Opinion dynamics on social networks

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Outline

Problem posing

We want to **model, simulate and control** an interacting system describing the evolution of individuals’ **opinions over a social network**

The objects of study are

- **Individual’s opinion**, which evolves according to the exchange of informations with other individuals.
- **The interaction network**, the social system ruling the interactions among individuals (Facebook, Twitter, personal network,...)
- **Influence by external factors**: how advertisement, political policies affect the global opinion?
In several social phenomena, a group of agents faces the problem of coordinating on the basis of mutual communication. The modeling of such scenarios has to comply with nature of protocols and customs governing the interaction among agents, limited or unreliable information transmission, and changing interaction topologies. The mathematical description of emerging collective phenomena and self-organization in system composed of large numbers of agents has gained increasing interest in various fields in biology, robotics and control theory, as well as sociology and economics.

Emergent behaviors in complex systems

The whole is greater than the sum of its parts
- Aristotle
**Examples** in socio-economy, biology and robotics are given by forcing *animals/humans/robots* to follow a specific path or to reach a desired zone...

... but also influencing *consumers* towards a given good, persuading *voters* during political elections, influencing *opinions* over social networks.
Example: Three zones models

Models of interacting agents are based on the *three zones model*\(^1\).

- **Repulsion** when too close to the others individual tends to move from that area.
- **Alignment**, where the individual tries to identify the possible direction of the group and to align with it.
- **Attraction** when individual is too far from the group he wants to get closer.

\(^1\)Aoki, I.,’82; Reynold ’87; A.Huth, C. Wissel,’92
Modelling of crowd dynamics can be applied to design optimal evacuation policies\(^2\)

How do we model interacting agents systems?

\(^2\)Albi, G., Bongini, M., Cristiani, E. Kalise, D. ’15.
The second principle of the dynamics (revisited)

\[ F = m \cdot a \]

- Newton observed the apple falling from the tree (this is actually just folklore!) and he inferred the second law of the dynamics (Principia Mathematica, 1687).

- Actually Galileo already notices even before (ca. 1590) that two falling masses acquire velocity in a simultaneous way (gravity acceleration).

Isaac Newton (1642-1726)

Galileo Galilei (1564-1642)
The second principle of the dynamics (revisited)

G. Galilei, (Il Saggiatore, 1623)

“Philosophy is written in that great book which ever lies before our eyes - I mean the universe - but we cannot understand it if we do not first learn the language and grasp the symbols, in which it is written. This book is written in the mathematical language, and the symbols are triangles, circles and other geometrical figures, without whose help it is impossible to comprehend a single word of it; without which one wanders in vain through a dark labyrinth.”
The (applied) mathematics of the centuries XVI-XX have been mainly addressed to the developments of this method of analysis of the physical phenomena. Most recently, scientists tried to extend these methods to sociology, finance, economy.

The term Econophysics was introduced initially by H. Eugene Stanley, initially to denote the usage of statistical mechanics tools for economic and finance analysis. Nowadays this term is used to identify a large class of studies which have developed new methodology, for the analysis of socio-economical processes.

The focus of our modelling will be systems of $N$ interacting agents whose evolution obeys to a revisited second principle of the dynamics,

$$\frac{d^2}{dt^2} x_i = \sum_{j=1}^{N} \omega_{ij} F(x_i, x_j),$$

and can be seen as a $N$-body problem.
Opinion dynamics on social networks

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Consensus, network & control
Modelling opinion dynamics
The network dynamics
The opinion dynamics

Modelling opinion dynamics & networks

Opinion leader

Individuals in social contact with an opinion leader

Mass Media

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Opinion dynamics on social network

- We consider problems where the collective behavior corresponds to the process of alignment, like in the opinion/consensus formation dynamics\(^3\).

- Interaction network represents also a relevant role, and it is understood as a dynamical process which can highly influence the behavior of the system.

- Different to the classical approach, here we are also interested in such problems in a constrained setting\(^4\).

- Classical examples in socio-economy are given by persuading voters to vote for a specific candidate, by influencing buyers towards a given good or asset.

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\(^3\) R. Hegselmann, U. Krause '02, G. Toscani '06, K. Sznajd-Weron, J. Sznajd '00

\(^4\) M. Caponigro, M. Fornasier, B. Piccoli, E. Trélat '13
Graphical representation of the connections of a Facebook’s profile.
We want to introduce some basic concepts of graph theory in order to describe a social network of \( N \) agents.

- We call graph the ordered pair \( G = \{V, \mathcal{E}\} \), defined by:
  - the set of vertices \( V = \{V_1, V_2, \ldots, V_N\} \),
  - the set of edges \( \mathcal{E} \subseteq V \times V \).

- We say that two edges \( V_i \) and \( V_j \) are directly connected in \( G \) if the edge \( e_{ij} = (V_i, V_j) \) is contained in \( \mathcal{E} \).

- The order of \( G \) is the number of vertices, i.e. \( |V| = N \).

- The size of \( G \) is the number of edges, i.e. \( 0 \leq |\mathcal{E}| \leq N^2 \).

- We will consider only simple graphs (like the one on the left, and not as the one on the right is not!): loops and multiple edges are not allowed.
In general, a graph $G$ is called a **direct graph**, since the edges have an orientation, e.g. if $e_{ij} \in E$ and $e_{ji} \notin E$ means that there $V_i$ is connected to $V_j$, but the converse is not necessarily true.

$G$ is called an **undirect graph** if there is no orientation on the vertices, or in other terms for every edge $e_{ij} \in E$ also $e_{ji} \in E$.

Left: **Direct graph** (Twitter)  
Right: **Undirect graph** (Facebook)

Source and more details at: [https://en.wikipedia.org/wiki/Graph_theory](https://en.wikipedia.org/wiki/Graph_theory)
Modelling opinions dynamics

- A graph $G$, of order $|V| = N$ and size $|E| = M$, is well characterized by its 
  *adjacency matrix*, $A \in \{0, 1\}^{N \times N}$, which tells who is connected with whom:

  \[
  A_{ij} = \begin{cases} 
  1, & e_{ij} \in E \\
  0, & e_{ij} \notin E
  \end{cases}
  \]

- As an example we report here the *adjacency matrix* of the below *undirect* and *direct graphs*:

  \[
  \begin{bmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  1 & 1 & 0
  \end{bmatrix} \quad \begin{bmatrix}
  0 & 1 & 1 \\
  1 & 0 & 1 \\
  1 & 1 & 0
  \end{bmatrix}
  \]

  - Left: Direct graph
  - Right: Undirect graph

- Note that in the case of *undirect graph* the adjacency matrix is always symmetric.
A weighted graph is a (simple) graph, $G$ with weights assigned to the edges, $\mathcal{E}$.

Hence the adjacency matrix of a weighted graph, is now defined as follows:

$$\Theta_{ij} = \begin{cases} \theta_{ij}, & e_{ij} \in \mathcal{E} \\ 0, & e_{ij} \notin \mathcal{E} \end{cases}$$

Write the weighted adjacency matrix associated to the left graph.
We want to embed the network framework into an opinion formation process.

We do the following modelling choices:

- vertexes $\mathcal{V}$ represent the agents of the system,
- edges $\mathcal{E}$ represent the social network,
- weight $\Theta$ represents the social influence among pair of agents.

Thus we associating to each vertex $V_i$ a value $w_i$ representing the opinion of the agent $i$.

**Drawbacks:**

- This framework is static, whereas the opinion formation is a dynamical process.
- In a social network connections are not fixed and they can change over time.

We introduce a dynamical system representing the evolution of the opinions and network.
Modelling opinion dynamics

A. Landmesser *refused to do the salute* during a nazi rally at the Blohm & Voss shipyard in Hamburg, on June 13, 1936.

[link to the full story]
We propose the following model for opinion formation

\[
\frac{d}{dt} w_i = \sum_{j=1}^{N} P_{ij}(t)(w_j - w_i), \quad i = 1, \ldots, N
\]

where,

- \( w_i \in [-1, +1] \) represents the opinion of the \( i \)-agent, where \(-1\) and \(+1\) represent two opposite opinions.

- \( P_{ij} \geq 0 \) is the communication function, quantifying the influence between agent \( i \) and \( j \).

The underlying process of such model represents a generic way to describe alignment, where opinion of agent \( i \) aligns toward the opinion of agent \( j \),

\[
w_i \rightarrow w_j
\]
Global consensus

Interactions which are sufficiently strong lead to global consensus, in the sense that all initial configurations of agents concentrate around an emerging limit state, the consensus $w^\infty$,

$$w_i(t) \to w^\infty.$$  

- The communication function plays a fundamental role, in order to understand under which conditions of global consensus is obtained.
We can consider the simplified situation where

- $N = 2$
- $P_{12}(t) \equiv 1$ and $P_{21}(t) \equiv 0$ for all $t \geq 0$, *(which is the associated graph?)*

\[
\begin{align*}
\frac{d}{dt} w_1 &= w_2 - w_1, & w_1(0) &= w_0^1 \\
\frac{d}{dt} w_2 &= 0, & w_2(0) &= w_0^2
\end{align*}
\]

In this case the system decouples and the solution reads

\[
\begin{align*}
w_1(t) &= e^{-t}w_1^0 + (1 - e^{-t})w_2^0 \\
w_2(t) &= w_2^0.
\end{align*}
\]

Note that for $t \to \infty$ we have that the opinion of 1 converges towards the opinion of 2,

$w_1(t) \to w_2^0$. 

Modelling opinion dynamics

We consider the simplified situation where

- $N = 2$
- $P_{ij}(t) \equiv 1$ for all $t \geq 0$, \textit{(which is the associated graph?)}

In matrix vector notation we have for $w = [w_1, w_2]^T$

$$\frac{d}{dt} w = Aw, \quad w(0) = \begin{bmatrix} +1 \\ -1 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & +1 \\ +1 & -1 \end{bmatrix}$$

Thus the solution can be computed explicitly as follows

$$w(t) = e^{-2t} w(0)$$

which tells that for $t \to \infty$ $w_1(t), w_2(t)$ converges to the average opinion 0.

Solve the general case where $w_1(0) = w_0^1$ and $w_2(0) = w_0^2$. 

In a general setting we assume the communication function \( P_{ij}(t) \) to be a nonlinear model of the agents’ positions, and the associated graph, \( \mathcal{G} \),

\[
P_{ij}(t) = P_{ij}(w(t), \mathcal{G}(t)), \quad w(t) = \{w_k(t)\}_k
\]

- Note that \( P_{ij}(t) \) can be seen as weights that naturally induce a direct graph structure on the set of agents.
- Hence we can define the graph induced by \( P_{ij} \), for any \( \varepsilon \geq 0 \) and \( t \geq 0 \) the graph \( \mathcal{P}_\varepsilon(t) \) as

\[
\mathcal{P}_\varepsilon = \{(i, j) \in \{1, \ldots, N\}^2 | P_{ij}(t) > \varepsilon\}.
\]

- Thus \( \mathcal{P}_0(t) \) is the set of edges \((i, j)\) for which the communication channel from \( i \) to \( j \) is active at time \( t \).
The communication function

In what follows, we will assume that interactions $P_{ij}(t) = P_{ij}(w(t), G(t))$ are ruled by two \textit{main mechanisms}:

- **Network based** interactions, where $P_{ij}$ is the \textit{adjacency matrix} of an associated graph, $G$, describing the set of \textit{direct connections} among agents,

$$P_{ij}(t) = P_{ij}(G(t))$$

- **Metric based** interactions, where the communication among agents $P_{ij}$ is a function of the \textit{relative distance}, $d_{ij} = |w_i - w_j|$.

$$P_{ij}(|w_i - w_j|)$$
We considered $N = 30$ agents with different adjacency matrix $P_{ij}$.
As an example we can consider *Bounded Confidence* type of interactions,

\[ P_{ij} = \frac{1}{N} \chi(|w_i - w_j| \leq C), \]

where agents interact only within a *confidence level* \( C \).

Left: \( C = 0.25 \), Center: \( C = 0.45 \) Right: \( C = 0.65 \)
Bibliography


