Remarks on Sequent Calculus

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1 Abstract

In the last section V of his thesis, after the proof of the Hauptsatz, Gentzen proved the equivalence between the main three types of formalization of the logical inference: the Hilbert-Ackermann system (H.A.), the Natural Deduction Calculus (N.D.), and the Sequent Calculus (S.C.). In this proof we can see, so to say, the birth of the same formalism of S.C., which is maybe the most important formalization of logical deduction ever provided. Also the handwritten version of the thesis, let's say *Ms.ULS*, contains a similar proof of equivalence, as we have learnt from the important researches made by Jan von Plato on the newly found Gentzen's texts. Admittedly, the last section of the thesis is normally rated "less important" than the other sections, but nonetheless it casts some important light on the emergence of the S.C., and more generally on some structural features of Gentzen's work. In the Thesis the equivalence proof proceeds through the following sequence of steps: i) a proof that every derivation within the H.A.-axiomatization can be transformed in an equivalent derivation of N.D.-calculus; ii) a proof that every N.D.-derivation can be transformed into an equivalent S.C.-derivation; iii) a proof that every S.C.-derivation can be transformed into an equivalent H.A.-derivation. The proof is conducted first for Intuitionistic logic and afterward for Classical logic. In this way, of course, the goal to prove the equivalence of all three calculi is accomplished. However, the main single component showing the origin of S.C. is *the translation of derivations built within N.D.-formalism into derivations built within the axiomatic logical calculus of Hilbert and Ackermann's book*. And it is interesting to note that in the pertinent part of *Ms.ULS* Gentzen provided a proof of the equivalence between N.D.-calculi and the H.A.-formalism by showing the possibility to translate every (classical) N.D.-derivation into an equivalent H.A.-derivation; in this way it is explicitly supplied a missing link which is only implicitly present, as a by-product of previous steps i)-iii), in the published version of the thesis. Gentzen proceedes as follows: given an N.D.-proof of, say A, one first lists all those assumptions which are not already discharged before the accomplishing of the inference leading to A. Let us indicate them by Γ. Then one substitutes A by $\Gamma \to A$. If A is an assumption, $A \to A$ takes its place. The steps of inference of N.D. are accordingly translated:

$$\frac{A \qquad B}{A\&B}\,\mathrm{I}\& \qquad \rightsquigarrow \qquad \frac{\Gamma \to A \qquad \Delta \to B}{\Gamma, \Delta \to A\&B}$$

Paired with the occurrence of the figure of *sequent*, here we see, probably for the first time, the disentangling of two meanings often conflated in the notion of implication: the propositional (object-language) connective, say $\supset$, and the (meta-level) notation

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for the formal derivability relation, say $\rightarrow$. Of course, in this step Gentzen was greatly helped by his work on Hertz-systems from the summer of 1931, which output his first published paper of 1932.

Beside trying to retrace the intricates threads leading to the proof of the equivalence, I mean to focus on the emergence of two paradigms in the conception of Cut. The paradigm of *structural reasoning*, which was preserved in the intermediate calculus $\mathbf{LDK}$ of *Ms.ULS* , where the Cut rule continues to play a fundamental role, and the *analytic* paradigm. In the latter paradigm analytic proofs were the new goal, and Gentzen was able to attain it thanks to the *Hauptsatz* proved for that "evolution" of $\mathbf{LDK}$-calculi which is constituted by the $\mathbf{LK}$-calculi. In the latter calculi, structural reasoning was sharply separated from logical meaning, and the general setting was purely inferential.