ALGEBRA LINEARE CON ELEMENTI DI GEOMETRIA

modulo: ELEMENTI DI GEOMETRIA (Prof. M. Spera)

Prova scritta del 6 Settembre 2011

(1) Nel piano euclideo reale, in cui sia fissato un riferimento cartesiano, ampliato proiettivamente, si determini la <u>parabola</u> $\mathcal{P}$ passante per $P: [1,1,1]$, <u>tangente</u> in P a $\tau: y = 2x-1$, tale che $d: y = \frac{1}{2}x + \frac{1}{2}$ sia un <u>diametro</u> e che passi infine per $O: [1,0,0]$. Se ne individui la forma canonica metrica, il fuoco e se ne abbozzi il grafico.

(2) Nello spazio euclideo, in cui sia fissato un riferimento cartesiano, si verifichi che le rette

$$r: \begin{cases} x = 1+t \\ y = 0 \\ z = 2t \end{cases} \qquad e \qquad r': \begin{cases} x+y-z=0 \\ x-y+z=0 \end{cases}$$

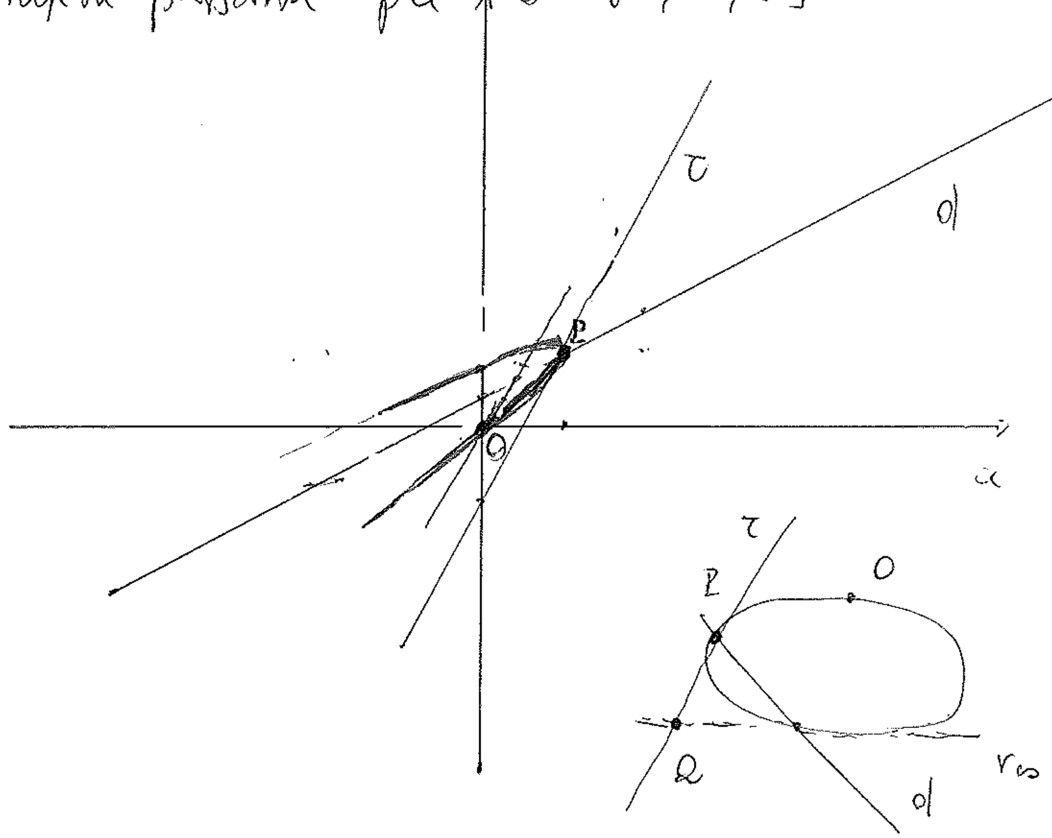
sono sghembe e se ne determini la <u>perpendicolare comune</u>.

Tempo a disposizione: 1h15m. Le risposte vanno adeguatamente giustificate

Elegeo
6 Settembre 2011

(1) Parabola $\mathcal{P}$ passante
per $P: [1,1,1]$,
tangente in P a $\tau: y = 2x - 1$
$d: \quad y = \frac{1}{2}x + \frac{1}{2}$ sia un diametro

e infine passante per $O: [1,0,0]$



(primo abbozzo...)

l'intersezione $Q: r_\infty \cap \tau$ è la direzione di τ, sicché

$Q = [0,1,2]$. $\mathcal{P}$ appartiene al fascio

di coniche bitangenti (forma non om.)

$$\tau \cdot r_\infty + \lambda d^2 = 0$$

$$(2x_1 - x_2 - x_0)x_0 + \lambda(x_1 + x_0 - 2x_2)^2 = 0$$

$$(2x - y - 1) + \lambda(x + 1 - 2y)^2 = 0$$

pass. per $O: (0,0)$ $\qquad -1 + \lambda = 0 \quad \Rightarrow \quad \boxed{\lambda = 1}$

$$\underbrace{2x-y-1}_{Y} + (\underbrace{x-2y+1}_{X})^2 = 0$$

$Y + X^2 = 0$

eq. c/ne

Asse = polare della direzione $\perp$ diametri

Direzione comune dei diametri: $D: [0, 2, 1]$

$D^\perp: [0, -1, 2]$

$$p_{D^\perp}: \quad (x_0\ x_1\ x_2)\begin{pmatrix}0 & 4 & -5\\ 4 & 2 & -4\\ -5 & -4 & 8\end{pmatrix}\begin{pmatrix}0\\ -1\\ 2\end{pmatrix} = 0$$

$$x^2+4y^2+\cancel{1}-4xy+2x-4y+2x-y-\cancel{1}=0$$

$$= x^2+4y^2-4xy+4x-5y=0$$

$$\tilde{A}: \begin{pmatrix}0 & 2 & -\frac{5}{2}\\ 2 & 1 & -2\\ -\frac{5}{2} & -2 & 4\end{pmatrix}$$

$$\underset{P}{\approx} A: \begin{pmatrix}0 & 4 & -5\\ 4 & 2 & -4\\ -5 & -4 & 8\end{pmatrix}$$

$$(x_0\ x_1\ x_2)\begin{pmatrix}\overbrace{-4-10}^{-14}\\ \underbrace{-2-8}\ -10\\ \underbrace{4+16}_{20}\end{pmatrix} = 0$$

$$-14x_0-10x_1+20x_2=0$$

$$-7x_0-5x_1+10x_2=0$$

$$-7-5x+10y=0$$

$$10y=5x+7$$

$$y=\frac{1}{2}x+\frac{7}{10} \quad (/\!/\ \delta\ r)$$

a asse

vertice $V = p \cap a$

$$2y = x+\frac{7}{5} \qquad x-2y=-\frac{7}{5}$$

$$2x-\frac{1}{2}x-\frac{7}{10}-1+(\underbrace{-\frac{7}{5}+1}_{-\frac{2}{5}})^2=0$$

$$\begin{array}{r}17\\ \times\ 5\\ \hline 85\end{array} \qquad \begin{array}{r}15\\ \times\ 7\\ \hline 105\end{array}$$

$$\frac{3}{2}x - \frac{17}{10} + \frac{4}{25} = 0$$

$$x = \frac{2}{3}\cdot\left(\frac{17}{10} - \frac{4}{25}\right) = \frac{2\cdot(85-8)}{3\cdot 50} = \frac{\cancel{2}}{3}\,\frac{77}{\underset{25}{\cancel{50}}} = \frac{77}{75}$$

$$y = \frac{1}{2}x + \frac{7}{10} = \frac{1}{2}\,\frac{77}{75} + \frac{7}{10} = \frac{77+105}{150} = \frac{182}{150} = \frac{91}{75}$$

$$\bar{V} = \left(\frac{77}{75}, \frac{91}{75}\right)$$

$$A = \begin{pmatrix} 0 & 4 & -5 \\ 4 & 2 & -4 \\ -5 & -4 & 8 \end{pmatrix}$$

$$\Omega = \det A = \underbrace{80+80}_{160} - 50 - \overset{128}{16\cdot 8}$$

$$= 110 - 128 = -18 = -9\cdot 2$$

$$\gamma = \operatorname{tr} A_{00} = 10$$

$$p = \sqrt{-\frac{\Omega}{\gamma^3}} = \sqrt{\frac{+18}{10^3}} = \frac{1}{10}\sqrt{\frac{18}{10}} = \frac{1}{10}\sqrt{\frac{9}{2}} = \frac{3}{10}\,\frac{\sqrt{2}}{2}$$

$$= \frac{3}{20}\sqrt{2}$$

F, H: $\quad V \pm \frac{p}{2}u \qquad u = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

$F \to -$

$H \to +$

(ragioni geometriche)

$$\begin{pmatrix} \frac{77}{75} \\ \frac{91}{75} \end{pmatrix} \pm \frac{3}{40}\sqrt{2}\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$\frac{3\sqrt{2}\sqrt{5}}{20\cdot 5} = \frac{3}{100}\sqrt{10}$$

$$= \begin{pmatrix} \frac{77}{75} \pm \frac{3\sqrt{2}}{20\sqrt{5}} \\ \frac{91}{75} \pm \frac{3\sqrt{2}}{20\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{77}{75} \pm \frac{3\sqrt{10}}{100} \\ \frac{91}{75} \pm \frac{3\sqrt{10}}{100} \end{pmatrix} = \begin{pmatrix} \frac{308 \pm 9\sqrt{10}}{300} \\ \frac{364 \pm 9\sqrt{10}}{100} \end{pmatrix}$$

$$308 + 9\sqrt{10} \sim 308 + 9\cdot 3 \sim 335$$

$$\begin{array}{r}77\\ \times\ 4\\ \hline 308\end{array} \qquad \begin{array}{r}91\\ \times\ 4\\ \hline 364\end{array}$$

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$$r: \begin{cases} x = 1 + t \\ y = 0 \\ z = 2t \end{cases} \qquad \text{dir } r: \left\langle \underset{\underline{u}}{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}} \right\rangle$$

$$r': \begin{cases} x + y - z = 0 \\ x - y + z = 0 \end{cases}$$

sghembe?

$$r' \begin{cases} x + y - z = 0 \\ 2x = 0 \end{cases}$$

$$\begin{cases} x = 0 \\ y = s \\ z = s \end{cases} \qquad \text{dir: } \left\langle \underset{\underline{v}}{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}} \right\rangle$$

$P: (1,0,0) \in r$

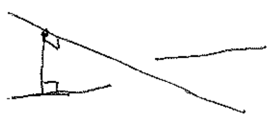
$Q: (0,0,0) \in r'$

$$\overrightarrow{QP} = (1,0,0)$$

$$\left| \overrightarrow{QP}, \underline{u}, \underline{v} \right| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} = -2 \neq 0$$

r, r' sono sghembe. (anche ovvio a priori...)

troviamo la perpendicolare comune



$$\overrightarrow{Q_s P_t} = \begin{pmatrix} 1+t \\ -s \\ 2t - s \end{pmatrix} \perp \begin{matrix} \underline{u} \\ \underline{v} \end{matrix}$$

$$\underline{P}_t \equiv \begin{pmatrix} 1+t \\ 0 \\ 2t \end{pmatrix}$$

$$Q_s \equiv \begin{pmatrix} 0 \\ -s \\ -s \end{pmatrix}$$

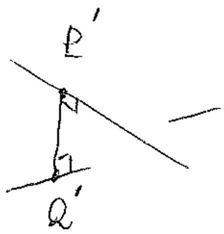
Si arriva al sistema:

$$\begin{cases} (1+t)\cdot 1 - s\cdot 0 + (2t-s)\cdot 2 = 0 \\ (1+t)\cdot 0 - s\cdot 1 + (2t-s)\cdot 1 = 0 \end{cases}$$

$$\begin{cases} 1+t + 4t - 2s = 0 \\ -s + 2t - s = 0 \end{cases} \qquad \begin{cases} 5t - 2s = -1 \\ t - s = 0 \end{cases} \qquad t = s$$

$$5t - 2t = -1 \qquad 3t = -1 \qquad t = -\frac{1}{3} \qquad s = -\frac{1}{3}$$

$$r \ni P' : \begin{pmatrix} 1-\frac{1}{3} \\ 0 \\ 2(-\frac{1}{3}) \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ 0 \\ -\frac{2}{3} \end{pmatrix}$$



$$r' \ni Q' : \begin{pmatrix} 0 \\ -\frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}$$

retta $P'Q'$: $\quad X = P' + \xi \overrightarrow{P'Q'} \qquad \overrightarrow{P'Q'} = \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$

retta per P' di dir $\overrightarrow{P'Q'}$

$$\begin{cases} x = \frac{2}{3} + \xi(-\frac{2}{3}) \\ y = 0 + \xi(-\frac{1}{3}) \\ z = -\frac{2}{3} + \xi(\frac{1}{3}) \end{cases} \qquad -\frac{1}{3} + \frac{2}{3} = \frac{1}{3}$$

$$\begin{cases} x = \frac{2}{3} - \frac{2}{3}\xi \\ y = 0 - \frac{1}{3}\xi \\ z = -\frac{2}{3} + \frac{1}{3}\xi \end{cases} \qquad \begin{cases} x = \frac{2}{3} + 2\eta \\ y = \eta \\ z = -\frac{2}{3} - \eta \end{cases}$$

$\eta \in \mathbb{R}$