GEOMETRIA a.a. 2008/09 Prova scritta dell'8/9/2009

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(1) Sia dato, nel piano (x, z), il ramo di iperbole $\mathcal{I}$ $z^2 - x^2 = 1$, $z > 0$. Dopo aver opportunamente parametrizzato la curva, se ne determinino la curvatura e l'evoluta (possibilmente in più modi).

(2) Si consideri la superficie Σ ottenuta ruotando il ramo di iperbole $\mathcal{I}$ dell'es. 1 attorno all'asse x. Se ne calcolino, in un pto generico, le curvature principali e la curvatura gaussiana.

(3) 

Σ

Dire se è possibile dotare la superficie Σ in figura di una metrica con curvatura $K \geq 0$.

È possibile far sì che il trasporto parallelo lungo una curva chiusa qualsiasi su Σ risulti identicamente nullo?

Tempo a disposizione 2h. Le risposte vanno adeguatamente giustificate.

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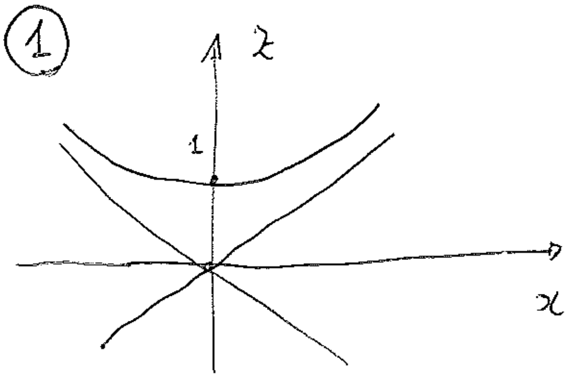
(1)



$$z^2 - x^2 = 1 \qquad z > 0$$

Evoluta. (1) inviluppo delle normali

$$\begin{cases} x = \operatorname{sh} t \\ z = \operatorname{ch} t \end{cases} \qquad \operatorname{ch}^2 t - \operatorname{sh}^2 t = 1$$

Eq. tangenti

$$f(x,z) = 0$$

$$f^0_x (x - x_0) + f^0_z (z - z_0) = 0$$

$$(-\cancel{2}x_0)(x - x_0) + \cancel{2}z_0 \cdot (z - z_0) = 0$$

Normali $\longrightarrow$

$$z_0 (x - x_0) + x_0 (z - z_0) = 0$$

$$\operatorname{ch} t\, (x - \operatorname{sh} t) + \operatorname{sh} t\, (z - \operatorname{ch} t) = 0$$

$$\operatorname{ch} t\, x - \operatorname{sh} t \operatorname{ch} t + \operatorname{sh} t\, z - \operatorname{sh} t \operatorname{ch} t = 0$$

$$\operatorname{ch} t\, x + \operatorname{sh} t \cdot z - \operatorname{sh} 2t = 0$$

$\frac{\partial}{\partial t}() = 0$

$$\operatorname{sh} t\, x + \operatorname{ch} t\, z - 2 \operatorname{ch} 2t = 0$$

$$\begin{cases} \operatorname{ch} t\, x + \operatorname{sh} t\, z = \operatorname{sh} 2t \\ \operatorname{sh} t\, x + \operatorname{ch} t\, z = 2 \operatorname{ch} 2t \end{cases}$$

$$x = \frac{\begin{vmatrix} \operatorname{sh} 2t & \operatorname{sh} t \\ 2\operatorname{ch} 2t & \operatorname{ch} t \end{vmatrix}}{\underbrace{\operatorname{ch}^2 t - \operatorname{sh}^2 t}_{=1}} = \operatorname{sh} 2t \operatorname{ch} t - 2\operatorname{ch} 2t \operatorname{sh} t = \dots \quad (+)$$

$$= -2 \operatorname{sh}^3 t$$

$$z = \begin{vmatrix} \operatorname{ch} t & \operatorname{sh} 2t \\ \operatorname{sh} t & 2\operatorname{ch} 2t \end{vmatrix} = 2 \operatorname{ch} 2t \operatorname{ch} t - \operatorname{sh} 2t \cdot \operatorname{sh} t$$

$$= 2 \operatorname{ch} 2t \cdot \operatorname{ch} t - 2 \operatorname{sh}^2 t \operatorname{ch} t$$

$$= 2 (\operatorname{ch} 2t - \operatorname{sh}^2 t) \operatorname{ch} t$$

$$= 2 (2\operatorname{ch}^2 t - 1 - \operatorname{sh}^2 t) \operatorname{ch} t$$

$$= 2 \operatorname{ch}^3 t$$

(+) $\operatorname{sh} 2t \operatorname{ch} t - 2\operatorname{ch} 2t \operatorname{sh} t$

$$= 2 \operatorname{sh} t \operatorname{ch}^2 t - 2[2\operatorname{ch}^2 t - 1] \operatorname{sh} t$$

$$= 2 \operatorname{sh} t \operatorname{ch}^2 t - 4 \operatorname{sh} t \operatorname{ch}^2 t + 2\operatorname{sh} t$$

$$= -2 \operatorname{sh} t \cdot \operatorname{ch}^2 t + 2\operatorname{sh} t$$

$$= 2 \operatorname{sh} t [\underbrace{1 - \operatorname{ch}^2 t}_{-\operatorname{sh}^2 t}]$$

$$= -2 \operatorname{sh}^3 t$$

$$\begin{cases} x = -2 \operatorname{sh}^3 t \\ z = 2 \operatorname{ch}^3 t \end{cases}$$

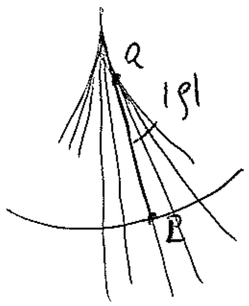
$$x^{\frac{1}{3}} = (-2)^{\frac{1}{3}} \operatorname{sh} t$$

$$z^{\frac{1}{3}} = 2^{\frac{1}{3}} \cdot \operatorname{ch} t$$

$$\boxed{z^{2/3} - x^{2/3} = 2^{2/3}}$$

Calcoliamo ρ in questo modo

$$\overline{PQ} = |\rho| = \sqrt{(-2sh^3t - sht)^2 + (2ch^3t - cht)^2}$$



$$= \sqrt{sh^2t\,[\underbrace{+2sh^2t+1}_{=\,2ch^2t-2+1\,=\,2ch^2t-1}]^2 + ch^2t\,[2ch^2t-1]^2}$$

$ch^2 - sh^2 = 1$

$$= (2ch^2t-1)^{3/2} \qquad (\text{segno! } +)$$

$$(\kappa = (2z^2-1)^{-3/2})$$

$sh^2 + ch^2 =$
$= ch^2 + ch^2 - 1$
$= 2ch^2 - 1$

Altro modo (standard)

$P: (sht, cht)$

$\dot{P} = (cht, sht) \qquad \|\dot P\|^2 = ch^2 + sh^2 = 2ch^2 - 1$

$\ddot{P} = (sht, cht)$

$i\dot{P} = (-sht \quad cht)$

$$Q = P + \frac{\|\dot P\|^2}{\underbrace{\langle i\dot P, \ddot P\rangle}_{=\,ch^2 - sh^2 = 1}}\, i\dot P \qquad \begin{cases} x = sht - \overbrace{(2ch^2t-1)}^{2sh^2t+1} sht = -2sh^3t \\ y = cht + (2ch^2t-1)\,cht \\ \quad = 2ch^3t \end{cases}$$

V

$$|k| = \frac{\| \dot{\underline{r}} \times \ddot{\underline{r}} \|}{\| \dot{\underline{r}} \|^3}$$

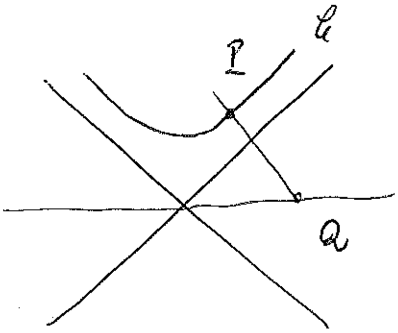
$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \operatorname{sh} t & \operatorname{sh} t & 0 \\ \operatorname{sh} t & \operatorname{ch} t & 0 \end{vmatrix} = \ldots = \underline{k}$$

$$= \left(\operatorname{ch}^2 t + \operatorname{sh}^2 t \right)^{-\frac{3}{2}}$$

$$= \left(2\operatorname{ch}^2 t - 1 \right)^{-\frac{3}{2}}$$

segno : +

(2)



gran normale

Normale a ℓ in P : $(\operatorname{sh} t, \operatorname{ch} t)$

$\boxed{\text{V. ex. 1}}$

$$\operatorname{ch} t\,(x - \operatorname{sh} t) + \operatorname{sh} t\,(z - \operatorname{ch} t) = 0$$

poniamo $z = 0$

$$\cancel{\operatorname{ch} t}\,(x - \operatorname{sh} t) - \operatorname{sh} t\,\cancel{\operatorname{ch} t} = 0$$

$$x - \operatorname{sh} t - \operatorname{sh} t = 0$$

$$x = 2\operatorname{sh} t \implies Q : (2\operatorname{sh} t, 0)$$

$$\overline{PQ}^2 = (2\operatorname{sh} t - \operatorname{sh} t)^2 + \operatorname{ch}^2 t$$

$$= \operatorname{sh}^2 t + \operatorname{ch}^2 t = 2\operatorname{ch}^2 t - 1$$

$$\overline{PQ} = (2\operatorname{ch}^2 t - 1)^{\frac{1}{2}}$$

$$k_2 = -(2\operatorname{ch}^2 t - 1)^{-\frac{1}{2}}$$

$\boxed{\text{V. ex. 1}}$ $$k_1 = (2\operatorname{ch}^2 t - 1)^{-\frac{3}{2}}$$

$$K = k_1 k_2 = -(2\operatorname{ch}^2 t - 1)^{-2}$$

$$\left(= -(2z^2 - 1)^{-2}\right)$$

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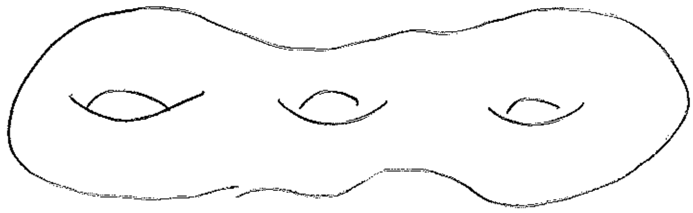
3



$K \geq 0$ ovunque implicherebbe

$$0 \leq \frac{1}{2\pi} \iint_\Sigma K = 2 - \underbrace{2 \cdot 3}_{6} = -4 < 0$$

assurdo.

Il trasporto $\parallel$ non può essere id. nullo

poiché $\iint_\Omega K = 0 \quad \forall \Omega$

$$\Rightarrow K \equiv 0 \quad \Rightarrow \underbrace{\frac{1}{2\pi} \int_\Sigma K}_{= -4} = 0$$

$0 = -4$!

ASSURDO