TOPOLOGIA E GEOMETRIA DIFFERENZIALE

(M. Spera, N. Sansonetto) Prova scritta del 16 febbraio 2011

① Dati, in $\mathbb{R}^2 \setminus (\{x=0\} \cup \{y=0\})$, $X = e^{-x}\frac{\partial}{\partial x}$, $Y = y^2 \frac{\partial}{\partial y} - \frac{\partial}{\partial x}$

$\alpha = \frac{1}{x}\,dx + xy\,dy$, $\beta = \frac{1}{xy}\,dx \wedge dy$,

Calcolare :
1. $[X, Y]$
2. $i_{[X,Y]}\,\alpha$
3. $i_X \circ i_X \circ \beta$

② Calcolare $\mathcal{L}_X \alpha$.
Dire se $d\alpha$ e β sono coomologhe

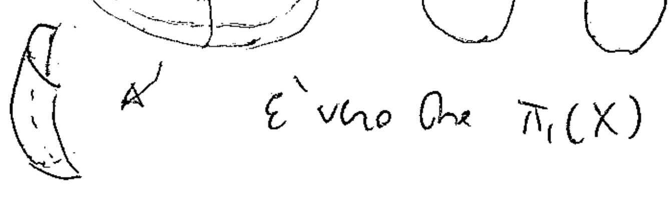
③ Determinare $f : \mathbb{R} \circlearrowleft$ tale che $X = f(x)\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$,
$Y = x\frac{\partial}{\partial x} + \frac{1}{x^2+x+1}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$ definiscano una distribuzione integrabile

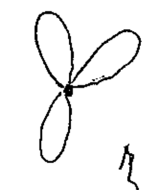
④ Determinare $H^*(\mathbb{R}^3 \setminus \{P, Q\})$
$\nwarrow \uparrow$ pti

⑤ Determinare il gruppo fondamentale di

$X =$  $(X = \mathbb{T}^2 \# S^2)$

Sugg.



È vero che $\pi_1(X) \cong \pi_1($  $)$?

bouquet

① $\quad X = e^{-x}\frac{\partial}{\partial x} \qquad Y = y^2\frac{\partial}{\partial y} - \frac{\partial}{\partial x}$

$$\alpha = \frac{1}{x}\,dx + xy\,dy \qquad \beta = \frac{1}{xy}\,dx\wedge dy \qquad \begin{matrix} x\neq 0 \\ y \neq 0 \end{matrix}$$

$$[X,Y] = \underbrace{\left[e^{-x}\frac{\partial}{\partial x}, y^2\frac{\partial}{\partial y}\right]}_{\substack{\parallel \\ 0}} - \left[e^{-x}\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right]$$

$$\left[e^{-x}\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right](f) = e^{-x}\frac{\partial^2}{\partial x}f - \frac{\partial}{\partial x}\left(e^{-x}\frac{\partial f}{\partial x}\right) =$$

$$= \cancel{e^{-x}\frac{\partial^2 f}{\partial x^2}} + e^{-x}\frac{\partial f}{\partial x} - \cancel{e^{-x}\frac{\partial^2 f}{\partial x^2}} = e^{-x}\frac{\partial}{\partial x}\cdot f$$

$$\Rightarrow \quad [X,Y] = -e^{-x}\frac{\partial}{\partial x}$$

Pertanto $\quad i_{[X,Y]}\alpha = \left(\frac{1}{x}\,dx + xy\,dy, -e^{-x}\frac{\partial}{\partial x}\right) =$

$$= -\frac{1}{x}e^{-x}\underbrace{\left(dx, \frac{\partial}{\partial x}\right)}_{\substack{\parallel \\ 1}} - xye^{-x}\underbrace{\left(dy, \frac{\partial}{\partial x}\right)}_{\substack{\parallel \\ 0}} = -\frac{1}{x}e^{-x}$$

$$\left(i_X \circ i_X \cdot \beta\right) = 0 \qquad \text{poiché} \quad i_X^2 = 0$$

(2) $\mathcal{L}_X = d\, i_X + i_X\, d$

$$d\alpha = d\left(\tfrac{1}{x}\,dx + xy\,dy\right) =$$

$$\underbrace{d(d\log x)}_{=0} + d(xy\,dy) = (dx\,y + x\,dy)\wedge dy$$

$$= y\,dx\wedge dy + x\,\underbrace{dy\wedge dy}_{=0}$$

$$= y\,dx\wedge dy$$

$$i_X\, d\alpha = i_X(y\,dx\wedge dy) = \left(y\,dx\wedge dy,\ e^{-x}\tfrac{\partial}{\partial x}\right)$$

$$= y\,e^{-x}\,dy$$

$$i_X\,\alpha = \left(\tfrac{1}{x}\,dx + xy\,dy,\ e^{-x}\tfrac{\partial}{\partial x}\right) = \frac{e^{-x}}{x}$$

$$d\left(\frac{e^{-x}}{x}\right) = \frac{-e^{-x}\cdot x - e^{-x}}{x^2}\,dx = -\frac{e^{-x}(1+x)}{x^2}\,dx$$

$$\mathcal{L}_X\,\alpha = -\frac{e^{-x}(1+x)}{x^2}\,dx + e^{-x}\,y\,dy$$

$$= e^{-x}\left[-\frac{1+x}{x^2}\,dx + y\,dy\right]$$

$d(d\alpha) = 0 \qquad d\beta = 0 \qquad$ (2-forma su $\mathbb{R}^2 \setminus (\{x=0\} \cup \{y=0\})$)

ora $\qquad H^2_{dR}(\mathbb{R}^2 \setminus (\{x=0\}\cup\{y=0\})) = 0$

(Poincaré)

om

$\Rightarrow [\beta] = [d\alpha] = 0$

(3) $$X = f(x)\frac{\partial}{\partial y} - x\frac{\partial}{\partial z} \qquad Y = x\frac{\partial}{\partial x} + \frac{1}{x^2+x+1}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

Dobbiamo imporre $[X,Y] \in \langle X, Y \rangle$ s.sp. vett gen da

$$[X,Y] = \overset{(1)}{\left[f\frac{\partial}{\partial y}, x\frac{\partial}{\partial x}\right]} - \overset{(2)}{\left[x\frac{\partial}{\partial z}, x\frac{\partial}{\partial x}\right]}$$

$$+ \overset{(3)}{\left[f\frac{\partial}{\partial y}, \frac{1}{x^2+x+1}\frac{\partial}{\partial y}\right]} - \overset{(4)}{\left[x\frac{\partial}{\partial z}, \frac{1}{x^2+x+1}\frac{\partial}{\partial y}\right]}$$

$$+ \overset{(5)}{\left[f(x)\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]} - \overset{(6)}{\left[x\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right]}$$

(1)

$$\left[f\frac{\partial}{\partial y}, x\frac{\partial}{\partial x}\right](g) = f\frac{\partial}{\partial y}\left(x\frac{\partial g}{\partial x}\right) - x\frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial y}\right) =$$

$$= \cancel{fx\frac{\partial^2 g}{\partial y \partial x}} \boxed{- x\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}} + \cancel{fx\frac{\partial^2 g}{\partial x \partial y}}$$

(2)

$$\left[x\frac{\partial}{\partial z}, x\frac{\partial}{\partial x}\right](g) = x\frac{\partial}{\partial z}\left(x\frac{\partial g}{\partial x}\right) - x\frac{\partial}{\partial x}\left(x\frac{\partial g}{\partial z}\right)$$

$$= \cancel{x^2\frac{\partial^2 g}{\partial z \partial x}} - x\frac{\partial g}{\partial z} - \cancel{x^2\frac{\partial^2 g}{\partial x \partial z}}$$

(3) $$\left[f(x)\frac{\partial}{\partial y}, \frac{1}{x^2+x+1}\frac{\partial}{\partial y}\right](g) = f\frac{\partial}{\partial y}\left(\frac{1}{x^2+x+1}\frac{\partial g}{\partial y}\right) - \frac{1}{(\;)}\frac{\partial}{\partial y}\left(f(x)\frac{\partial g}{\partial y}\right)$$

$$= f\frac{1}{(\;)}\frac{\partial^2 g}{\partial y^2} - f\frac{1}{(\;)}\frac{\partial^2 g}{\partial y^2} = 0$$

④ $$\left[x \frac{\partial}{\partial z} , \frac{1}{x^2+x+1} \frac{\partial}{\partial y} \right](q) = .. 0$$

⑤ $$\left[f(x) \frac{\partial}{\partial y} , \frac{\partial}{\partial z} \right](q) = \dots 0$$

⑥ $$\left[x \frac{\partial}{\partial z} , \frac{\partial}{\partial z} \right](q) = \dots 0$$

$\Rightarrow$

$$[X, Y] = -x f' \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}$$

$$\Rightarrow \quad x f' = f \qquad \frac{f'}{f} = \frac{1}{x} \qquad f \neq 0$$

$$d \log|f| = \frac{1}{x} dx$$

$$d \log|f| = d \log|x|$$

$$\log|f| = \log|x| + \mathbb{R}$$

$$|f| = C_1 \cdot |x| \qquad x \neq 0 \qquad C_1 > 0$$

$$f = \pm c x \qquad x \neq 0 \qquad c > 0$$

(4) $H^*(\underbrace{\mathbb{R}^3 \setminus \{P, Q\}}_{X})$

w_P, w_Q : d. di flusso del campo elettrico generato dalla carica in P (Q)

$H^0 = \mathbb{R}$ (X connesso)

$H^1 = H^3 = 0$] X sempl. connesso, X non compatto

$H^2 \cong \mathbb{R} \oplus \mathbb{R}$ — intuitivamente chiaro cf. teor. di Gauss

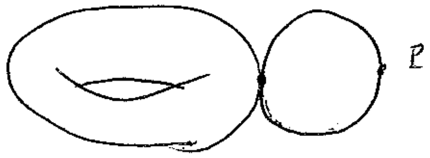
Mayer-Vietoris
$U = \mathbb{R}^3 - \{P\}$
$V = \mathbb{R}^3 - \{Q\}$
$U \cup V = \mathbb{R}^3$
$U \cap V = X$

$H^2 = \langle [w_P], [w_Q] \rangle$

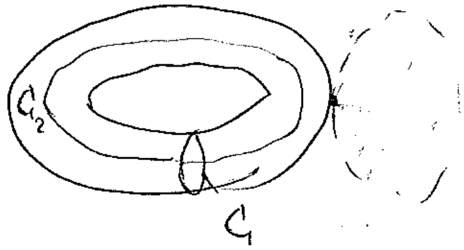
$$\begin{array}{ccccc} U \cup V & & U \amalg V & & U \cap V \\ H^1(\mathbb{R}^3) & \leftarrow & H^1(U) \oplus H^1(V) & \leftarrow & H^2(X) \\ \| & & \mathbb{R} \oplus \mathbb{R} & & \\ 0 & & & & \\ H^2(\mathbb{R}^3) & & & & \\ \| & & & & \\ 0 & & & & \end{array}$$

$\Rightarrow H^2(X) \cong \mathbb{R}^2$

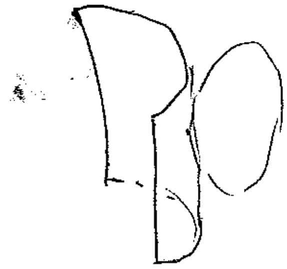
(5) $X =$



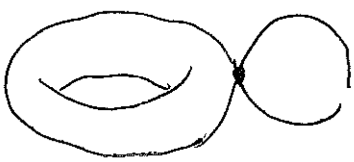
$U = X - \{C_1, C_2\}$



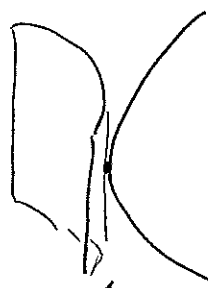
$\simeq$



$V = X - P$



$U \cap V =$



è semplicemente connesso

$\Rightarrow \pi_1(X) = \pi_1(U) * \pi_1(V) = \mathbb{Z} * \mathbb{Z}^2 \neq \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$

$\pi_1(U) = \mathbb{Z}$, $\pi_1(V) = \mathbb{Z}^2$

intuitivamente chiaro