What is continuous optimization?

Classical optimization:

\[
\min_{x \in \mathbb{R}^n} f(x)
\]

\[
\text{s.t. } \quad g_i(x) \leq 0, \quad h_j(x) = 0
\]

being discrete optimization.

\(x^*\) allows to apply results like compactness.

\(x^* \in \mathbb{R}^n\) if cont. and existence of

on projection and separation arguments:

\[
x^* = \arg\min_{x \in \mathbb{R}^n} f(x)
\]

\[
\nabla f(x^*) = \nabla g_i(x^*) \cdot A = 0,
\]

Here \(g_i(x) = 0\), \(i = 1, \ldots, m\) and \(A = \partial g_i\nabla.

\(A\)

\(x^*\) is a local minimum:

\[
f(x^* + y) \geq f(x^*)
\]

\[
y = 0
\]

\[
\nabla f(x^*) \cdot y = 0
\]

\[
\nabla g_i(x^*) \cdot y = 0
\]
For any we obtain \( \delta(x) \) so that

\[ \nabla f(x) ^T \delta(x) = 0. \]

Thus, \( y(x) = \delta(x) ^T \) is orthogonal to \( y(x) \) and we have \( \partial y(x) \). In finite dimensions, we have

\[ (A^T A)^{-1} = \text{range}(A^+) \]

for all \( x \) in range \( A \in \mathbb{R}^{n \times m} \).

Thus, \( y(x) \in \text{range}(\nabla h(x)) \), so \( y(x) = y(x) \) for some \( x \in \mathbb{R}^n \). It is unique since

\[ 0 = \nabla h(x) (\alpha - \alpha) = 0 \]

for \( \alpha \in \mathbb{R}^n \). Since \( \text{range}(\nabla h(x)) = \mathbb{R}^m \),

Neither compactness nor the relation for operators nor differentiability is given in the case where \( X \) is infinite-dimensional.

Examples of \( \mu(x) \) where \( X \) are finite-dimensional are control of objects, process shape optimization.

Typical \( \mu(x) \) for those cases are the following...
\[
\begin{align*}
\text{Solution:} \\
\end{align*}
\]
The first approach is the two-point method. Introduce the function: \( f(x) = x^2 + h(x)^2 \).

Thus, \( f''(x) = 2x + 2h(x)h'(x) \), with \( h(x) = x - x^2 \).

Then, \( x^2 \) is a local minimum. The saddle point is

\[ (x^*, x^*) : L(x^2, x^2) = 0 \]

Consequently, \( (x^*, x^*) = L(x^2, x^2) \), yielding

\[ \text{Consider the following first point. A possible}
\]

\[ \text{lagrange function is}
\]

\[ L(x, \lambda) = 4(1-x^2) + \lambda \left[ x - (x^2, x^2) \right] \]

\[ \text{for } L : \mathbb{R}^2 \rightarrow \mathbb{R}, \text{ differentiation in } \mathbb{R}^2 \text{-space.}
\]

\[ \text{Consider}
\]

\[ \psi(y) = \int_0^1 y(x(t)) \, dt
\]

Thus,

\[ \frac{d}{dx} \left[ \phi(x) \right] = \lim_{h \to 0} \frac{\psi(x + h) - \psi(x)}{h}
\]

\[ \text{is the Cockeian}
\]

\[ \text{derivative of } \phi(x) \text{ at } x \text{ if } x \text{ is a fixed point.} \]
Let's consider a linear and bounded operator, then it is Fredholm, all.

\[ \int_0^1 \left[ \frac{d}{dx} g(x(s)) \right] ds = \int_0^1 \frac{d}{ds} g(x(s)) ds = \int_0^1 g'(x(s)) \frac{dx}{ds} ds. \]

Note that \( f, g, h \) are all \( \mathcal{C}^1 \).

Since the dual space is the dual to \( \mathcal{C}^0 \), i.e., \( \mathcal{C}^0 \)

the space of integrals. Hence, \( \frac{d}{ds} g \) and \( \frac{d}{dx} f \)

\[ \frac{d}{dx} [f(x)] = \left[ \frac{d}{ds} f(y(s)) ds, u \right]_{u \in \mathcal{C}^0} = \int_0^1 g'(x(s)) h(s) ds. \]

Differentiating the Laplace for we obtain

\[ \frac{\partial}{\partial x} L(x) = \int_0^1 h_x(x - f(x,s)) ds = 0 \quad \forall h_x \]

\[ \Rightarrow x = \xi (x, u) \]

\[ L(x, y) = f(x(t)) - \int_0^1 \lambda x - \lambda f(x, u) ds \]

\[ L(x, y) = f(x(t)) - \int_0^1 \lambda x - \lambda f(x, u) ds + (\partial x)(t) - \lambda x(t) \]

\[ \frac{\partial}{\partial x} \left( \int_0^1 \lambda x - f(x, u) h_x ds \right) \]

\[ \Rightarrow \frac{\partial}{\partial x} \left( \int_0^1 \lambda x - f(x, u) h_x ds \right) = 0 \]
Take now a variation $h_x$ st. $h_x(t) \neq 0$ but

$$h_x = 0 \text{ at } h((0,1))$$

we obtain

$$h_x(k(t)) + x(t) = 0.$$ 

And for the opposite case,

$$-\varphi(s) = \left. \frac{1}{h_x} \right| \begin{array}{l} \psi_x(k(s)) \mid \psi(s) \end{array}.$$

Taking a variation in $u$ yields

$$\theta = k(s) f_u(k(s), u(s)) \psi(s).$$

The system is known as FMP. It consists of a forward and backward eq for $x$ and $u$ as well as an eq for $u$. Numerically, one possibility to solve the eq's are as follows:

1. Pick $u^0$. Solve $x^k = f(x^k, u^k)$, $x(0) = x_0$ for $x^k$. Solve

$$-x^k = x^k \psi_x(x^k, u^k), \quad x^k(s) = -\varphi(y(x^k(s)),$$

Update $u^{k+1} = u^k - \Delta u^k f_u(x^k, u^k)$

This scheme is independent of the normal method for the ODEs.

**Why does it work?** The pb min $y(x(s))$ s.t. $x = f(x, u)$

$$x(0) = x_0$$

is an unconstrained pb for $u$, i.e., for any

given $u$ we can copy $x = x(u)$ through time

constraints.
Therefore, we could also write

$$u_i = \frac{1}{\theta} u_k - \frac{1}{\theta} \frac{\partial^2}{\partial u^2} \mathcal{L}(x(T_i u))$$

and solve this unconstrained by a gradient descent, i.e.,

$$u_k = u_k - \frac{1}{\theta} \frac{\partial}{\partial u} \mathcal{L}(x(T_i u))$$

This requires to compute the gradient of $\mathcal{L}$ with respect to $u_i$.

$$\frac{\partial}{\partial u} \mathcal{L}(x(T_i u)) = \frac{\partial}{\partial u} \delta u_i \mathcal{L}(x(T_i u)) \frac{\partial}{\partial u} \mathcal{L}(x(T_i u))$$

for a variable at time $T_i$.

Now, solving $\delta u_i$ can be done by applying the ODE

$$\frac{\partial x}{\partial u} = \frac{x_0 + \int \frac{\partial}{\partial u} \mathcal{L}(x(T_i u)) \frac{\partial}{\partial u} \mathcal{L}(x(T_i u))}{\mathcal{L}(x(T_i u))} dt$$

we obtain

$$\mathcal{L}(x(T_i u)) = \int \delta u_i \mathcal{L}(x(T_i u)) \frac{\partial}{\partial u} \mathcal{L}(x(T_i u)) dt$$

Consider $x = \mathcal{L}(x(T_i u))$ and $x_i = \frac{\partial}{\partial u} \mathcal{L}(x(T_i u))$. Then

$$\frac{\partial x_i}{\partial u} = \frac{\partial}{\partial u} \mathcal{L}(x(T_i u)) \frac{\partial}{\partial u} \mathcal{L}(x(T_i u))$$

and because

$$\frac{\partial^2 x}{\partial u^2} = -\frac{\partial}{\partial u} \mathcal{L}(x(T_i u)) \frac{\partial}{\partial u} \mathcal{L}(x(T_i u))$$

For $\lambda(T_i u) = \theta \frac{\partial}{\partial u} \mathcal{L}(x(T_i u))$ we therefore get

$$\frac{\partial^2 x}{\partial u^2} = -x_i \frac{\partial}{\partial u} \mathcal{L}(x(T_i u)) \frac{\partial}{\partial u} \mathcal{L}(x(T_i u))$$
This leads to a coupled system of PDEs:

$$-\Delta \psi = u, \quad -\Delta p = \text{const} \cdot \psi$$

Next, we can solve in an iterative way:

$$u \rightarrow \psi \rightarrow p$$

Shape Optimization

Consider the general problem with \( f(t) \) where \( K \) is a suitable class of compact subsets in \( \mathbb{R}^d \) with equal boundary. There are two methods: topological and shape optimization. In the topological one also other \( e \) etc. could be obtained. In a shape variation problem, \( v: \Omega \rightarrow \mathbb{R}^d \) is a velocity field given by \( \nabla \psi = \nabla(v(x)) \) and consider

$$x \in \Omega \quad \text{and} \quad v(x) = \int_0^1 \nabla \psi(x+t \xi) dt.$$ 

Then we can define \( \Delta(t) = f(v(x)) \) and distribute

$$\Delta(\omega, \lambda) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\Omega(t+\epsilon)} \Delta(t) \, dx$$

This is a nonlinear differential equation of the form:

$$\frac{d}{dt} \Delta(t) = \text{nonlinear}.$$
The solution to this typically consists as follows:

\[ u(t, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i \omega t}}{\omega^2 + c^2} \hat{f}(\omega) d\omega, \]

where \( u(t, x) = V(x(t)) \), \( x(0) = x_0 \), and \( c \) is a constant.

Differentiating \( x(t) \) with respect to \( t \), we get:

\[ \frac{dx(t)}{dt} = \frac{dV(x(t))}{dx} \frac{dx(t)}{dt} = V'(x(t)) \frac{dx(t)}{dt}, \]

which simplifies to:

\[ \frac{dx(t)}{dt} = \frac{dV(x(t))}{dx} = \frac{dV(x(t))}{dx(t)}, \]

indicating that we can express \( x(t) \) in terms of \( V(x(t)) \).

We require the duration of \( t = \tau \),

where \( x(t) = x_0 + \int_{0}^{\tau} V(x(t)) dt = \frac{dV(x(t))}{dx} + \frac{dV(x(t))}{dx(t)}, \]

and for the duration of \( M \), we use the discrete expansion:

\[ M = \sum \left( \begin{array}{c}
-\frac{1}{2} & 1 \\
-1 & 2
\end{array} \right) x_k \]

\[ g(x(t)) = V(x(t)) \]

\[ \frac{dV(x(t))}{dx} = \frac{dV(x(t))}{dx(t)}, \]

only if \( K = i_k \Rightarrow \delta = 1 \rightarrow \text{no} \)
\[- \frac{d I(x_0)}{dt} \bigg|_{t=0} = \left( \nabla g(x_0) V(x_0) + g(x_0) \text{div} V(x_0) \right) dx_0 \]

\[- \int_{\Gamma} \nabla g(x) \cdot V(x) \, ds = \int_{\Gamma} \nabla g \cdot V \, n \, ds. \]

\[- \text{Step: always defined only on the boundary action of } V! \]

\[\text{Extension to conducting case} \]

\[\sum_{i=1}^{n} \mathcal{I}_i(x_0) \text{ s.t. } -V \cdot \nabla \mathcal{I}_i = 0 \quad \text{in } \Omega \]

\[\mathcal{I}_i = \begin{cases} a_i & x \in \Omega \\ 0 & x \in \Gamma \end{cases} \]

We achieve by \( e(y, 0) = 0 \) the PDT and use again a perturbation of \( \Omega \) as before. Then, we have that \( y(y) \) is the sol. to \( e(y, 0) \), \( \eta = 0 \), and

\[0 = \frac{\partial e}{\partial y}(y + \eta, 0) y' + \frac{\partial e}{\partial y}(y, 0) y' \]

In the example we have \( e(y, 0) = \int_{\Omega} \frac{a_2}{2} V \nabla y \, \nabla \, y' \, d\Omega \]

\[\text{Let e.g.,} \]

\[\frac{\partial e}{\partial y}(y, 0) y' = \int_{\Omega} a \nabla y \, \nabla y' \, d\Omega + \int_{\Gamma} (g_1 - g_2) \nabla y \, \nabla y' \, d\Gamma \]

\[\frac{\partial e}{\partial n}(y, 0) = \frac{1}{2} \int_{\Gamma} (g_1 - g_2) \nabla y \, n \, ds \]

as before.
Numerical approach

The idea is to parameterize the evolution of the boundary:

\[ \frac{\partial \Phi}{\partial t} + \nabla \cdot \mathbf{V} \Phi = 0 \]

where \( \Phi: \mathbb{R}^d \times [0, T] \rightarrow \mathbb{R}^n \) is a differentiable function, e.g., a smoothed signed distance function \( \Phi(x, t) = \text{dist}(x, \partial \Omega(t)) \in \mathbb{R}^n \).

We choose \( \mathbf{V} = \mathbf{n} \), \( n \in \mathbb{R}^n \), \( n = \frac{\nabla \Phi}{| \nabla \Phi |} \), \( | \nabla \Phi | \neq 0 \),

It now \( \Phi = \Phi(t, \mathbf{n}) \), \( \mathbf{n} \) is fixed, we obtain for \( \Phi \in \mathbb{R}^n \)

\[ \begin{align*}
\frac{\partial \Phi}{\partial t} = 0 &= \mathbf{V} \cdot \nabla \Phi + \mathbf{V} \times \nabla \mathbf{V} \cdot \mathbf{V} = \mathbf{V} \cdot \nabla \Phi + \left( \mathbf{V} \times \frac{\nabla \Phi}{| \nabla \Phi |} \right) \cdot \mathbf{V} \\
\text{the evolution of } \Phi \text{ is governed by a PDE.}
\end{align*} \]

We have seen that usually \( \frac{\partial \Phi}{\partial n} \) is required to evaluate the shape derivative. Here \( n \) is the normal on \( \partial \Omega \). \( \mathbf{n} \cdot \mathbf{V} \cdot \nabla \Phi = 0 \)

Since \( \mathbf{V} \cdot \nabla \Phi \) is orthogonal on its level curve, we get

since \( \mathbf{n} \) points towards \( \partial \Omega \) we have

\[ \mathbf{n} \cdot \mathbf{V} = \frac{\nabla \Phi}{| \nabla \Phi |} \cdot \mathbf{V} = 0 \quad \forall \mathbf{x} \in \partial \Omega \]

Since we are only interested in shrinking \( \partial \Omega \) in normal direction we...
Write $V = V_n \cdot h$ and substitute for $h$.

Base for $i$: 

$$
\theta = 2 + p + \nabla \cdot V_n \cdot n \\
\nu = 2 + p + \frac{\nabla \cdot \nabla \cdot V_n}{\nabla \cdot \nabla} = 2 + p + 1 \frac{x + y}{\nabla \cdot \nabla}
$$

Thus for $V_0 = 0$, we have 

$$
\frac{\partial}{\partial \phi} (V_0 \cdot \phi) = \int \sum g \cdot V_0 \, ds
$$

Add the following method to update the stage $n$:

$$
\sum_{n=1}^{n} \int \phi(x, x_n) = 0 \\
2 \phi + V_0 \cdot \nabla \phi = 0 \quad \text{on } (t_{k+1} - t_k) \text{ }
$$

Thus the method is now how to choose $V_0$. Clearly we want with this condition to have $V_n = \sum \phi_{V_n}$ so 

This is clearly achieved as for $V_n = \phi - g$. 

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