Chapter 2

Diffusion equations

2.1 Separation of variables: intervals

Diffusion equation is a linear partial differential equation, since the functions related to u in the equations (u_t and Δu) are both linear. Recall for the linear ordinary differential equations:

$$\frac{dP}{dt} = kP, \text{ and } D\frac{d^2P}{dx^2} = kP, \tag{2.1}$$

it is well-known that e^{kt} and $A\cos(\sqrt{k/D}tx) + B\sin(\sqrt{k/D}x)$ are the solutions of the respective equations. Indeed in the second case, the exponential function $e^{kitx/D}$ is a complex solution. Thus we can make a lucky guess for the solution of the diffusion equation:

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2}, \tag{2.2}$$

that the solution is an exponential function

$$P(t,x) = e^{at+bx}, \tag{2.3}$$

for some constants a, b, which could be complex numbers. By substitution, we find that the function in (2.3) is a solution of (2.2) if $a = Db^2$, and depending on the values of b, we obtain three families of solutions:

$$P(t,x) = e^{Db^2t}e^{bx}, \quad b > 0, \tag{2.4}$$

$$P(t,x) = e^{Db^2t}e^{-bx}, \quad b > 0, \tag{2.5}$$

$$P(t,x) = e^{-Db^2t}e^{-bix}, \quad b > 0. \tag{2.6}$$

The first two families of solutions are not reasonable solutions, since when $x \to \infty$ (or $-\infty$) $P(t,x) \to \infty$, which implies an unlimited growth at $x = \infty$ (or $-\infty$). The third family of solutions are complex, but their real and imaginary parts are both solutions of (2.2) too:

$$P(t,x) = e^{-Db^2t}\cos(bx), \quad \text{and } P(t,x) = e^{-Db^2t}\sin(bx). \tag{2.7}$$

These solutions are little more "reasonable" as they are bounded as $x \to \infty$, but still they do not satisfy natural boundary conditions $P, P_x \to 0$ as $x \to \infty$.

Nevertheless solutions with above forms are solutions of diffusion equations, and we notice that they are in form of $P(t,x) = U(t)V(x)$. This motivates us to consider the solutions which is separable on the two variables t and x:

$$u(t,x) = U(t)V(x). \tag{2.8}$$

With the form of u in (2.8), the diffusion equation (2.2) becomes

$$U'(t)V(x) = DU(t)V''(x), \quad \text{and } \frac{U'(t)}{U(t)} = D\frac{V''(x)}{V(x)} = Dk,$$

where k is a constant independent of both t and x. Thus U satisfies an equation:

$$U'(t) = DkU(t), \ \ t > 0, \tag{2.9}$$

and

$$U(t) = C_1 e^{Dkt}; \tag{2.10}$$

and V satisfies

$$V''(x) = kV(x). \tag{2.11}$$

The solutions of (2.11) are $C_2 e^{\sqrt{k}x}$ with any complex number k. Thus we obtain solution of (2.2):

$$Ce^{Dkt}e^{\sqrt{k}x}, \tag{2.12}$$

just like the three families of solutions we obtain by making lucky guess. To obtain more specific solutions, we must take the boundary and initial conditions into considerations.

We start with the one-dimensional diffusion equation on an interval with length L:

$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2}, \qquad x \in (0,L). \tag{2.13}$$

Here the individuals of the the species inhabits a one dimensional patch $(0,L)$, and the diffusion coefficient of the population is $D > 0$. To get a definite solution of the problem, we need to have an initial population distribution:

$$P(0,x) = P_0(x), \qquad x \in (0,L), \tag{2.14}$$

and a boundary condition. We assume that the exterior is hostile so homogeneous Dirichlet boundary condition is satisfied:

$$P(t,0) = P(t,L) = 0. \tag{2.15}$$

We look for solution with form $P(t,x) = U(t)V(x)$. Then we obtain (2.9) and (2.11), and (2.15) implies that

$$V(0) = V(L) = 0. \tag{2.16}$$

(2.11) and (2.16) become a boundary value problem for an ordinary differential equation, which is a special case of *Sturm-Liouville problems*. We shall discuss this kind of problems in a separate section next.

2.2 Eigenvalues and eigenfunctions of boundary value problems

We consider the Dirichlet boundary value problem:

$$y'' = ky, \quad x \in (0, L), \quad y(0) = y(L) = 0. \tag{2.17}$$

We can see that $y(x) = 0$ is always a solution of the equation. So we shall look for solutions other than the trivial zero solution. From elementary ODE knowledge, we know that the general solution of $y'' = ky$ is one of the following three forms:

$$c_1 e^{-\sqrt{k}t} + c_2 e^{\sqrt{k}t}, \quad \text{if } k > 0, \tag{2.18}$$

$$c_1 + c_2 t, \quad \text{if } k = 0, \tag{2.19}$$

$$c_1 \cos(\sqrt{-k}t) + c_2 \sin(\sqrt{-k}t), \quad \text{if } k < 0. \tag{2.20}$$

One important characteristic of a boundary value problem is that it may not have a solution, while the initial value problem always has a solution and the solution is unique if all terms in the equation are differentiable. For (2.17), there is no solution when $k > 0$. Indeed if there is a solution, it must be $y(t) = c_1 e^{-\sqrt{k}t} + c_2 e^{\sqrt{k}t}$ for some constants c_1 and c_2, and we can solve the constants by

$$0 = y(0) = c_1 + c_2, \quad 0 = y(L) = c_1 e^{-\sqrt{k}L} + c_2 e^{\sqrt{k}L}.$$

But this system of equations is unsolvable: if one gets $c_1 = -c_2$ from the first equation and plugs it into the second one, one has $(e^{-\sqrt{k}L} - e^{\sqrt{k}L})c_2 = 0$, and one must have $c_2 = 0$ and $c_1 = 0$. Thus (2.17) has only zero solution when $k > 0$. It is also easy to check that (2.17) has only zero solution when $k = 0$. So the only meaningful case is when $k < 0$.

In this case, (2.17) may still have only the zero solution, but for some number $k < 0$, it can have a solution $y(t)$ which is not the zero solution. (Note that from the linear principle, if $y(t)$ is a solution of (2.17), so is $c \cdot y(t)$ for any constant c, and if $y_1(t)$ and $y_2(t)$ are solutions of (2.17), so is $y_1(t) + y_2(t)$.) So for k which (2.17) has non-zero solutions, we call these numbers **eigenvalues**, and the solutions of (2.17) **eigenfunctions**. (In fact, historically the solutions of (2.17) were first called eigenvalues and eigenfunctions.)

Now let's search for the eigenvalues and eigenfunctions of (2.17): the eigenfunction must be $y(t) = c_1 \cos(\sqrt{-k}t) + c_2 \sin(\sqrt{-k}t)$ for some constants c_1 and c_2; by the boundary conditions, we have

$$0 = y(0) = c_1, \quad 0 = y(L) = c_1 \cos(\sqrt{-k}L) + c_2 \sin(\sqrt{-k}L);$$

so $c_1 = 0$ and if c_2 is not zero, $\sin(\sqrt{-k}L) = 0$; thus $\sqrt{-k}L = m\pi$, the eigenvalue k must be

$$k = -\frac{m^2\pi^2}{L^2}, \tag{2.21}$$

and the eigenfunctions associated with $k = -\dfrac{m^2\pi^2}{L^2}$ are $c \sin\left(\dfrac{m\pi t}{L}\right)$. Therefore (2.17) has a sequence of eigenvalues: $k_m = -\dfrac{m^2\pi^2}{L^2}$, and the set of eigenfunctions are the sine functions which can "fit in" the interval $(0, L)$. When the interval is $(0, \pi)$ (so $L = \pi$), the expression of eigen-pairs is much simpler:

$$k_m = -m^2, \;\; f_m(t) = \sin(mt). \tag{2.22}$$

Back to (2.13) and (2.15), we find a sequence of solutions:

$$u_m(t,x) = exp\left(-D\frac{m^2\pi^2}{L^2}t\right)\sin\left(\frac{m\pi x}{L}\right), \;\; m \text{ is a positive integer.} \tag{2.23}$$

Because of the linear principle, the general solution of (2.13) is

$$u(t,x) = \sum_{m=1}^{\infty} c_m exp\left(-D\frac{m^2\pi^2}{L^2}t\right)\sin\left(\frac{m\pi x}{L}\right), \;\; t > 0, \; x \in (0, L). \tag{2.24}$$

Example 2.1.

$$y'' + ky = 0, \;\; y(0) + y'(0) = y(\pi) = 0. \tag{2.25}$$

This problem is same as previous example except the boundary conditions. So we still consider the cases of $k < 0$, $k = 0$ and $k > 0$. When $k < 0$, $y(t) = c_1 e^{-\sqrt{-k}t} + c_2 e^{\sqrt{-k}t}$, from the boundary conditions, we get $y(0)+y'(0) = c_1+c_2-\sqrt{-k}c_1+\sqrt{-k}c_2 = 0$ and $y(\pi) = c_1 e^{-\sqrt{-k}\pi}+c_2 e^{\sqrt{-k}\pi} = 0$, So (c_1, c_2) satisfies the equations

$$\begin{pmatrix} 1-\sqrt{-k} & 1+\sqrt{-k} \\ e^{-\sqrt{-k}\pi} & e^{\sqrt{-k}\pi} \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.26}$$

The equation has non-zero solution only if $det\begin{pmatrix} 1-\sqrt{-k} & 1+\sqrt{-k} \\ e^{-\sqrt{-k}\pi} & e^{\sqrt{-k}\pi} \end{pmatrix} = 0$. The determinant equals to $(1-p)e^{p\pi} - (1+p)e^{-p\pi}$ if we denote $p = \sqrt{-k}$, thus $det = 0$ is equivalent to $e^{2p\pi} = \dfrac{1+p}{1-p}$. The functions $f_1(p) = e^{2p\pi}$ and $f_2(p) = \dfrac{1+p}{1-p}$ have no common points when $p > 0$ (in fact, they only intersect at $p = 0$. thus $e^{2p\pi} = \dfrac{1+p}{1-p}$ is not possible for $p > 0$, and there is no eigenvalue such that $k < 0$. When $k = 0$, $y(t) = c_1 + c_2 t$, from the boundary conditions, we get $y(0)+y'(0) = c_1+c_2 = 0$ and $y(\pi) = c_1 + c_2\pi = 0$, since the determinant of the matrix $\begin{pmatrix} 1 & 1 \\ 1 & \pi \end{pmatrix}$ is not zero, then only solution for (c_1, c_2) is $(0, 0)$. So 0 is not an eigenvalue either.

For $k > 0$, $y(t) = c_1\cos(\sqrt{k}t) + c_2\sin(\sqrt{k}t)$. The boundary conditions: $y(0) + y'(0) = c_1 + c_2\sqrt{k} = 0$, $y(\pi) = c_1\cos(\sqrt{k}\pi) + c_2\sin(\sqrt{k}\pi) = 0$. The matrix is $\begin{pmatrix} 1 & \sqrt{k} \\ \cos(\sqrt{k}\pi) & \sin(\sqrt{k}\pi) \end{pmatrix}$, and the determinant of the matrix is 0 if $\sin(\sqrt{k}\pi) - \sqrt{k}\cos(\sqrt{k}\pi) = 0$, or $\tan(\pi p) = p$ for $p = \sqrt{k}$. The graphs of $f_1(p) = \tan(\pi p)$ and $f_2(p) = p$ intersect at a sequence of points, p_m, $m = 1, 2, 3, \cdots$, and $0.5 < p_1 < 1.5$, $1.5 < p_2 < 2.5$, $m - 0.5 < p_m < m + 0.5$. $k_m = p_m^2$ are the eigenvalues of the problem (2.25), and the eigenfunctions are $y_m(t) = \sqrt{k_m}\cos(\sqrt{k_m}t) - \sin(\sqrt{k_m}t)$.

2.3 Fourier series

To completely solve (2.13)-(2.15), we need to determine the constants c_m in (2.24) by setting $t = 0$ in (2.24):

$$u_0(x) = \sum_{m=1}^{\infty} c_m \sin\left(\frac{m\pi x}{L}\right), \quad x \in (0, L). \tag{2.27}$$

The expression in the right-hand side of (2.24) is the Fourier series of $u_0(x)$ on $(0, L)$. Fourier series is widely used in mathematical physics and signal process. All sine and cosine functions are considered to be regular smooth waves, and Fourier series convert any noise (or signal) into a sum of more regular waves, and that is a big help for the signal processing. Here we collect a few facts of Fourier series.

Theorem 2.2. *Let $f(x)$ and $f'(x)$ be piecewise continuous functions on the interval $[0, L]$. Then $f(x)$ can be expanded in either a pure sine series*

$$f(x) = \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right), \tag{2.28}$$

where $b_m = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right) dx$, $m = 1, 2, \cdots$, or a pure cosine series

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos\left(\frac{m\pi x}{L}\right), \tag{2.29}$$

where $a_m = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\left(\frac{m\pi x}{L}\right) dx$, $m = 0, 1, 2, \cdots$.

The Fourier series is similar to the Taylor series

$$f(x) = \sum_{m=0} \frac{f^{(n)}(0)}{n!} x^n, \tag{2.30}$$

but instead of expanding the function in polynomials, Fourier series expands a function in sine or cosine functions. But Taylor series converges to the original function only in a small neighborhood of $x = 0$ (though some time it is large), and the neighborhood depends on the function f. Fourier series converges to the original functions for all functions and on the whole interval $(0, L)$. (But it may not converge to $f(x)$ at $x = 0$ and $x = L$.)

Theorem 2.2 shows all sound wave can be decomposed into several regular waves plus a small noise. The approximation of Fourier series is sine or cosine Fourier polynomials:

$$f(x) \approx \sum_{m=1}^{N} b_m \sin\left(\frac{m\pi x}{a}\right), \quad f(x) \approx \frac{a_0}{2} + \sum_{m=1}^{N} a_m \cos\left(\frac{m\pi x}{a}\right). \tag{2.31}$$

Now we can complete the solution formula of (2.13)-(2.15):

$$
\begin{aligned}
&u(t,x) = \sum_{m=1}^{\infty} c_m exp\left(-D\frac{m^2\pi^2}{L^2}t\right)\sin\left(\frac{m\pi x}{L}\right), \\
&\text{where } c_m = \frac{2}{L}\int_0^L u_0(x)\sin\left(\frac{m\pi x}{L}\right)dx, m \geq 1, \;\; t>0, \; x \in (0,L).
\end{aligned} \tag{2.32}
$$

Example 2.3.

$$
\begin{cases}
u_t = u_{xx}, & t>0, \;\; x \in (0,\pi), \\
u(t,0) = u(t,\pi) = 0, \\
u(0,x) = \sin x.
\end{cases} \tag{2.33}
$$

From the solution formula (2.32), we get

$$
u(t,x) = \sum_{m=1}^{\infty} c_m e^{-m^2 t}\sin(mx), \text{ and } c_m = \frac{2}{\pi}\int_0^{\pi}\sin x\sin(mx)dx. \tag{2.34}
$$

However $\displaystyle\int_0^{\pi}\sin x\sin(mx)dx = 0$ for any $m \geq 2$, and $\displaystyle\int_0^{\pi}\sin^2 x dx = \pi/2$. Therefore the solution is simply

$$
u(t,x) = e^{-t}\sin x. \tag{2.35}
$$

Example 2.4. For the homogeneous Neumann boundary value problem

$$
\begin{cases}
u_t = Du_{xx}, & t>0, x \in (0,L), \\
u_x(t,0) = u_x(t,L) = 0, \\
u(0,x) = u_0(x),
\end{cases} \tag{2.36}
$$

the solution is

$$
\begin{aligned}
&u(t,x) = \frac{c_0}{2} + \sum_{m=1}^{\infty} c_m exp\left(-D\frac{m^2\pi^2}{L^2}t\right)\cos\left(\frac{m\pi x}{L}\right), \\
&\text{where } c_m = \frac{2}{L}\int_0^L u_0(x)\cos\left(\frac{m\pi x}{L}\right)dx, \;\; m \geq 0, \;\; t>0, \; x \in (0,L).
\end{aligned} \tag{2.37}
$$

The derivation of this formula is left as exercise.

For these examples, the asymptotic behavior of the solution can be read from the formulas. For homogeneous Dirichlet problem (2.13)-(2.15), the asymptotic limit is

$$
\lim_{t\to\infty} u(t,x) = 0, \text{ and the asymptotic profile is } u(t,x) \sim c_1 exp\left(-\frac{\pi^2 t}{L^2}\right)\sin\left(\frac{\pi x}{L}\right). \tag{2.38}
$$

On the other hand, for homogeneous Neumann problem (2.36), the asymptotic limit is

$$
\lim_{t\to\infty} u(t,x) = \frac{c_0}{2}, \text{ and the asymptotic profile is } u(t,x) \sim \frac{c_0}{2} + c_1 exp\left(-\frac{\pi^2 t}{L^2}\right)\cos\left(\frac{\pi x}{L}\right). \tag{2.39}
$$

The constant c_0 above is $(2/L)\displaystyle\int_0^L u_0(x)dx$, thus the limit is

$$\frac{1}{L}\int_0^L u_0(x)dx, \tag{2.40}$$

the average value of the initial value $u_0(x)$. The limits of both cases are also solutions of diffusion equation, and in fact they are solutions which are independent of time t. Such solutions are **equilibrium solutions**. For a general reaction-diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u + f(x,u), & t>0,\ x\in\Omega, \\ u(0,x) = u_0(x), & x\in\Omega, \\ B(u) = 0, & t>0,\ x\in\partial\Omega, \end{cases} \tag{2.41}$$

where $B(u)$ is the boundary condition, the equilibrium solutions satisfy

$$\begin{cases} d\Delta u + f(x,u) = 0, & x\in\Omega, \\ B(u) = 0, & x\in\partial\Omega. \end{cases} \tag{2.42}$$

A solution $u_1(x)$ of (2.42) is a solution of (2.41) in the sense that the function $U_1(t,x) = u_1(x)$ is a solution of (2.41) with $u(0,x) = u_1(x)$. It is not just a coincidence that for two examples we have considered so far, the limit is always an equilibrium solution. We will show in next section that the limit of a diffusion equation is usually an equilibrium solution.

Equilibrium solutions of diffusion equation are easier to find, and that will help us understand the limit of solution when the solution itself is not so easy to find. The equilibrium equation for (2.13)-(2.15) is

$$u'' = 0,\ \ u(0) = u(L) = 0. \tag{2.43}$$

The general solution for $u'' = 0$ is $u(x) = ax + b$, and from the boundary conditions, we obtain $u(x) = 0$ is the only equilibrium solution. The equilibrium equation for (2.36) is

$$u'' = 0,\ \ u'(0) = u'(L) = 0. \tag{2.44}$$

For this problem $u(x) = b$ for any $b \in \mathbf{R}$ is an equilibrium solution. However the limit in (2.36) is the average value of the initial value $u_0(x)$ only. Why does the solution of diffusion equation selects this particular equilibrium solution, but not the others? We will answer this question in next section.

2.4 Relaxation to equilibrium solutions

We have obtained analytic solutions of 1-d diffusion equation in previous sections. The solutions are in a form of infinite series. However in general, even such formulas cannot be obtained when the spatial dimension is higher and the domain is arbitrary. (Later we will formulas of solutions of diffusion equations for some special cases.) Here we use some integral formulas to show some qualitative properties of solutions of general diffusion equation.

As a calculus warmup, we consider the diffusion equation with no-flux boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\Delta u, \quad t > 0,\, x \in \Omega, \\ u(0, x) = u_0(x), \quad x \in \Omega, \\ \nabla u(t, x) \cdot \mathbf{n}(x) = 0, \quad t > 0,\, x \in \partial\Omega. \end{cases} \tag{2.45}$$

From the biological modelling point of view, the species only randomly moves in Ω, the growth rate is zero, and the boundary is "sealed". Thus the total population should not change since no one moves in or out, and no one is born or dead. Indeed that is the basic character of the equation, and it also explains why the limit of the solution of (2.36) is the average value of the initial value function.

Proposition 2.5. *Suppose that $u(t, x)$ is the solution of* (2.45)*, then*

$$\int_\Omega u(t, x)dx = \int_\Omega u_0(x)dx. \tag{2.46}$$

Proof. Integrating the left hand side of the equation in (2.45) over Ω, we obtain

$$\int_\Omega \frac{\partial u}{\partial t}(t, x)dx = \frac{d}{dt}\left(\int_\Omega u(t, x)dx\right), \tag{2.47}$$

and integrating the right hand side, we have

$$D\int_\Omega \Delta u(t, x)dx = D\int_{\partial\Omega} \nabla u(t, x) \cdot \mathbf{n}(x)ds = 0, \tag{2.48}$$

from the divergent theorem and the no-flux boundary condition. Therefore from (2.47) and (2.48), we find that

$$\frac{d}{dt}\left(\int_\Omega u(t, x)dx\right) = 0, \tag{2.49}$$

which implies (2.46). □

Next we show that the solution $u(t, x)$ always tends to a limit constant c, and because of (2.46), that constant must be

$$\frac{1}{|\Omega|}\int_\Omega u_0(x)dx. \tag{2.50}$$

A rigorous proof of this fact needs many more advanced mathematical knowledge, so we only sketch the ideas here. We multiply the equation in (2.45) by the function $u(t, x)$,

$$u\frac{\partial u}{\partial t} = Du\Delta u. \tag{2.51}$$

Similar to the proof of Proposition 2.5, we integrate the left hand side of (2.51):

$$\int_\Omega u\frac{\partial u}{\partial t}dx = \int_\Omega \frac{\partial}{\partial t}\left(\frac{1}{2}u^2\right)dx = \frac{d}{dt}\int_\Omega \left(\frac{1}{2}u^2\right)dx = \frac{1}{2}\frac{d}{dt}\int_\Omega [u(t, x)]^2dx, \tag{2.52}$$

and the right hand side:

$$D\int_\Omega u\Delta u dx = D\int_{\partial\Omega} u(\nabla u\cdot n)ds - D\int_\Omega \nabla u\cdot\nabla u dx. \tag{2.53}$$

Here we use Green's identity (a consequence of divergence theorem):

$$\int_\Omega u\Delta v dx = \int_{\partial\Omega} u(\nabla v\cdot n)ds - \int_\Omega \nabla u\cdot\nabla v dx. \tag{2.54}$$

From the boundary condition, $\int_{\partial\Omega} u(\nabla u\cdot n)ds = 0$, thus from (2.52) and (2.53), we find

$$\frac{1}{2}\frac{d}{dt}\int_\Omega [u(t,x)]^2 dx = -D\int_\Omega |\nabla u(t,x)|^2 dx. \tag{2.55}$$

Let $E(t) = \displaystyle\int_\Omega [u(t,x)]^2 dx$. Then (2.55) shows that $E'(t) < 0$ for all $t > 0$ and obviously $E(t) > 0$ for all $t > 0$. Thus $\lim_{t\to\infty} E(t) \geq 0$ exists, and as $t\to\infty$, $E'(t)\to 0$, so

$$\lim_{t\to\infty}\int_\Omega |\nabla u(t,x)|^2 dx = 0. \tag{2.56}$$

(2.56) implies $\lim_{t\to\infty}|\nabla u(t,x)| = 0$ for any $x\in\Omega$, thus the limit of $u(t,x)$ when $t\to\infty$ must be a function with zero gradient, thus the limit function must be a constant $u(x) = c$. Since $c|\Omega| = \displaystyle\int_\Omega u(x)dx = \lim_{t\to\infty}\int_\Omega u(t,x)dx = \int_\Omega u_0(x)dx$, then $c = \int_\Omega u_0(x)dx/|\Omega|$. (Notice that we implicitly assume $u(t,x)$ has a limit $u(x)$ when $t\to\infty$. While such a limit indeed exists, the proof of its existence is beyond the scope of this notes, that is why this is only a sketch of proof.) Nevertheless, we put our findings into another proposition:

Proposition 2.6. *Suppose that $u(t,x)$ is the solution of* (2.45)*, then $u(t,x)$ tends to a constant function $u(x) = \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u_0(x)dx$.*

Similar result can also be shown for Dirichlet problem, in which the limit function is $u = 0$.

From this property of diffusion equation, we can get a conclusion that the diffusion equation will make the initial value flatter and bring it eventually to an equilibrium state without spatial pattern. This can also be indicated from the differential equation itself. Let the average of the initial value be c. For the points where $u(t,x) > c$, the function is likely to be convex down, with $\Delta u < 0$, thus the function value will decrease over these points; and for the points where $u(t,x) < c$, the opposite occurs. This is called **smothering effect** of diffusion. In the following we use a Maple program to demonstrate the smothering effect and the asymptotic tendency to the equilibrium. We consider

$$\begin{cases} u_t = 4u_{xx}, & t>0,\quad x\in(0,4),\\ u(t,0) = u(t,4) = 0, & \\ u(0,x) = f(x), & \end{cases} \tag{2.57}$$

where $f(x)$ is given by the formula $f(x) = \begin{cases} 100, & x \in [1,3], \\ 0, & x \in [0,1) \bigcup (3,4]. \end{cases}$. From (2.32), we get the formula of the solution of (2.57):

$$
\begin{aligned}
&u(t,x) = \sum_{m=1}^{\infty} c_m exp\left(-\frac{m^2\pi^2}{4}t\right)\sin\left(\frac{m\pi x}{4}\right), \\
&\text{where } c_m = \frac{1}{2}\int_0^4 f(x)\sin\left(\frac{m\pi x}{4}\right)dx, m \geq 1, \quad t > 0, \ x \in (0,4).
\end{aligned}
\tag{2.58}
$$

The coefficients c_m in the Fourier series can easily be calculated, but we will let Maple to do that. Here is the Maple program `diffusion.mws` and some of the results:

```
> restart;
> with(plots): n:='n':
> f:=x->100*Heaviside(x-1)-100*Heaviside(x-3);
> plot(f(x), x=0..4);
> c:=n->0.5*int(f(x)*sin(n*Pi*x/4),x=0..4): c(n);
```

$$200.0\frac{\text{cos(1/4 n Pi) - cos(3/4 n Pi)}}{\text{n Pi}}$$

```
>uapprox1:=(x,t)->sum(c(n)*sin(n*Pi*x/4)*exp(-4*n^2*t), n=1..30);
```

$$\text{uapprox1 :=(x,t) - >} \sum_{n=1}^{30} \text{c(n) sin(1/4 n Pi x) exp(-4 n}^2\text{ t)}$$

```
> tvals:=seq(0.07/20*i,i=0..20):
> toplot:=[seq(uapprox1(x,t),t=tvals)]:
> plot(toplot,x=0..4);
> densityplot(uapprox1(x,t),x=0..4,t=0..0.2,colorstyle=HUE);
> animate(uapprox1(x,t), x=0..4, t=0..0.1,frames=50);
```

This is an animation generated by approximating the solution by the partial sum with $n = 30$ of the series solution. The approximation is accurate since the remainder is an exponential decaying function with smaller exponent. Note that the initial condition $f(x)$ can be written as the difference of two Heaviside functions, which is defined as

$$H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases} \tag{2.59}$$

It is an interesting experiment to observe the changes of the spatial patterns of $u(t,x)$ when t gets forward. Here are a few snap shots of the animation:

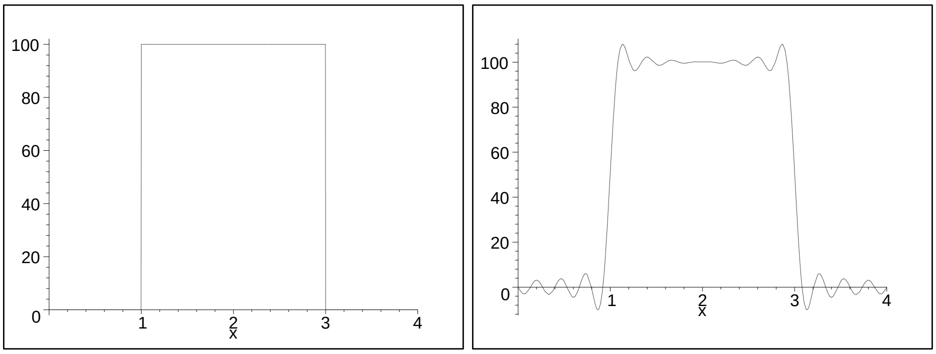

Figure 2.1: **(a)** $f(x)$; **(b)** partial sum of Fourier series of $f(x)$

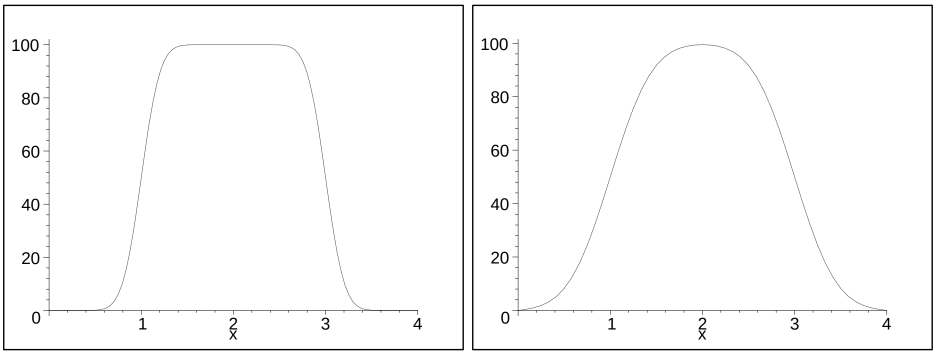

Figure 2.2: **(a)** $u(0.002, x)$; **(b)** $u(0.010, x)$

Because of the truncation of the series solution, $u(t, x)$ above indeed is the solution of (2.57) with initial data

$$\widetilde{f}(x) = \sum_{m=1}^{30} c_m \sin\left(\frac{m\pi x}{4}\right) dx, \quad c_m = \frac{200}{m\pi}\left[\cos\left(\frac{m\pi}{4}\right) - \cos\left(\frac{m\pi}{4}\right)\right]. \tag{2.60}$$

Figure 2.1 is a comparison of f and $\widetilde{f}$. From the last section, we know that when $t \to \infty$, $u(t, x) \to 0$ exponentially fast. However the road to distinction is worth a more careful looking—it can be dismantled into the following stages:

1. **Forming a plateau** From $t = 0$ (Figure 2.1(b)) to $t = 0.002$ (Figure 2.2 (a)), we can observe that the initial data has been smoothed by the diffusion equation, and all the small wiggles in $u(0, x)$ are gone at $t = 0.002$. On the other hand, although $u(0, x)$ is not a constant for $x \in (1, 3)$, $u(0.002, x)$ is almost a constant on an interval slightly smaller than $(1, 3)$. So in the initial stage of the evolution of $\widetilde{f}$ under diffusion, a plateau (or flat core) of height 100 forms, with a sharp but smooth drop-off to 0 near $x = 1$ and $x = 3$.

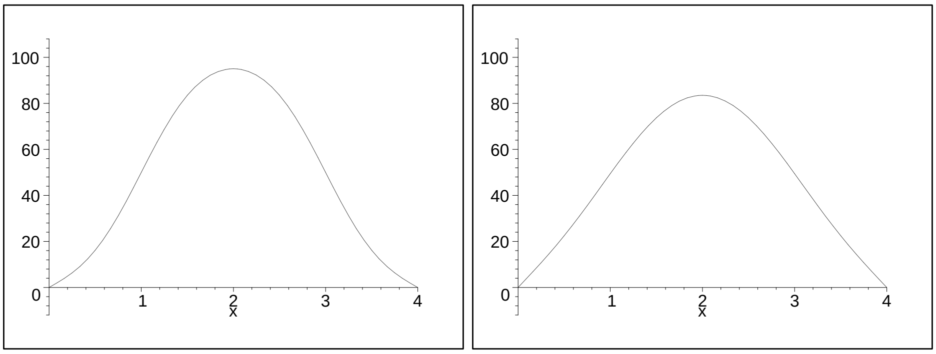

Figure 2.3: **(a)** $u(0.020, x)$; **(b)** $u(0.040, x)$

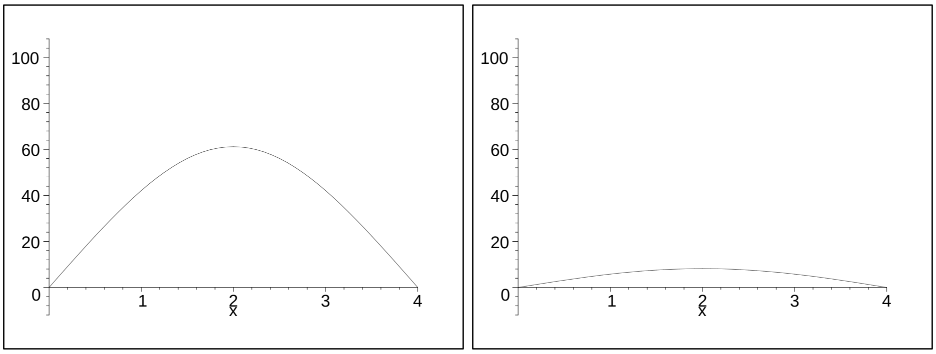

Figure 2.4: **(a)** $u(0.100, x)$; **(b)** $u(0.600, x)$

2. **Erosion of the plateau** From $t = 0.002$ (Figure 2.2 (a)) to $t = 0.01$ (Figure 2.2 (b)), the flat core keeps shrinking though the top of the core is still at the level near 100. At $t = 0.01$, the erosion finally dissolves the whole flat top, and the maximum point of $u(t, \cdot)$ at $x = 2$ looks more like a regular local maximum point. The graph of $u(0.01, x)$ is close to a bell-shape. In that time interval, the inflection points of $u(t, \cdot)$ are still near $x = 1$ and $x = 3$, but at $t = 0.01$, the interface between $u = 0$ and $u = 100$ is no longer as sharp as when $t = 0.002$.

3. **Bell getting round** In the next stage, smoothing effect brings the concave part of the graph down, and the convex part of the graph up. And at the same time, the convex part is shrinking as population loss via boundary gets bigger. At $t = 0.04$, the graph becomes almost concave for all x, and the bell-shape becomes an arch. (see Figure 2.3)

4. **Collapse of the arch** In the final stage, the arch is close to the leading term of the series: $c_1 exp(-\pi^2 t/4)\sin(\pi x/4)$, and an exponential collapse is obvious. After $t = 1$, the graph can hardly be seen. (see Figure 2.4)

2.5 One dimensional chemical mixing problem

Here we consider a simplified version of the chemical mixing problem which is discussed in Chapter 1: A tube with length L contains salt water. The cross-section (yz-direction) of the tube is so small so we can assume that the concentration of the salt water is same for any point on a cross-section. Let the concentration of the salt water in the tube be $c(t,x)\ kg/m^3$, $t > 0$ and $0 < x < L$. We assume that the diffusion of the salt is one-dimensional in the tube, and the velocity of fluid is ignorable, thus $c(t,x)$ satisfies the diffusion equation:

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}. \tag{2.61}$$

At the left hand side of the tube, a salt water solution with constant concentration $C_0\ kg/m^3$ enters the tube at a rate of $R_0\ m^3/s$, and on the right hand side of the tube, the mixed solution is removed at the rate of $R_0\ m^3/s$. We also assume that the area of the cross section of the tube is A. Let A_1 and A_2 be the cross sections at $x = 0$ and $x = L$ correspondingly. From the Fick's law, the total in-flux at $x = 0$ is

$$\int_{A_1} \mathbf{J}\cdot(-\mathbf{n})ds = \int_{A_1} [-Dc_x(t,0)]\cdot(1)ds = -DAc_x(t,0) = C_0R_0, \tag{2.62}$$

and the total out-flux at $x = L$ is

$$\int_{A_2} \mathbf{J}\cdot\mathbf{n}ds = \int_{A_1} [-Dc_x(t,L)]\cdot(1)ds = -DAc_x(t,0) = C(t,L)R_0, \tag{2.63}$$

thus we obtain the boundary conditions

$$\frac{\partial c(t,0)}{\partial x} = -\frac{C_0R_0}{DA}, \text{ and } \frac{\partial c(t,L)}{\partial x} = -\frac{c(t,L)R_0}{DA}. \tag{2.64}$$

To simplify the notations, we define

$$B = \frac{C_0R_0}{DA}, \text{ and } E = \frac{R_0}{DA}. \tag{2.65}$$

Thus the initial-boundary value problem is

$$\begin{cases} \dfrac{\partial c}{\partial t} = D\dfrac{\partial^2 c}{\partial x^2}, & t > 0,\ \ x \in (0,L), \\ \dfrac{\partial c(t,0)}{\partial x} = -B, & \dfrac{\partial c(t,L)}{\partial x} + Ec(t,L) = 0, \\ c(t,x) = c_0(x), & t > 0,\ \ x \in (0,L). \end{cases} \tag{2.66}$$

Before attempting to solve the equation, we could find the equilibrium solutions, which satisfy equation:

$$\begin{cases} D\dfrac{\partial^2 c}{\partial x^2} = 0, & x \in (0,L), \\ \dfrac{\partial c(0)}{\partial x} = -B, & \dfrac{\partial c(L)}{\partial x} + Ec(L) = 0. \end{cases} \tag{2.67}$$

From a simple calculation, we find the unique equilibrium solution:

$$c_*(x) = \frac{B}{E} + B(L-x) = C_0 + \frac{C_0 R_0}{DA}(L-x), \tag{2.68}$$

which is a linear positive function with negative slope on $[0, L]$, and $c_*(L) = C_0$. To solve the equation, we define $b(t, x) = c(t, x) - c_*(x)$. Then $b(t, x)$ satisfies

$$\begin{cases} \dfrac{\partial b}{\partial t} = D\dfrac{\partial^2 b}{\partial x^2}, \quad t > 0, \ \ x \in (0, L), \\ \dfrac{\partial b(t,0)}{\partial x} = 0, \ \ \dfrac{\partial b(t,L)}{\partial x} + Ec(t, L) = 0, \\ b(t, x) = c_0(x) - c_*(x), \quad t > 0, \ \ x \in (0, L). \end{cases} \tag{2.69}$$

The reason we consider (2.69) instead of (2.66) is that the boundary conditions in (2.69) is homogeneous, thus we can use the technique of separation of variables. We assume that $b(t, x) = U(t)V(x)$. Then

$$\frac{U'(t)}{U(t)} = D\frac{V''(x)}{V(x)} = Dk, \tag{2.70}$$

$U(t) = e^{Dkt}$, and $V(x)$ satisfies

$$V'' - kV = 0, \ \ x \in (0, L), \ \ V'(0) = 0, \ \ V'(L) + EV(L) = 0. \tag{2.71}$$

This is a half-Neumann and half-Robin boundary condition. We use the method in Section 2.2 to solve (2.71).

If $k > 0$, then $V(x) = c_1 e^{-\sqrt{k}x} + c_2 e^{\sqrt{k}x}$. From the boundary conditions, we get

$$det \begin{pmatrix} -\sqrt{k} & \sqrt{k} \\ (E - \sqrt{k})e^{-\sqrt{k}L} & (E + \sqrt{k})e^{\sqrt{k}L} \end{pmatrix} = 0, \tag{2.72}$$

and k must satisfy

$$e^{2\sqrt{k}L} = \frac{\sqrt{k} - E}{\sqrt{k} + E}. \tag{2.73}$$

But the function on the left is greater than 1 when $k > 0$, and the one on the right is always less than 1 when $k > 0$. So (2.73) has no solution, thus there are no positive eigenvalues. Similarly $k = 0$ is not eigenvalue either.

If $k < 0$, $V(x) = c_1 \cos(\sqrt{-k}x) + c_2 \sin(\sqrt{-k}x)$. From the boundary conditions, we get $c_2 = 0$ and $c_1[\cos(\sqrt{-k}L) - \sqrt{-k}\sin(\sqrt{-k}L)] = 0$. Thus k satisfies

$$\cot(\sqrt{-k}L) = \sqrt{-k}. \tag{2.74}$$

From the graph of the functions $f_1(x) = \cot(Lx)$ and $f_2(x) = x$, there are exactly one intersection point x_m of f_1 and f_2 in $((m-1)\pi/L, m\pi/L)$ for any positive integer $m > 0$. Then this $k_m = -x_m^2$ is the m-th eigenvalue of (2.71), and the corresponding eigenfunction is $\cos(\sqrt{-k_m}x)$. Thus the solution of (2.69) is

$$b(t, x) = \sum_{m=1}^{\infty} b_m e^{-Dx_m^2 t}\cos(x_m x), \ \text{ where } x_m \in (\frac{(m-1)\pi}{L}, \frac{m\pi}{L}) \text{ satisfies } \cot(x_m L) = x_m. \tag{2.75}$$

Hence the solution is the sum of the equilibrium solution and an exponential decaying Fourier series, and the solution converges to the equilibrium solution exponentially fast. It shows that in equilibrium state, the exiting solution also has the concentration C_0, same as the incoming solution, thus the amounts of salt entering and exiting the tube are balanced, but the concentration of solution in the tube is higher than C_0, and the concentration at $x = 0$ is the highest.

The example in this section can be thought as a model of a continuously polluted river. The left end point $x = 0$ is the source of the pollution where the pollutant enters the river at a constant rate, and the right end point $x = L$ is where the river merges to a bigger river or the ocean. Our mathematical result shows that the concentration of pollutant in a continuously polluted river is even higher than that in the source. As a consequence, we can see that at the confluence of two polluted rivers, the pollution can be most serious, since the concentration there is much higher than the ones of either branches. We should be cautious that we ignore the effect of drifting here, which could make a big difference for rivers.

2.6 Critical patch size and bifurcations

The simplest population growth model is the Malthus model $dP/dt = aP$ for a positive constant $a > 0$. In this section we consider Dirichlet boundary value problem of diffusive Malthus equation

$$\begin{cases} \dfrac{\partial P}{\partial t} = D\dfrac{\partial^2 P}{\partial x^2} + aP, \ \ t > 0, \ x \in (0, L), \\ u(t, 0) = u(t, L) = 0, \\ u(0, x) = u_0(x), \ \ x \in (0, L). \end{cases} \tag{2.76}$$

We have seen that population are at a loss from the hostile boundary, but on the other hand, the population will increase through linear reproduction. So the loss and gain are in a competition. Let's see what factor will determine the outcome of this battle.

Equation (2.76) can be solved in a similar way as (2.13)-(2.15). In fact, the form of the solution only changes slightly: adding a term e^{at} to reflect the exponential growth. The solution of (2.76) is

$$\begin{aligned} &u(t, x) = \sum_{m=1}^{\infty} c_m exp\left(at - D\frac{m^2\pi^2}{L^2}t\right) \sin\left(\frac{m\pi x}{L}\right), \\ &\text{where } c_m = \frac{2}{L}\int_0^L u_0(x) \sin\left(\frac{m\pi x}{L}\right) dx, m \geq 1, \ \ t > 0, \ x \in (0, L). \end{aligned} \tag{2.77}$$

However the fate of the solution may be different: since at is added to the exponent, now the exponent may be a positive one, then $u(t, x)$ will have a part which is exponentially increasing. In fact, when the index m in the sum increases, the exponent gets more negative, thus whether $u(t, x)$ is exponentially increasing is determined by the first exponent coefficient $a - (D\pi^2/L^2)$. We have

$$\begin{aligned} &\text{when } a > \frac{D\pi^2}{L^2} \text{ or } L > \sqrt{\frac{D}{a}}\pi, \ \ \lim_{t\to\infty} u(t, x) = \infty, \\ &\text{when } a < \frac{D\pi^2}{L^2} \text{ or } L < \sqrt{\frac{D}{a}}\pi, \ \ \lim_{t\to\infty} u(t, x) = 0. \end{aligned} \tag{2.78}$$

If we assume that the growth rate a and D are intrinsic parameters determined by the nature of species and environment, then the size of the habitat L will play the deciding role here. The number $L_0 = \sqrt{D/a}\pi$ is called the *critical patch size*, as the population cannot survive if the habitat is too small (the outgoing flux wins over the growth), but the population will not only survive but thrive to an exponential growth if the habitat is large enough (growth outpaces the emigration).

We say that a bifurcation occurs at $L = L_0$ as the equilibrium solution $u = 0$ changes stability type from stable for $L < L_0$ to unstable for $L > L_0$. We notice that at $L = L_0$, the equation has other equilibrium solutions $u(x) = \sin(\pi x/L)$, the eigenfunction associated with first eigenvalue $k_1 = -\pi^2/L^2$. We will have more discussion on bifurcation problems when the growth rate is nonlinear. From (2.78), we can also use a or D as bifurcation parameter instead of L, sometime that is more convenient since we do not need to change the domain, but we get equivalent results.

The concept of critical patch size is related to habitat fragmentation in ecological studies. Habitat fragmentation[1] is the breaking up of a continuous habitat, ecosystem, or land-use type into smaller fragments, which is considered to be one of several spatial processes in land transformation. It is commonly used in relation to the fragmentation of forests. Habitat fragmentation is mainly caused by human activities such as logging, conversion of forests into agricultural areas and suburbanization, but can also be caused by natural processes such as fire. From our mathematical result in this section, if the the fragmentation of habitat limits the spatial movement of the plants and animals, then the species can become extinct.

2.7 Separation of variables: rectangles

Most of this chapter, we have exclusively considered one dimensional spatial domain, for the simplicity of the mathematical analysis. However any interesting application happens in 2-D or 3-D spaces. In this section, we discuss the separation of variables in a two-dimensional rectangle.

We first consider a linear diffusion reaction equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) + \lambda u, & t > 0,\ (x,y) \in R = (0,a)\times(0,b), \\ \nabla u \cdot \mathbf{n} = 0, & (x,y) \in \partial R, \\ u(0,x,y) = u_0(x,y),\ (x,y) \in R, & \end{cases} \tag{2.79}$$

where $D, \lambda, a, b > 0$, and ∂R is the boundary of rectangle R. From the method of separation of variables, we assume that

$$u(t,x,y) = U(t)V(x,y). \tag{2.80}$$

Then similar to one-dimensional case, we obtain

$$U'(t) = (Dk + \lambda)U(t), \tag{2.81}$$

[1]From TEMS, Terrestrial Ecosystem Monitoring Sites, `http://www.fao.org/gtos/tems/`

and

$$\begin{cases} \dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = kV, & t > 0,\ (x, y) \in R = (0, a) \times (0, b), \\ \nabla V \cdot \mathbf{n} = 0, & (x, y) \in \partial R. \end{cases} \tag{2.82}$$

For (2.82), we need a further separation of variables:

$$V(x, y) = W(x)Z(y), \tag{2.83}$$

Then we obtain that

$$\frac{W''(x)}{W(x)} + \frac{Z''(y)}{Z(y)} = k, \tag{2.84}$$

so both $W''(x)/W(x)$ and $Z''(y)/Z(y)$ must be constants. On the other hand, the Neumann boundary condition implies that $W'(0) = W'(a) = 0$ and $Z'(0) = Z'(b) = 0$. Therefore W and Z satisfy

$$W''(x) = k_1 W(x),\ \ x \in (0, a),\ \ W'(0) = W'(a) = 0, \tag{2.85}$$

$$Z''(y) = k_2 Z(y),\ \ y \in (0, b),\ \ Z'(0) = Z'(b) = 0, \tag{2.86}$$

and

$$k = k_1 + k_2. \tag{2.87}$$

Both (2.85) and (2.86) are familiar eigenvalue problems in one dimensional space which have been studied in Chapter 2. The eigenvalues and eigenfunctions of (2.85) are

$$k_{1n} = -\frac{n^2\pi^2}{a^2},\ \ W_n(x) = \cos\left(\frac{n\pi x}{a}\right),\ \ n = 0, 1, 2, \cdots, \tag{2.88}$$

and the eigenvalues and eigenfunctions of (2.86) are

$$k_{2m} = -\frac{m^2\pi^2}{b^2},\ \ Z_m(y) = \cos\left(\frac{m\pi y}{b}\right),\ \ m = 0, 1, 2, \cdots. \tag{2.89}$$

Therefore the eigenvalues and eigenfunctions of (2.82) are

$$k_{n,m} = -\frac{n^2\pi^2}{a^2} - \frac{m^2\pi^2}{b^2},\ \ V_{n,m}(x, y) = \cos\left(\frac{n\pi x}{a}\right) \cdot \cos\left(\frac{m\pi y}{b}\right), \tag{2.90}$$

where $n, m = 0, 1, 2, \cdots$. Thus we find that the solution of (2.79) is

$$u(t, x, y) = \sum_{n=0,m=0}^{\infty} c_{n,m} e^{(Dk_{n,m}+\lambda)t} \cos\left(\frac{n\pi x}{a}\right) \cdot \cos\left(\frac{m\pi y}{b}\right), \tag{2.91}$$

where $c_{n,m}$ can be determined by the initial conditions.

The spatial patterns of the eigenfunctions for rectangle are much more complex than those of one-dimensional. Here we consider a square $R = (0, \pi) \times (0, \pi)$, then the first five distinctive eigenvalues are (see the contour graph of eigenfunctions below)

$$k_{0,0} = 0,\ \ k_{0,1} = k_{1,0} = -1,\ \ k_{1,1} = -2,\ \ k_{2,0} = k_{0,2} = -4,\ \ k_{2,1} = k_{1,2} = -5, \tag{2.92}$$

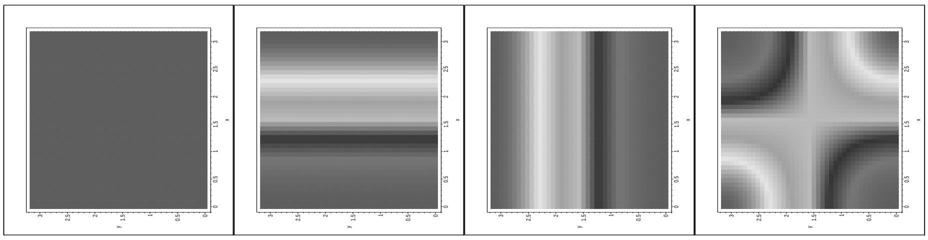

Figure 2.5: (From left to right) (a) $V_{0,0} = 1$; (b) $V_{0,1} = \cos y$; (c) $V_{1,0} = \cos x$; (d) $V_{1,1} = \cos x \cdot \cos y$.

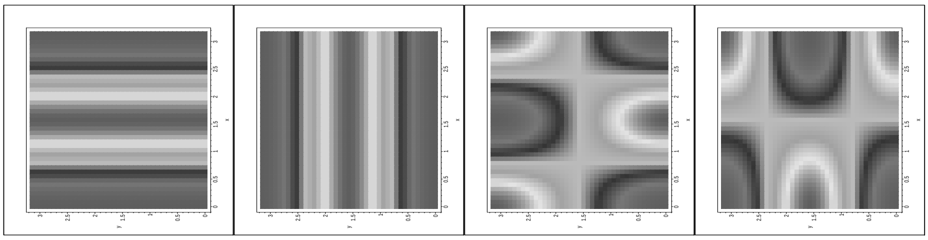

Figure 2.6: (From left to right) (a) $V_{0,2} = \cos(2y)$; (b) $V_{2,0} = \cos(2x)$; (c) $V_{1,2} = \cos x \cdot \cos(2y)$; (d) $V_{2,1} = \cos(2x) \cdot \cos y$.

We notice that many eigenvalues $k_{m,n}$ have multiplicity more than 1, for example $k_{m,n} = k_{n,m} = -(m^2 + n^2)$. In such case, the eigenspace is of higher dimensional, so more possible spatial patterns are generated. For example, $k_{1,0} = k_{0,1} = -1$, and the eigenspace is $span\{\cos x, \cos y\}$, thus $k_1 \cos x + k_2 \cos y$ is a spatial pattern for any k_1 and k_2. In particular, the patterns generated by $\cos x \pm \cos y$ are symmetric with respect to one of the diagonal lines.

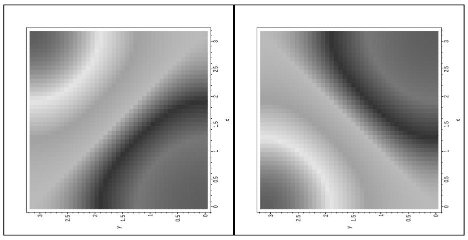

Figure 2.7: (From left to right) (a) $\cos x + \cos y$; (b) $\cos y - \cos x$.

The solution (2.91) is always an exponential growth if $\lambda > 0$. Indeed, from the equation, the no-flux boundary condition prevents the emigration through the boundary, and $\lambda > 0$ implies a positive growth rate, thus the total population in R has an exponential growth. We can consider corresponding Dirichlet boundary problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) + \lambda u, & t > 0,\ (x,y) \in R = (0,a) \times (0,b), \\ u(t,x,y) = 0, & (x,y) \in \partial R, \\ u(0,x,y) = u_0(x,y),\ (x,y) \in R. \end{cases} \tag{2.93}$$

Similar to the method above, the solution is

$$u(t,x,y) = \sum_{n=1,m=1}^{\infty} b_{n,m} e^{(Dk_{n,m}+\lambda)t} \sin\left(\frac{n\pi x}{a}\right) \cdot \sin\left(\frac{m\pi y}{b}\right), \tag{2.94}$$

where $b_{n,m}$ can be determined by the initial conditions, and $k_{n,m}$ is defined in (2.90). Similar to last section, the persistence/extinction of the population can be determined:

$$\begin{aligned} &\text{when } \frac{\lambda}{D\pi^2} > \frac{1}{a^2} + \frac{1}{b^2}, \ \lim_{t\to\infty} u(t,x) = \infty, \\ &\text{when } \frac{\lambda}{D\pi^2} < \frac{1}{a^2} + \frac{1}{b^2}, \ \lim_{t\to\infty} u(t,x) = 0. \end{aligned} \tag{2.95}$$

Chapter 2 Exercises

1. Find the eigenvalues and eigenfunctions of homogeneous Neumann boundary problem

$$y'' = ky, \ \ x \in (0,L), \ \ y'(0) = y'(L) = 0. \tag{2.96}$$

2. Find the eigenvalues and eigenfunctions of periodic boundary problem

$$y'' = ky, \ \ x \in (0,\pi), \ \ y(0) = y(\pi), \ \ y'(0) = y'(\pi). \tag{2.97}$$

3. Find the eigenvalues and eigenfunctions of Robin boundary problem ($b > 0$)

$$y'' = ky, \ \ x \in (0,L), \ \ y'(0) = by(0), \ \ y'(L) = -by(L). \tag{2.98}$$

4. Find the eigenvalues and eigenfunctions of the problem

$$y'' + 2y' + ky = 0, \ \ x \in (0,\pi), \ \ y(0) = y(\pi) = 0. \tag{2.99}$$

5. In Example 2.1, use Maple to find the numerical values of the first three eigenvalues, and plot the graphs of the corresponding eigenfunctions.

6. Show that the solution of

$$\begin{cases} u_t = Du_{xx}, & t > 0, \ \ x \in (0,L), \\ u_x(t,0) = u_x(t,L) = 0, \\ u(0,x) = u_0(x), & x \in (0,L), \end{cases} \tag{2.100}$$

is

$$\begin{aligned} &u(t,x) = \frac{c_0}{2} + \sum_{m=1}^{\infty} c_m exp\left(-D\frac{m^2\pi^2}{L^2}t\right) \cos\left(\frac{m\pi x}{L}\right), \\ &\text{where } c_m = \frac{2}{L}\int_0^L u_0(x)\cos\left(\frac{m\pi x}{L}\right) dx, \ m \geq 0, \ \ t > 0, \ x \in (0,L). \end{aligned} \tag{2.101}$$

7. (a) Find the solution of

$$\begin{cases} u_t = 4u_{xx}, & t > 0, \ \ x \in (0,1), \\ u_x(t,0) = u_x(t,1) = 0, \\ u(0,x) = f(x), \end{cases} \tag{2.102}$$

where $f(x)$ is given by the formula $f(x) = \begin{cases} 1, & x \in [0, 0.5], \\ -1, & x \in (0.5, 1]. \end{cases}$

(b) Modify the `Maple` program in Section 2.4 to simulate the solution of (2.102).

8. Find the solutions of equilibrium equation

$$u'' = 0, \ \ x \in (0,1), \ \ u(0) + 3u'(0) = 5, \ \ u(1) - 5u'(1) = 7. \tag{2.103}$$

9. Find the solutions of equilibrium equation

$$u'' + 3u' = 0, \ \ x \in (0,1), \ \ u(0) = 5, \ \ u'(1) = 4. \tag{2.104}$$

10. Suppose that $u(t,x)$ is the solution of

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\Delta u, & t > 0, \ \ x \in \Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \\ \nabla u(t,x) \cdot \mathbf{n}(x) + a \cdot u(t,x) = 0, & t > 0, \ \ x \in \partial\Omega. \end{cases} \tag{2.105}$$

where $a > 0$. Show that the mean-squared value $I(t) = \displaystyle\int_\Omega [u(t,x)]^2 dx$ is strictly decreasing unless $u(t,x) \equiv 0$.

11. Suppose that $u(t,x)$ is the solution of the convective-diffusion equation: ($D > 0$ and $V > 0$)

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2} - V\dfrac{\partial u}{\partial x}, & x \in (0,\pi), \\ u(0,x) = u_0(x), & x \in (0,\pi), \\ \dfrac{\partial u}{\partial x}(t,0) = \dfrac{\partial u}{\partial x}(t,\pi) = 0. \end{cases} \tag{2.106}$$

Show that the total population $\displaystyle\int_0^1 u(t,x)dx$ is a constant if $u_0(x) = \sin x$ but not a constant if $u_0(x) = \cos(x)$.

12. Suppose that in the chemical mixing problem in Section 2.5, the salt solution drifts to the right with velocity V. Then $c(t,x)$ satisfies the convective-diffusion equation:

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} - V\frac{\partial c}{\partial x}, \tag{2.107}$$

with $D, V > 0$. Suppose that $c(t,x)$ satisfies the same initial and boundary conditions in (2.66). Find the equilibrium solution(s) in this situation, and explain the effect of the drifting to the related pollution problem.

13. Suppose that in the chemical mixing problem in Section 2.5, the salt is consumed in a chemical reaction at a rate k. Then $c(t,x)$ satisfies a linear diffusion equation:

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} - kc, \tag{2.108}$$

with $D, k > 0$. Suppose that $c(t,x)$ satisfies the same initial and boundary conditions in (2.66). Find the equilibrium solution(s) in this situation, and discuss the possible spatial patterns of equilibrium solution.

14. Determine the critical patch for a diffusive Malthus equation with Robin boundary conditions:

$$\begin{cases} \dfrac{\partial P}{\partial t} = D\dfrac{\partial^2 P}{\partial x^2} + aP, \;\; t > 0, \; x \in (0, L), \\ u_x(t,0) = bu(t,0), \; u_x(t,L) = -bu(t,L), \\ u(0,x) = u_0(x), \;\; x \in (0,L). \end{cases} \tag{2.109}$$

15. Consider the doubly periodic boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) + \lambda u, \;\; t > 0, \; (x,y) \in R = (0,a) \times (0,b), \\ u(0,y) = u(a,y), \; u_x(0,y) = u_x(a,y), \\ u(x,0) = u(x,b), \; u_y(x,0) = u_y(x,b), \\ u(0,x,y) = u_0(x,y), \; (x,y) \in R. \end{cases} \tag{2.110}$$

Find the eigenvalues and eigenfunctions of the problem, and the series representation of the solution.

16. Consider a convective-diffusion equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + 4\dfrac{\partial u}{\partial x} + 8u, \;\; t > 0, \; x \in (0, L), \\ u(t,0) = 0, \; u(t,L) = 0, \\ u(0,x) = u_0(x), \;\; x \in (0,L). \end{cases} \tag{2.111}$$

Find the solution of the equation with the following steps:

(a) Use separation of variables method to show that if $u(t,x) = U(t)V(x)$ is a solution, then for some constant k, U and V satisfy

$$U'(t) = kU(t), \;\; V''(x) + 4V'(x) + 8V(x) = kV(x), \;\; V(0) = V(L) = 0.$$

(b) Find the eigenvalues and eigenfunctions of

$$V''(x) + 4V'(x) + 8V(x) = kV(x), \;\; V(0) = V(L) = 0.$$

(Hint: treat the cases of $k > 4$, $k = 4$ and $k < 4$ separately.)

(c) Find the solution of the equation in a series form.
(Hint: $c_n = \dfrac{2}{L}\displaystyle\int_0^L e^{2x} u_0(x) \sin\left(\frac{n\pi x}{L}\right) dx$.)

(d) Determine the critical patch L_0 of the problem.

(e) Describe the population distribution qualitatively when $L > L_0$.

(f) Suppose that the species lives in a river with length L, and the convection is due to the drifting of the river. Explain your mathematical results in this context.