# Opinion dynamics on social networks

**Giacomo Albi**

www.giacomoalbi.com

*Department of Computer Science,*
*University of Verona, Italy*

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# Outline

## Problem posing

We want to model, simulate and control an interacting system describing the evolution of individuals' opinions over a social network

The objects of study are

- Individual's opinion, which evolves according to the exchange of informations with other individuals.
- The interaction network, the social system ruling the interactions among individuals (Facebook, Twitter, personal network,. . . )
- Influence by external factors: how advertisement, political policies affect the global opinion?

# Modelling opinion dynamics

We propose the following model for opinion formation

$$\frac{d}{dt} w_i = \sum_{j=1}^{N} P_{ij}(t)(w_j - w_i), \qquad i = 1, \dots, N$$

where,

- $w_i \in [-1, +1]$ represents the opinion of the $i$-agent, where $-1$ and $+1$ represent two opposite opinions.
- $P_{ij} \geq 0$ is the communication function, quantifying the influence between agent $i$ and $j$.

The underling process of such model represents a generic way to describe *alignment*, where opinion of agent $i$ aligns toward the opinion of agent $j$,

$$w_i \to w_j$$

# The communication function

In a general setting we assume the communication function $P_{ij}(t)$ to be a *nonlinear model* of the agents' positions, and the *associated graph*, $\mathcal{G}$,

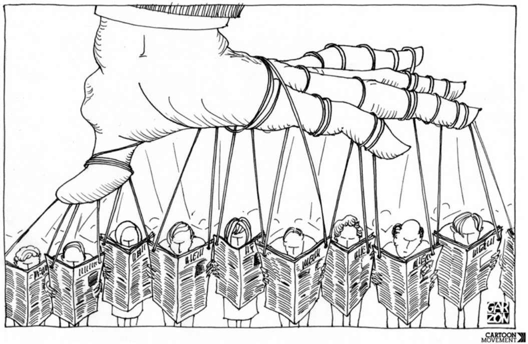
$$P_{ij}(t) = P_{ij}(w(t), \mathcal{G}(t)), \qquad w(t) = \{w_k(t)\}_k$$

- Note that $P_{ij}(t)$ can be seen as weights that naturally induce a direct graph structure on the set of agents.
- Hence we can define the *graph induced* by $P_{ij}$, for any $\varepsilon \geq 0$ and $t \geq 0$ the graph $\mathcal{P}_\varepsilon(t)$ as

$$\mathcal{P}_\varepsilon = \{(i,j) \in \{1, \ldots, N\}^2 | P_{ij}(t) > \varepsilon\}.$$

- Thus $\mathcal{P}_0(t)$ is the the set of edges $(i,j)$ for which the communication channel from $i$ to $j$ is active at time $t$.

# Control of the opinion dynamics



# Control of the opinion dynamics



We consider the evolution of the network $\mathcal{G}(t)$ and the opinion dynamics, and we introduce the following *optimal control problem*

$$\min_u J(w,u) := \frac{1}{2}\int_{t_0}^{t_f}\left[\frac{1}{N}\sum_{j=1}^N |w_j(s)-w_d|^2 + \nu|u(s)|^2\right]\,ds,$$

subject to

$$\frac{d}{dt}w_i = \frac{1}{N}\sum_{j=1}^N P_{ij}(w(t),\mathcal{G}(t))(w_j-w_i) + uQ_i(w(t),\mathcal{G}(t)),$$

$$w_i(0) = w_i^0,$$

- $\nu > 0$ is a *regularization parameter* expressing the strength of the control in the overall dynamics,
- $w_d \in [-1,1]$ is the *target opinion*.
- $Q(t)$ is a *selective function*, indicating the aimed action of the control.

*Remark:* We explicit the normalizing term $1/N$ outside $P_{ij}(t)$.

# Optimization problem: general setting

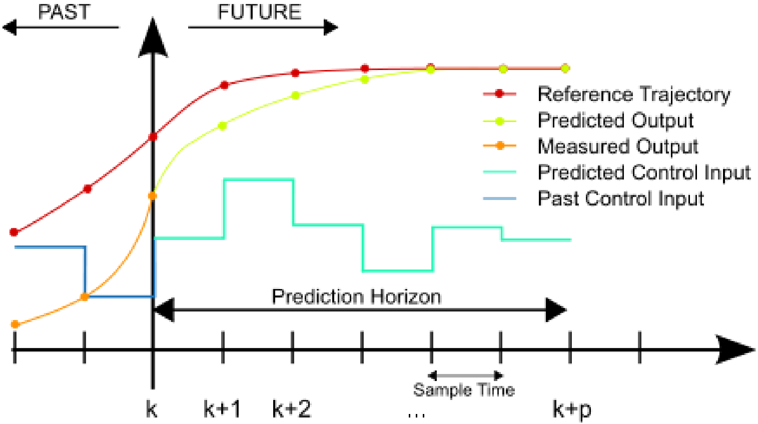
$$
\begin{aligned}
&\min_u J(w,u)\\
&\text{s.t.}\quad \frac{d}{dt}w_i = F_i(w,u), \qquad i=1,\dots,N.
\end{aligned}
$$

- *Standard solution* via Pontryagin's maximum principle, dynamic programming.[1].
- The associated adjoint variable has to be solved *backwards in time* over the full time interval $[t_0, t_f]$.
- The main *drawback* is that for large values of $N$ the computational effort is prohibitive.
- A standard methodology, when dealing with such complex system, is based on *model predictive control*.

[1] E. D. Sontag, '98, M. Krstic, I. Kanellakopoulos, and P. Kokotovic, '95

# Model Predictive Control (MPC)





# Model Predictive Control (MPC)



The problem is approximated on $[t_0, t_f]$ as follows:

1. Split $[t_0, t_f]$ in $M$ time intervals of length $\Delta t$ and let $t^n = \Delta t\ n$. Assume the control piecewise constant on each time interval.
2. Compute the sequence of controls $\bar{u} = (u^n, \dots, u^{n+p-1})$ on the *predictive horizon* interval $[t^n, t^{n+p}]$ by solving for a state $\bar{w}_i$

$$\dot{w}_i = \frac{1}{N}\sum_{j=1}^{N} P_{ij}(w_j - w_i) + u, \qquad w_i(t^n) = \bar{w}_i,$$

$$\bar{u} = \operatorname{argmin} \int_{t^n}^{t^{n+p}} \frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{2}|w_j - w_d|^2 + \frac{\nu}{2}|u|^2\right) ds$$

3. Having the control $u^n$ on the interval $[t^n, t^{n+1}]$, evolve $w_i$ as

$$\frac{d}{dt}w_i = \frac{1}{N}\sum_{j=1}^{N} P_{ij}(w_j - w_i) + u^n$$

   to obtain the new state $\bar{w}_i = w_i(t^{n+1})$.
4. Repeat this procedure until we reach $n\Delta t = t_f$.

# Instantaneous control (IC)



Let consider the *MPC strategy* on the single time interval $[t^n, t^{n+1}]$ with *first oder semi-implicit discretization*,

$$w_i^{n+1} = w^n + \frac{\Delta t}{N} \sum_{j=1}^{N} P_{ij}^n (w_j^n - w_i^n) + \Delta t u^n Q_i^n,$$

$$u^n = \operatorname{argmin} J(u^n) := \frac{1}{N} \sum_{j=1}^{N} \frac{1}{2} |w_j^{n+1} - w_d|^2 + \frac{\nu}{2} |u^n|^2$$

- *minimizing* $J(u^n)$ w.r.t. $u^n$[2] the above coupled system we have

$$u^n = \frac{\Delta t}{N\nu + \Delta t^2 \sum_j (Q_j^n)^2} \sum_{j=1}^{N} \left( w_d - w_j^n \right) Q_j^n + O(\Delta t^2/\nu)$$

- This value of the control is called *instantaneous control*.

[2] find $u$ such that $\nabla_u J(u) = 0$.

# Instantaneous control (IC)

- *Scaling* $\nu$ as follows $\nu = \kappa \Delta t$ we obtain

$$u^n = \frac{1}{N\kappa + \Delta t \sum_j \left(Q_j^n\right)^2} \sum_{j=1}^{N} \left(w_d - w_j^n\right) Q_j^n + O(\Delta t/\kappa)$$

- Thus, inserting back $u^n$ into the *controlled opinion dynamics* and let $\Delta t$ go to 0, we have

$$\frac{d}{dt} w_i = \frac{1}{N} \sum_{j=1}^{N} P_{ij}(t)(w_j - w_i) + \frac{1}{\kappa N} \sum_{j=1}^{N} (w_d - w_j) Q_j(t) Q_i(t),$$

- Note that for *uniform selective function* $Q_j(t) \equiv 1$ we have

$$\frac{d}{dt} w_i = \frac{1}{N} \sum_{j=1}^{N} P_{ij}(t)(w_j - w_i) + \frac{1}{\kappa}(w_d - \bar{w})$$

which show that we are controlling the *average opinion* of the system.

# Numerical Test: Setting



- We consider a population of $N = 100$ agents/nodes representing an undirected evolving graph with $\gamma = 30$.
- A *preferential attachment* mechanism has been considered with $\alpha = 0.01$.
- The *communication function* has been chosen as

$$P_{ij}(w(t); \mathcal{G}(t)) = H(w_i, w_j)K(c_i, c_j),$$

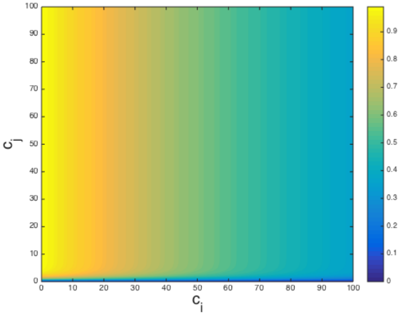
where

$$K(c_i, c_j) = e^{-\lambda c_i}(1 - e^{-\beta c_j}) \qquad H(w_i, w_j) = \chi(|w_i - w_j| \leq C),$$

- In what follows we chose $\lambda = 1/100, \beta = 1, C = 0.4$.
- For the *time discretization* we assumed $\Delta t = 5 \times 10^{-2}$, with final time $t_f = 50$.

# Numerical Test: Connectivity influence

- The function $K(\cdot,\cdot)$ measures the influence with respect to the *connectivity level*
- It accounts that agents with a *large number of connections* are more difficult to influence and, at the same time, they have more influence over other agents.



# Numerical Test: Control term

- For the control we account the following penalization parameter $\kappa = 0.1$.
- We want to promote the desired opinion $w_d = 0.8$.
- We consider a *selective function* as follows

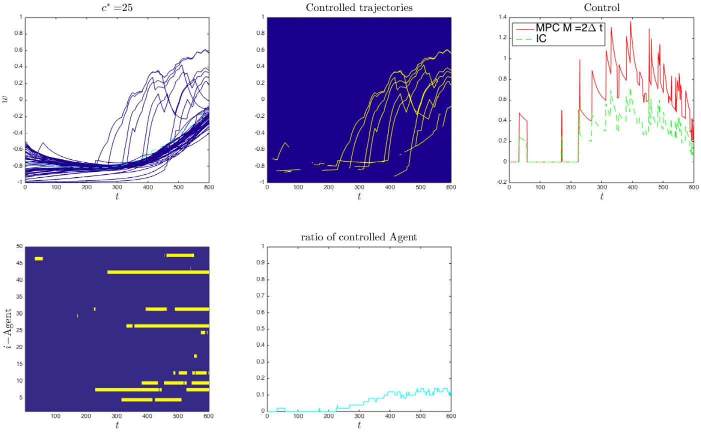
$$Q_i(t) = \chi(c_i(t) > c_*)$$

  where $c_*$ is a fixed connectivity level.
- In this way we want to model a selective action of the control, aiming to the *more connected agents*.

A different choice is to consider as *selective action* $Q(\cdot)$ a function which concentrates on a certain percentage of the agents that are the farthest from the *desired opinion* $w_d$.

# Numerical examples: $c^* = 25$





# Numerical examples: $c^* = 10$

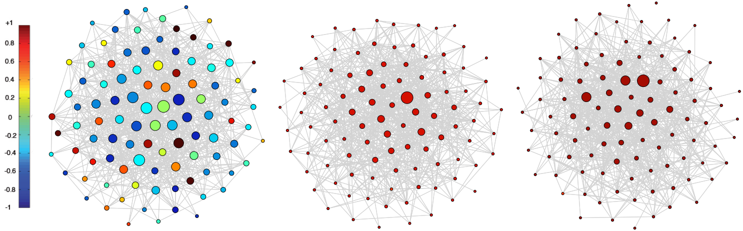



Figure : Evolution of opinion and connection degree of $\mathcal{G}^{100}$ at times $t = 0, 25, 50$ and control parameter $\kappa = 0.1$.

# Numerical examples: $c^* = 20$

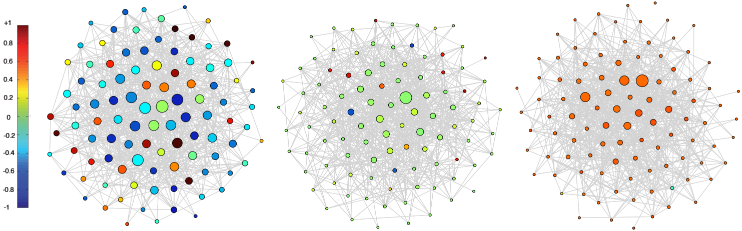



Figure : Evolution of opinion and connection degree of $\mathcal{G}^{100}$ at times $t = 0, 25, 50$ and control parameter $\kappa = 0.1$.

# Numerical examples: $c^* = 30$

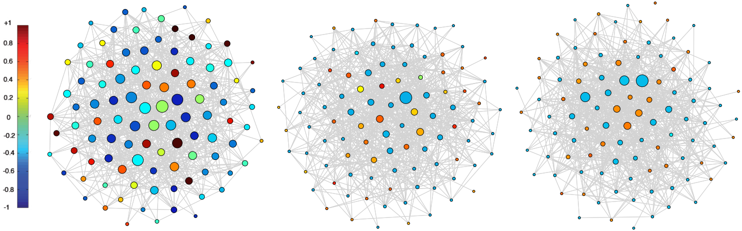
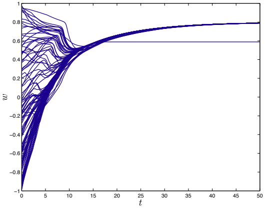


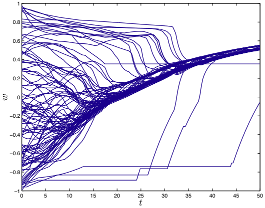

Figure : Evolution of opinion and connection degree of $\mathcal{G}^{100}$ at times $t = 0, 25, 50$ and control parameter $\kappa = 0.1$.

# Numerical examples: evolution constrained opinions

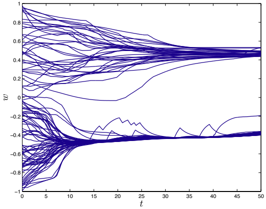





(a) $c^* = 10$

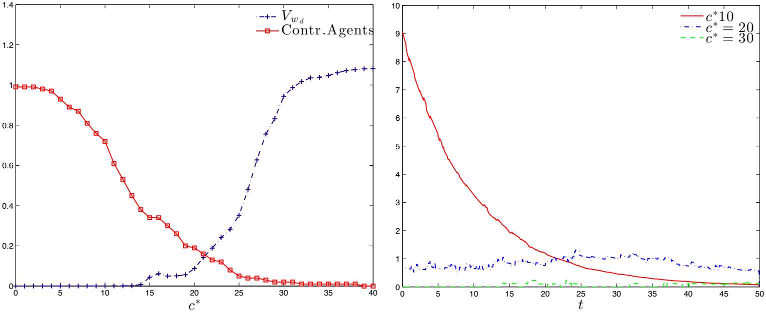



(b) $c^* = 20$




(c) $c^* = 30$

# Numerical examples: remarks




- *Left*: the red squared plot indicates the size of the set of controlled agent at the final time $t_f$ in dependence on $c^*$ whereas the blue line indicates the measure of consensus $V_{w_d} = \dfrac{1}{N-1} \sum_{i=1}^{N} (w_i(t_f) - w_d)^2$.
- *Right*: values of the control $u$ at each time step for $c^* = 10, 20, 30$.

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