Geometry of Waves

Palpable Machines

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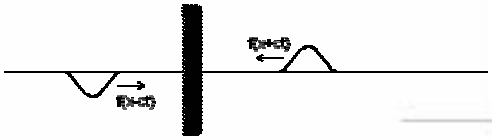
Waveguides Derived

(Smith, Cook, et al, 83+)

- Wave-equation: $\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0$
- General form of solution (d'Alembert):

$$y(x,t) = f_1(x+ct) + f_2(x-ct)$$



Standing and Traveling Waves

- Standing Waves via sums of traveling waves

Left

Both

Right

Standing and Traveling Waves

- Standing Waves via sums of traveling waves

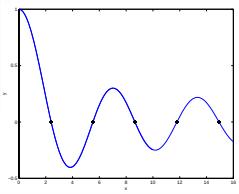
Left

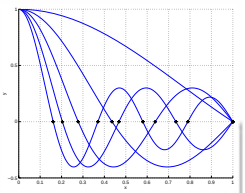
Both

Right

Bessel Function

- $J_0(k|r|) = 0$ are frequencies

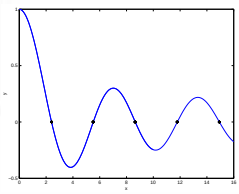


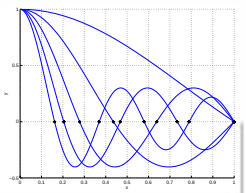


- Where is my analytic formula?

Bessel Function

- $J_0(k|r|) = 0$ are frequencies

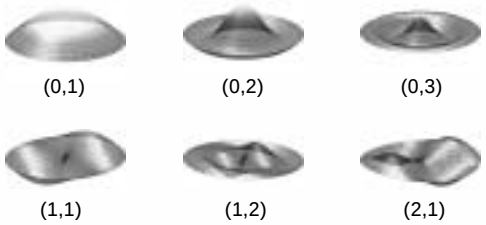




- ~~Where is my analytic formula?~~ ☹

"After 120 years of acoustics, the zeros or frequencies are still not understood." (Zelditch'03)

Isolated Membrane Oscillations



(0,1) (0,2) (0,3)

(1,1) (1,2) (2,1)

(Keller & Rubinov'60)

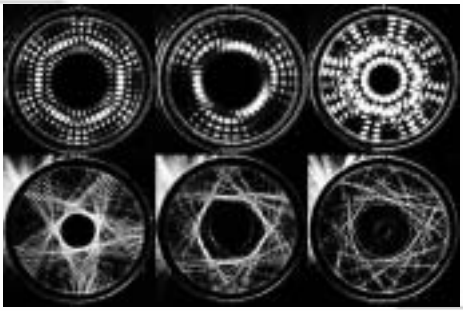
How Big A Mistake?

- Discovered that the error even for low frequencies is small

m	n	error
0	1	2.0%
	2	0.4%
	3	0.2%
	4	0.1%
1	1	1.0%
	2	0.3%
	3	0.1%
	4	0.1%

(m^{th} Bessel, n^{th} zero)

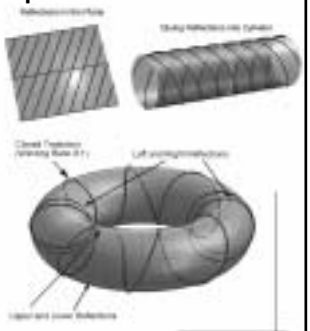
Schlieren Visualization of Liquid Tank



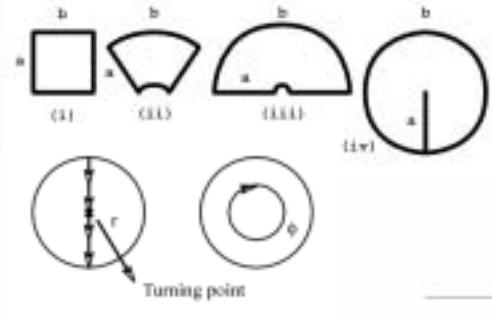
(Chinnery, Humphrey and Beckett'97)

Topology of a Square Drum

- Glue all reflecting edges together
- ~~Discontinuous lines with difficult connectivity book-keeping~~
- Straight lines and immediate connectivity



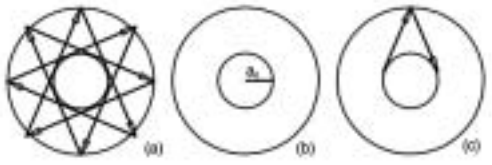
Homeomorphism: Cartesian to Cylindrical Coordinates



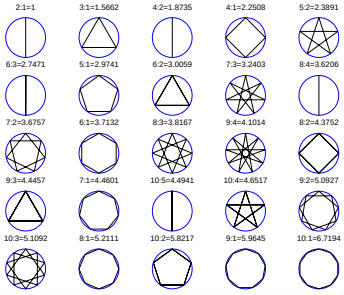
(Keller & Rubinov'60)

Paths in a Circular Room

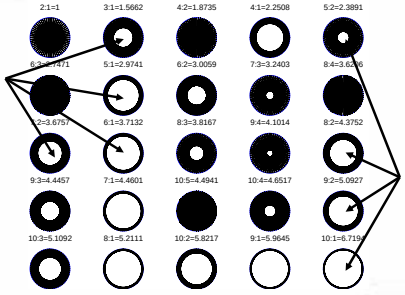
- "Resonance" is a path in a room that returns to its starting point



Resonant Paths



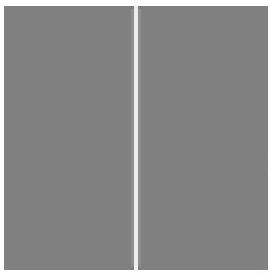
Unreachable Spots in Circular Rooms



(Gbur et al. '02)

Gouy Phase

- Phase shift through focal point
- But not for wave-lengths longer than focal length!
- "Along the axis of symmetry, at points where the intensity is zero, the phase is discontinuous."
- Short wavelengths, it's pi/2. Very large wavelengths it vanishes. Nobody understands the in-between.



(Bergervoet)

What a catastrophe: Caustic

- "Morning Coffee Caustic"
- Cardioid-like shape
- Focal regions of reflected wave rays

(Picture courtesy of Erin Panttaja)

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- Cardioid-like shape
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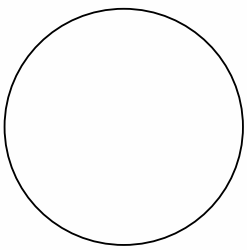
(Picture courtesy of Erin Panttaja)

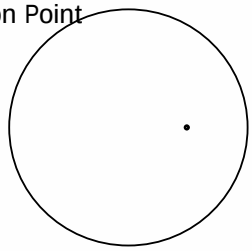
Construction

•Domain



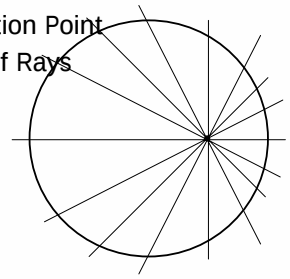
Construction

- Domain
- Interaction Point



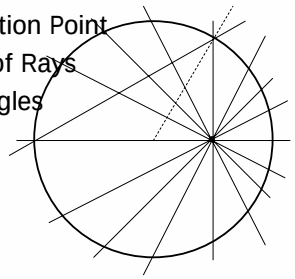
Construction

- Domain
- Interaction Point
- Pencil of Rays



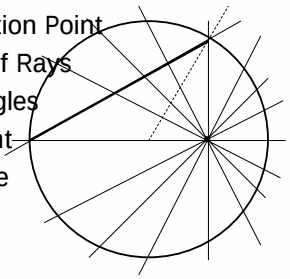
Construction

- Domain
- Interaction Point
- Pencil of Rays
- Ray Angles

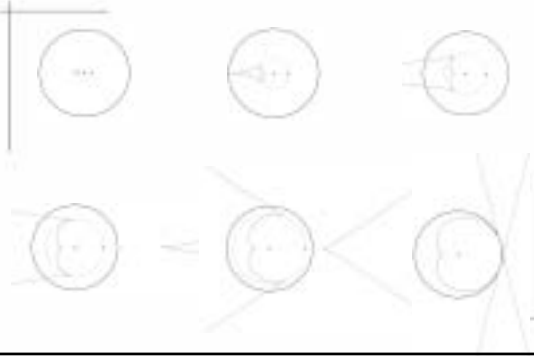


Construction

- Domain
- Interaction Point
- Pencil of Rays
- Ray Angles
- Segment
- Distance



Holditch's Caustic (First Order)

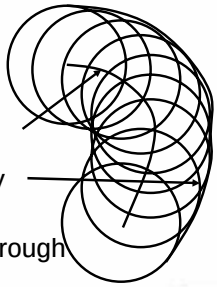


Cusp (Semi-cubical)

Lifted Cusp

Huygens & Malus

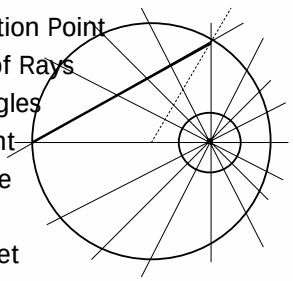
- Final front as sum of circular expansions
- Front normal to ray
- Maintain normal through
- reflections



Construction

- Domain
- Interaction Point
- Pencil of Rays
- Ray Angles
- Segment
- Distance

- Front Set



Construction

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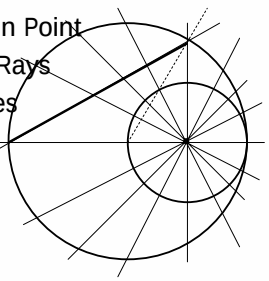
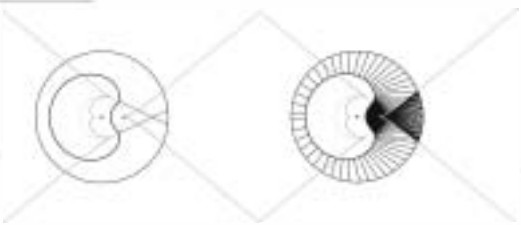


Waves and Rays



Pearcey Diffraction

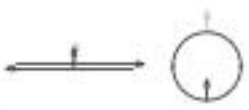
(Berry)

Lineland

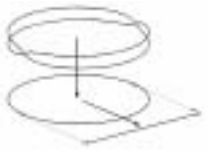
The Topology

Traveling on the Line is a Loop

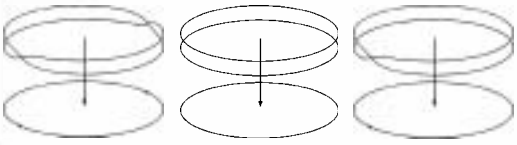
- Lift double line to loop



- Lift reflection states to double-loop



Double Loops indicate Dynamical Properties



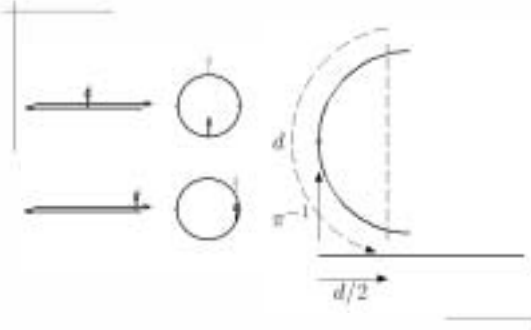
Dirichlet Neumann Mixed

- Integration properties
- Frequency properties

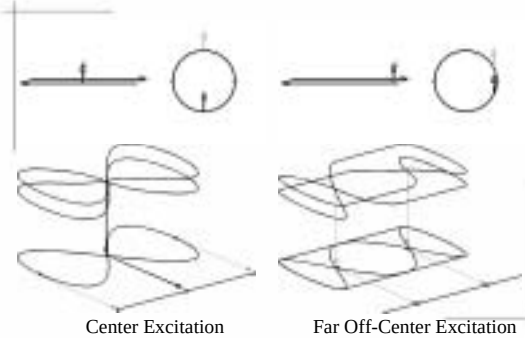
Interactions in Lineland

Interaction Modeling

A Notion of Coincidence



Excitation Points in 1-D



Center Excitation Far Off-Center Excitation

1-D Topology Examples

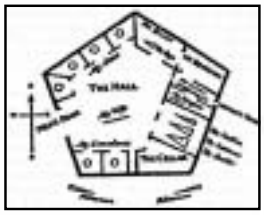
- Strings & Tubes
- Marimba Bars
  - Same topology
  - Different propagation
- Wine Glasses
- Tibetan Singing Bowls
  - "Similar" topology
  - "Same" dynamics
- Circular cylinder = string

Flatland

Some Membranes and Plates are

Nice Rooms in 2-D

(but less nice than in 1-D)



(Abbott'1884)

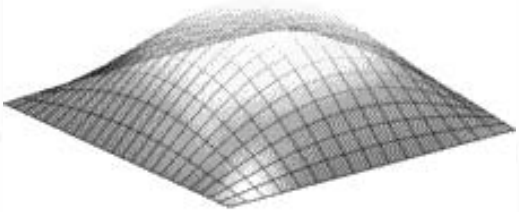
2-D Outline

- Background
  - Numerical Efficiency
    - Why we want to walk on lines
- The topology
  - How we walk on lines in 2-D
  - On Drums, round and square
    - Why 3-D can be nicer than 2-D
    - Why 2-D is tricky
  - How we interact in 2-D

Flatland

Background

Rooms in Flatland can be expensive



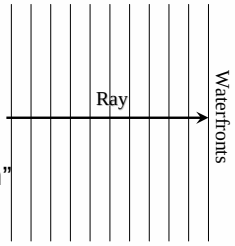
- Mesh Methods: $O(n^2)$

Can we have a Flat room that's cheap?

- The cost of walking on a line is independent of its size!
- Are there ways to walk a flat room via lines?

Rays and Wavefronts

- Short Wavelengths localize well
- Ray Assumption
- “Short-wave Asymptotics”
- “Hamiltonian Room”
- We’ll run with it and see...



Flatland

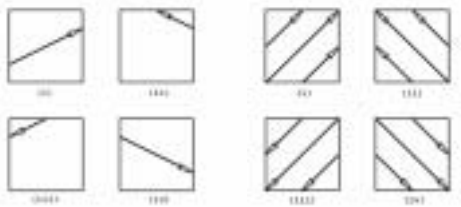
The Topology

Rectangular Flatland

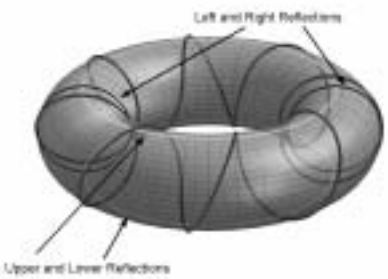
<u>On Square Drums</u>

Covering Space of a Square Drum

Single Ray Ray Family



“Path-connected” orbit on a Torus



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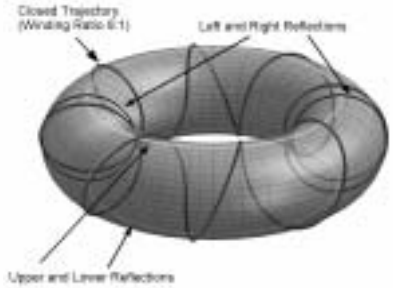


Modes and Orbits on Tori

- Winding around torus for closed orbits: length of orbit
- Index counting the number of “special points” (boundary reflections): phase changes there.
- -> Resonance (Chazarain’74, Duistermaat-Guillemin’75)

Circular Flatland

On Circular Drums

Drum!

- Circular Membrane
- Uniform (or non-uniform) thickness

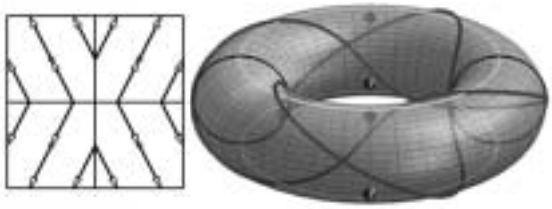
Interacting in Flatland

Interaction Modeling

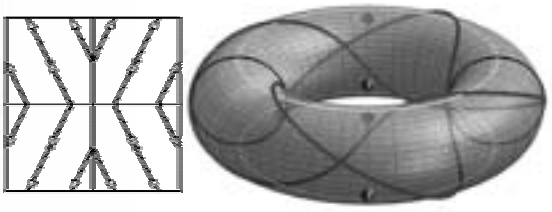
Similar paths on torus

- Torus does not itself provide reference
- Mirror symmetric
- Rotationally symmetric

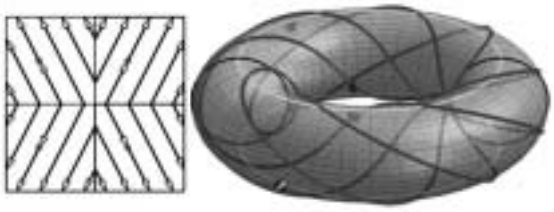
Center Excitation



Center Excitation



Off-Center Excitation (1)



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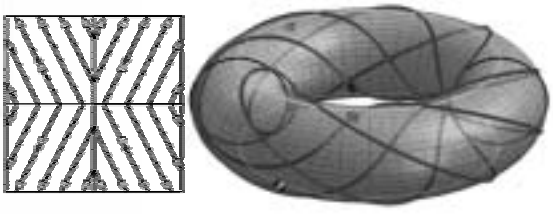
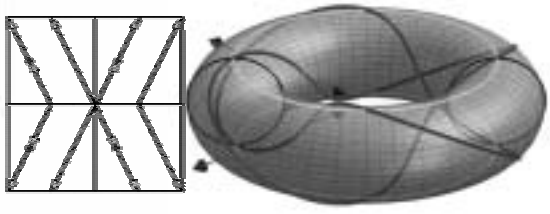
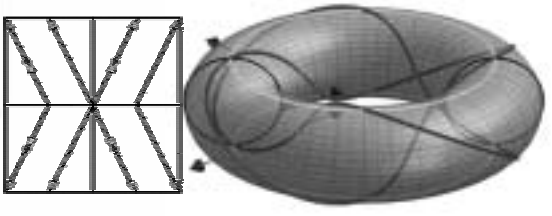


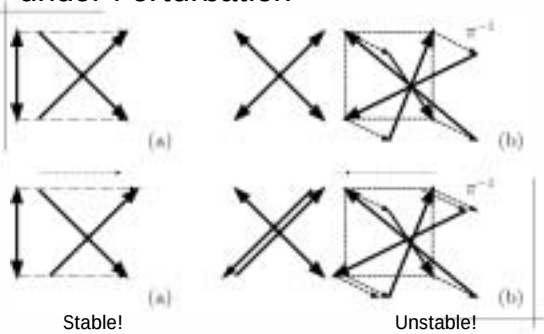
Off-Center Excitation (2)



Off-Center Excitation (2)



Stability of Coincidence Knots under Perturbation



Stable! Unstable!

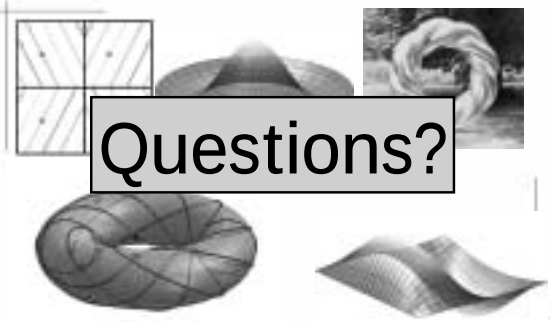
Demo

- Excitation on different points of the Torus

Open Problems

- Relationship of “topological spectrum” and conventional spectrum
- What is the proper treatment of excitations on caustic? (unknown)
- Can asymptotic assumption be removed? (hard)
- Domain shape (guitar, violin top plates, …) (Kac ’66: “Can we hear the shape of a drum?” hard!)

The end (for now)…

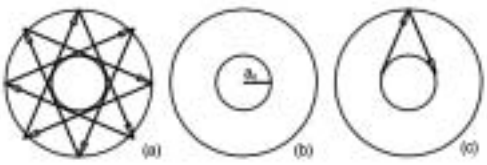


Nice Room in Flatland

> “At last, to complete a series of minor indiscretions, at a meeting of our Local Speculative Society held at the palace of the Prefect himself, - some extremely silly person having read an elaborate paper exhibiting the precise reasons why Providence has limited the number of Dimensions to Two.”
>
> – Edwin A. Abbott,
> Flatland: A Romance of Many Dimensions

Burning Regions in a Circular Room

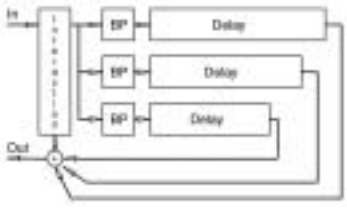
- Caustics is formed by ray family reflecting at one angle



If you know the Resonance, correct your stride

(Essl&Cook’02-04)

- You may have to skip a step to get the Resonance…



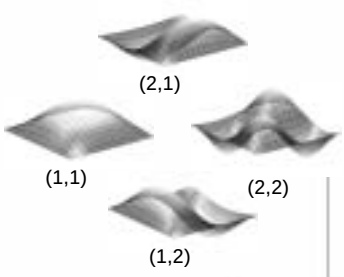
Why bother?

- Actual science to be had
- Faster and more furious
- Talks with pretty pictures
- (And cool new math)

(Robinson)

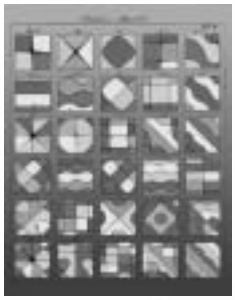
Vibration of a Rectangular Drum

- Frequencies function of the square-root of dimensions
- Shape sinusoidal



Chladni Figures - Square Drum

- Sand on membrane and proper finger placement
- -> Chladni figures



(Chladni, Heller)

What to do?

- We lie, cheat, steal.

What to do?

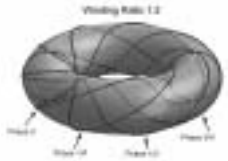
- We ~~lie, cheat,~~ steal.
- Triple-A
  - "Adjust"
  - Approximate
  - Asymptotics
    - Expansions of Bessel function known
    - Expansions of "Short-Wave" Asymptotics known
    - Same!

High Mode Troubles

- High modes dense
- High modes distribution "look random"
- Berry-Tabor conjecture: Use Poisson tail! (Like Schroeder does for room-acoustics)
- Alternative: Low-resolution mesh (Banded Waveguide Mesh, Serafin & Smith '01)

Interaction Points (Covering Space)

- Need reference to boundary (how close to reflection)
- Load at four mirror symmetric points (string we had two)



Toroidal Topologies

Introduction:

- Rectangular Membrane
  - Ray methods & Topology

Challenge:

- Circular Membranes
  - Bessel, Asymptotics and other catastrophes

Application:

- Interaction Modeling

Nice Rooms in Flatland

> "I call our world Flatland,
> not because we call it so,
> but to make its nature clearer to you,
> my happy readers,
> who are privileged to live in Space."
>
> - Edwin A. Abbott,
> Flatland: A Romance of Many Dimensions, 1884

A Rough Route Through Flatland

- What's so cool about studying flat rooms?
- Nice Rooms in 1-D
  - The lines and circles of strings, tubes, bars, shells and bells
- Nice Rooms in 2-D
  - Square, round and on doughnuts (2-D)

What makes a nice room?

- Symmetric
- Simple
- Nice sounding
- Efficiently computable

What's so cool about studying flat rooms?

- Interactive Applications:
  - Music Performance
  - Games
  - Motion Pictures (Especially CG)
- Requirements:
  - Interactive Rates (44100 iterations/second)
  - Commodity Hardware
  - Proper Numerical Behavior
  - Ease of Use for Nonexperts

Publications

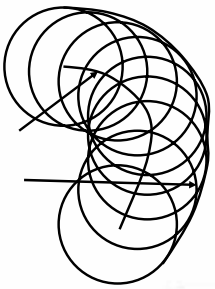
- Banded Waveguides
  - Solid Bars: Essl & Cook'99, Essl & Cool'00
  - Glasses & Bowls: Essl & Cool'02, Serafin et al'02
  - Membranes: Essl'02, Kapur et al'02, Essl et al'04
  - Interactions: Essl'04 submitted

Circular Rooms in Flatland

But…
It's harder to live in a round room in flatland…

Huygens in Flatland

- Initial disturbance
- Final front as sum of circular expansions



(Veselov'02)

What's so hard about it?

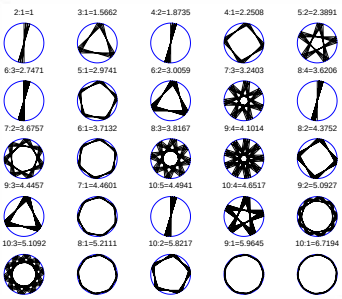
- Huygens' Principle
  - Proven to not hold in 1-D, 2-D (us!) and 2*n-D (all even dimensional rooms)
- Only odd dimensional rooms of 3 or more are generically nice!

(Watson'22)

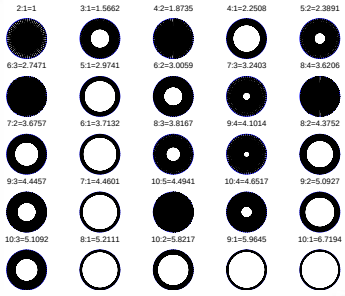
What's so hard about it?

- Restful places in a round room are the zeros of the …
- Bessel function
  - "Special" means rich and difficult in certain situations, unfortunately this is one ☹

Many paths sound the same



Covering of Ray Families



The end (for now)...