Mathematical logic

Problems 2

8. Let $\mathcal{B}$ be a Beth structure for minimal logic, and η a variable assignment.(*)

a) Prove for every proposition symbol R:
$\forall k\; I(R,k)=0 \Leftrightarrow \forall k\; k \not\Vdash R[\eta]$.
In particular, for $\mathcal{B}$ to be a Beth structure for intuitionistic logic, it does not matter whether one requires
$\forall k\; I(\bot,k)=0$ or $\forall \eta\, \forall k\; k \not\Vdash \bot[\eta]$.

b) Prove that each of the following items implies the next:

i) $\mathcal{B}$ is a Beth structure for intuitionistic logic

ii) $\forall \eta\, \forall k\; k \Vdash \bot \to A[\eta]$ for every formula A (here again of predicate logic)

iii) $\forall \eta\, \forall k\; k \Vdash \mathrm{EFQ}\,[\eta]$

(*) As usual k stands for a node of the underlying complete binary tree $\{0,1\}^*$.

9. Let $\mathcal{B}$ be a Beth structure of intuitionistic logic, η a variable assignment, k a node and A a formula. Prove:

a) $k \Vdash \neg A[\eta] \Leftrightarrow \forall k' \geq k \; k' \nVdash A[\eta]$

b) $k \Vdash \neg\neg A[\eta] \Leftrightarrow \forall k' \geq k \, \neg \forall k'' \geq k' \; k'' \nVdash A[\eta]$

10. Prove $\nvdash_i \mathrm{TND}_R$ for every relation symbol R by finding a Beth structure for intuitionistic logic in which $() \nVdash R\underline{x} \vee \neg R\underline{x}$.
(Hint: Use 9.a and a Beth structure that has been used in a lecture)

11. Deduce the axioms of symmetry and transitivity from the other equality axioms in minimal logic.

12. Let $\mathcal{L}$ be the language of groups.
a) Prove that there is no $\mathcal{L}$-axiom system the axioms of which are precisely the finite groups
b) Prove that if a sentence of $\mathcal{L}$ is valid in all finite groups, then it must also be valid in an infinite group.

13. Let $\mathcal{L} = \{0, 1, +, -, \times, ^{-1}; =\}$ be the language of fields, and A a sentence of $\mathcal{L}$. Prove that if A is valid in all fields of characteristic 0, then there must be $n \in \mathbb{N}$ such that A is valid in all fields of characteristic $> n$.

(Recall that the characteristic of a field K is defined as
$\mathrm{char}(K) = \min\{\ell \in \mathbb{N} : \ell \geq 2, \ell \cdot 1_K = 0\}$
if this set has an element, and
$\mathrm{char}(K) = 0$ if $\ell \cdot 1_K \neq 0$ for all $\ell \in \mathbb{N}, \ell \geq 2$.
Here $\ell \cdot 1_K = 1 + \dots + 1$ with ℓ summands.)

14. Show that the class of fields of finite characteristic is <u>not</u> axiomatisable, i.e. that there is <u>no</u> axiom system the models of which are <u>exactly</u> the fields of finite characteristic.

(Hint: Use 13. above.)