o) \in$ possibleUpdates (s,i)

   then s is safety-uncontrollable.

Compute the progress-controllable states of Plant

1. Every state in Target is progress-controllable.
2. For all states s,

   if there exists an input i
   for all states s' and outputs o
   if $(s',o) \in$ possibleUpdates (s,i)
   then s' is progress-controllable

   then s is progress-controllable.

Typical Exam Questions

A. Convert between the following system representations:

1. Mathematical input-output definition
2. Transition diagram
3. Block diagram

B. Apply the following algorithms on state machines:

1. Product construction
2. Subset construction
3. Check for existence of a simulation
4. Minimization
5. Compute controllable states

C. Explain the following concepts:

1. Memory-free vs. finite-state vs. infinite-state
2. Equivalence/refinement vs. simulation vs. bisimulation
3. Safety vs. progress control