Opinion dynamics on social networks

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Outline

Problem posing

We want to model, simulate and control an interacting system describing the evolution of individuals' opinions over a social network

The objects of study are

- Individual's opinion, which evolves according to the exchange of informations with other individuals.
- The interaction network, the social system ruling the interactions among individuals (Facebook, Twitter, personal network,. . .)
- Influence by external factors: how advertisement, political policies affect the global opinion?

Modelling opinion dynamics

We propose the following model for opinion formation

$$\frac{d}{dt} w_i = \sum_{j=1}^{N} P_{ij}(t)(w_j - w_i), \qquad i = 1, \dots, N$$

where,

- $w_i \in [-1, +1]$ represents the opinion of the i-agent, where -1 and $+1$ represent two opposite opinions.
- $P_{ij} \geq 0$ is the communication function, quantifying the influence between agent i and j.

The underling process of such model represents a generic way to describe *alignment*, where opinion of agent i aligns toward the opinion of agent j,

$$w_i \to w_j$$

The communication function

In a general setting we assume the communication function $P_{ij}(t)$ to be a *nonlinear model* of the agents' positions, and the *associated graph*, $\mathcal{G}$,

$$P_{ij}(t) = P_{ij}(w(t), \mathcal{G}(t)), \qquad w(t) = \{w_k(t)\}_k$$

- Note that $P_{ij}(t)$ can be seen as weights that naturally induce a direct graph structure on the set of agents.
- Hence we can define the *graph induced* by P_{ij}, for any $\varepsilon \geq 0$ and $t \geq 0$ the graph $\mathcal{P}_\varepsilon(t)$ as

$$\mathcal{P}_\varepsilon = \{(i,j) \in \{1,\ldots,N\}^2 | P_{ij}(t) > \varepsilon\}.$$

- Thus $\mathcal{P}_0(t)$ is the the set of edges (i,j) for which the communication channel from i to j is active at time t.

Unconditional consensus under symmetry

Theorem (S. Motsch- E. Tadmor '14)

Given the opinion model

$$\frac{d}{dt}w = -\mathcal{L}_P(t)w, \qquad w(0) = w^0$$

with *symmetric communication* P. Then the following concentration estimate holds

$$\mathbb{V}_{w(t)} \leq \exp\left(-2\int_0^t \lambda_2(\mathcal{L}_P(s))\, ds\right)\mathbb{V}_{w(0)}.$$

In particular, if the interactions remain "strong enough" so that

$$\lim_{t\to\infty}\int_0^t \lambda_2(\mathcal{L}_P(s))\, ds = +\infty,$$

then there is convergence toward consensus, i.e. $w(t) \to w^\infty = \bar{w}(0)$.

Unconditional consensus under strong connectivity

Let $\mathcal{P}_\varepsilon(t)$ be the graph induced by the *communication function* $P_{ij}(t)$, we recall that

$$\mathcal{P}_\varepsilon = \{(i,j) \in \{1,\dots,N\}^2 | P_{ij}(t) > \varepsilon\}.$$

Theorem (Haskovec, '15)

Given the opinion model

$$\frac{d}{dt} w = -\mathcal{L}_P(t) w, \qquad w(0) = w^0$$

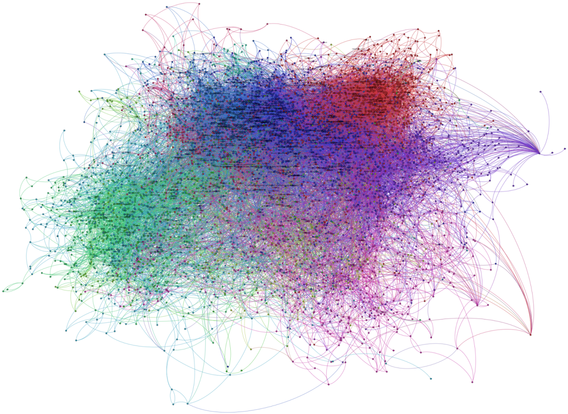
with *associated graph* $\mathcal{P}_\varepsilon(t)$. If there exists an $\varepsilon > 0$ and a *strongly connected graph* $\mathcal{G}$ on the set of agents on which the system spends an infinite amount of time, i.e.

$$\mathcal{T} = \{t \geq 0 | \mathcal{P}_\varepsilon(t) \equiv \mathcal{G}\}, \qquad \ell_1(\mathcal{T}) = +\infty.$$

Then there is convergence toward consensus, i.e. $w(t) \to w^\infty$.

where ℓ_1 is the Lebesgue measure.

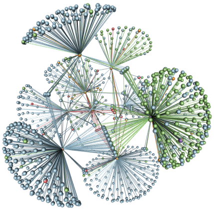
Network evolution



Cross references inside an Enciclopedia

Network evolution

- Analysis of complex network is an extremely prolific research field, which have been continuously growing due to the large variety of applications[1].
- One of the main problems is to understand how to have a general process which restitutes *realistic description* of certain type of networks.
- To this end several classifications have been introduced according to different *network proprieties*.

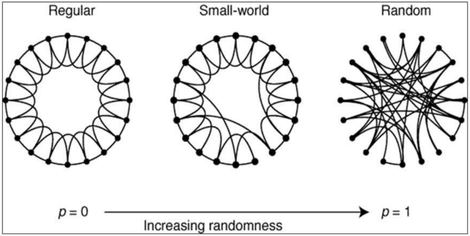


[1] Barabasi, A. Réka '99; Reny,A. Erdös, P., '59

Network evolution

- *Complex network* generated by a random process are usually called random graph.
- A common feature of some social network is the small-world proprieties, where most nodes are not neighbors of one another, but most *nodes can be reached* from every other node by a *small number of steps*.
- In Small-world network, the typical distance L between two nodes[2], follows the rule

$$L = \log(N).$$



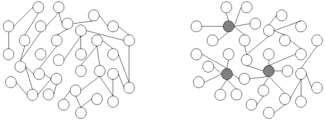
[2] The *distance* between two nodes is defined as $L(V_i, V_j) = \min_{\Gamma_{ij}} |\Gamma_{ij}|$, where $|\Gamma_{ij}|$ is the length of the path Γ_{ij} between the nodes

Network evolution

- Another interesting property arising in several network is the *scale invariance*, these types of network are called scale-free network.
- More precisely, the *fraction of nodes in the network having c connections* to other nodes, $\mathbb{P}(c)$, for large values of c, follows a power-law distribution

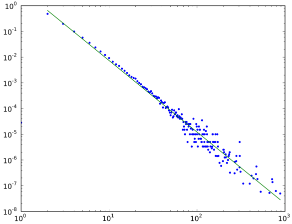
$$\mathbb{P}(c) \sim c^{-\gamma}$$

where $\gamma > 0$ is a positive parameter.



(a) Random network

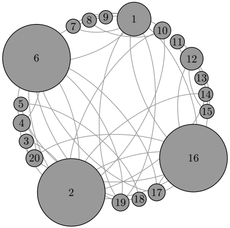
(b) Scale-free network

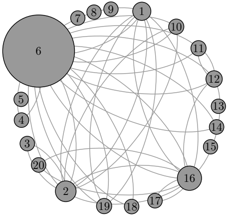


Network evolution: Preferential attachment

We want to model the situation of a network evolving in time, i.e $\mathcal{G}(t)$.

- We will consider a network with a *fixed number* of N vertexes,
- and a fixed number of edges $\mathcal{E}$, *switching over time*.
- We consider a preferential attachment process, for the creation and death of the edges, also known as *richer get richer*.





Network evolution: Preferential attachment

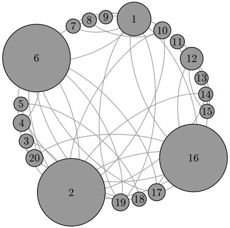
Preferential attachment

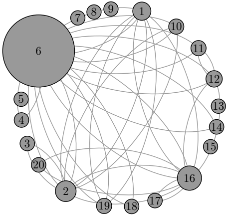
1. Consider a *random initial state* of the (undirect) network $\mathcal{G}^N(0)$
2. *Remove* at each time step an edge, which is chosen randomly
3. At the same time, select a node $i \in \{1, \dots, N\}$ among all the possible nodes of $\mathcal{G}^N$ with probability

   $$\Pi_\alpha(c_i) = \frac{c_i + \alpha}{\sum_{j=1}^N (c_j + \alpha)}, \qquad i = 1, \dots, N,$$

   where $\alpha > 0$ is an attraction coefficient.
4. With the same computed probability Π_α select a second node $j \in \{1, \dots, N\}, j \neq i$ and connect it with the node j.
5. Repeat the process for each time step.

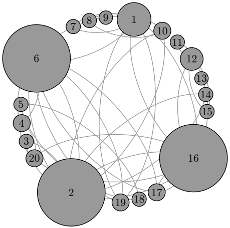
Network evolution: Preferential attachment

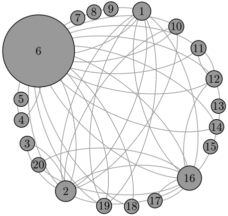




Left: initial configuration of the sample network $\mathcal{G}^{20}$. Right: a simulation of the network $\mathcal{G}^{20}$ after 10 time steps of *preferential attachment process*. The diameter of each node is proportional to its degree of connection.

Network evolution: Preferential attachment





Left: initial configuration of the sample network $\mathcal{G}^{20}$. Right: a simulation of the network $\mathcal{G}^{20}$ after 10 time steps of *preferential attachment process*. The diameter of each node is proportional to its degree of connection.

Preferential attachment: proprieties

- In the case of fix number of edges we have

$$\Pi_\alpha(c_i) = \frac{c_i + \alpha}{\sum_{j=1}^{N}(c_j + \alpha)} = \frac{c_i + \alpha}{2E + N\alpha}$$

  where $|\mathcal{E}| = E$.

- Let $\pi(c, t)$ indicates the *probability of having c connections* at time t, thus we have:
  1. $\sum_c \pi(c, t) = 1$,
  2. $\sum_c c\pi(c, t) = \gamma$

- The process can be written in terms of evolution of a *master equation*

$$\frac{d}{dt}\pi = S_\alpha \pi$$

  where S_α is the *transition matrix*, and for every t $\pi(t) = [\pi(0, t), \pi(1, t), \ldots, \pi(E, t)]$ is a *probability vector* over the states of c.

Preferential attachment: Master equation

Master equation

$$\frac{d}{dt}\pi(c,t) = \frac{D}{E}\left[(c+1)\pi(c+1,t) - c\pi(c,t)\right] + \frac{2D}{2E+N\alpha}\left[(c-1+\alpha)\pi(c-1,t) - (c+\alpha)\pi(c,t)\right].$$

where $D > 0$ determines the relaxation velocity of the network toward an asymptotic degree distribution $\pi_\infty(c)$.
On the right-hand side:

- *the first and the third terms* account the rate of gaining a node of degree c.
- *the second and fourth terms* the rate of losing a node of degree c.

Preferential attachment: Master equation

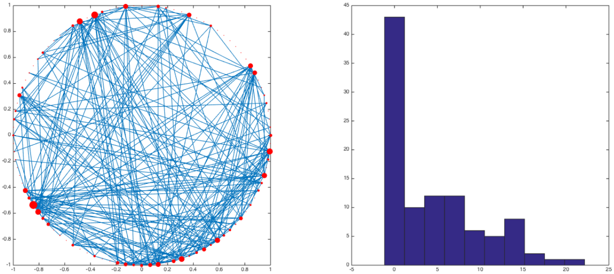
- The main advantage of the graph generated through the *preferential attachment* is the possibility to recover the scale-free properties.
- Indeed, if $\gamma = 2E/N \geq 1$ with attraction coefficient $\alpha \ll 1$ then for $t \to \infty$ $\pi_\infty(c)$ obeys a power-law

$$\pi_\infty(c) = \left(\frac{\alpha}{\gamma}\right)^\alpha \frac{\alpha}{c}.$$

- When $\alpha \gg 1$ we lose this feature, and the selection occurs in a *non preferential way* and the asymptotic degree distribution obeys the Poisson distribution

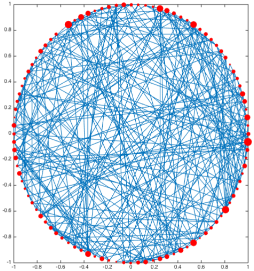
$$\pi_\infty(c) = \frac{e^{-\gamma}}{c!}\gamma^c.$$

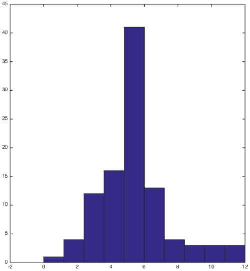
Preferential attachment: simulation



$$N = 100, \quad \gamma = 10, \quad \alpha = 0.01$$

Preferential attachment: simulation





$$N = 100, \quad \gamma = 10, \quad \alpha = 100$$

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