A note on unbounded Metric Temporal Logic over dense time domains

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Abstract: We investigate the consequences of removing the infinitary axiom and rules from a previously defined proof system for a fragment of propositional metric temporal logic over dense time.

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1 Introduction

This short note is a sequel of [BM04]. In [BM04] we introduced a complete Gentzen-style proof system for a fragment MTL_∞ of propositional metric temporal logic over dense time. The fragment allows use of the unbounded temporal operator $\Box_{[m,\infty[}$, where $\Box_{[m,\infty[}\alpha$ roughly means that α continually holds from time m onwards. The proof system extends a similar system presented in [MPT00].

The language of MTL_∞ comprises a *relational part* and a *labelled part* that is built upon *temporal formulas*. Among the relational axioms there is an infinitary one that is crucial for most properties. The system is essentially cut-free and is strong enough to prove formulas expressing a general temporal induction principle.

In addition to the infinitary axiom there are some infinitary rules for relational formulas only (no infinitary rule for temporal formulas is present).

In this note we investigate the consequences of removing the infinitary axiom and rules.

2 Preliminaries

We introduce a finitary proof system to deal with unbounded temporal operators. We call such a system MTL$'$ (see the Appendix). The language of MTL$'$ is the finitary fragment of the language of MTL$_\infty$ (see [BM04] for details), hence infinitary disjunction is not anymore present. Moreover we replace the infinitary axiom 4. in [BM04] by

4$'$. $\forall x(f(0) < x \to \exists y(f(y) = x))$,

and we retain all the other relational axioms.

The semantics, the finitary rules of the sequent calculus and the first order translation of l–formulas are the same as in [BM04] (see the Appendix).

In particular, we deal with structures like

$$\mathbf{A} = \langle A, <^{\mathbf{A}}, f^{\mathbf{A}} : A \to A, 0^{\mathbf{A}}, \sigma : Var \to A, \tau : A \to 2^{At} \rangle,$$

where $\mathbf{A}_F = \langle A, <^{\mathbf{A}}, f^{\mathbf{A}}, 0^{\mathbf{A}} \rangle$ satisfies the relational axioms and σ, τ are arbitrary maps. We say that $\mathbf{A}_F$ is the *frame* of $\mathbf{A}$. We also say that $\mathbf{A}$ is *based* on its frame.

It is straightforward to prove that every provable sequent in MTL$'$ is *valid* (true in all structures).

Remark 2.1. Contrary to what shown for MTL$_\infty$, in MTL$'$ the sequent S_t defined by

$$\vdash t : \alpha \wedge \Box_{[0,\infty[}(\alpha \to \Box_{[0,1]}\alpha) \to \Box_{[0,\infty[}\alpha$$

(representing a possible formalization of an instance of the axiom schema of induction) is not anymore valid. For, let $\lhd$ be the order on $A = (\mathbb{Q}^{\geq 0} \times \{0\}) \cup (\mathbb{Q} \times \{1\})$ defined by

$$(p, i) \lhd (q, j) \;\Leftrightarrow\; i < j \text{ or } (i = j \text{ and } \; p < q)$$

and let $\mathbf{A} = \langle A, \lhd, F, (0, 0), \sigma, \tau \rangle$, where:

a. $F : (q, i) \mapsto (q + 1, i)$, for $i \in \{0, 1\}$.

b. σ is any assignment of values to variables;

c. τ is any mapping from A to 2^{At} such that, for some proposition symbol p and for all $q \in \mathbb{Q}^{\geq 0}$,

$$p \in \tau((q, i)) \;\Leftrightarrow\; i = 0.$$

It is straightforward to check that the relational axioms are true in $\mathbf{A}$, but $\mathbf{A}$ does not satisfy the sequent S_0 given by $\vdash 0 : p \wedge \Box_{[0,\infty[}(p \to \Box_{[0,1]}p) \to \Box_{[0,\infty[}p$.

As a consequence of the previous remark and of the soundness of MTL$'$, we see that no induction rule implying provability of the sequent S_0 defined in Remark 2.1 can be derived in MTL$'$. This implies that MTL$'$ does not obey any reasonable form of induction.

3 Completeness

We state the following:

Theorem 3.1. *(Completeness) For each sequent* $\Gamma \vdash \Delta$ *one of the following holds:*

1. *there exists a proof of* $\Gamma \vdash \Delta$ *in MTL′ whose cuts are r-cuts on relational axioms or l-cuts on atomic l-formulas only;*
2. *there exists a structure* $\mathbf{A}$ *as above such that* $\mathbf{A} \models \gamma$ *and* $\mathbf{A} \not\models \delta$ *for all* $\gamma \in \Gamma$ *and all* $\delta \in \Delta$.

Proof. See the proof of Theorem 3.1 in [BM04]. □

In the sequel we investigate the structure of a countermodel $\mathbf{A} = \langle A, \triangleleft, F, 0, \sigma, \tau \rangle$ obtained as in the proof of the previous Theorem from an unprovable sequent $\Gamma \vdash \Delta$. By definition, $\mathbf{A}$ is a model of all relational axioms. Furthermore, $\mathbf{A}$ is obtained from a first order predicative structure $\mathbf{A}_0$ for a countable language (see the proof of Theorem 3.1 in [BM04]). Hence, by the downward Löwenheim-Skolem theorem of first order predicate logic, we can assume $\mathbf{A}$ being countable. In particular, the frame $\mathbf{A}_F$ of $\mathbf{A}$ is a countable model of the theory of dense linear ordering with least element and no greatest element. By $\aleph_0$–categoricity of such a theory, we can identify A with the nonnegative rationals $\mathbb{Q}^{\geq 0}$ and $\triangleleft$ with the usual order relation $<$ on rational numbers.

There are two possibilities for the function F:

Case 1. (*standard frame*) the sequence $(F^n(0))_{n \in \mathbb{N}}$ is unbounded in $\mathbb{Q}^{\geq 0}$;

Case 2. (*nonstandard frames*) the sequence $(F^n(0))_{n \in \mathbb{N}}$ is bounded in $\mathbb{Q}^{\geq 0}$.

In Case 1., an easy back–and–forth argument shows that $\mathbf{A}_F$ is, up to isomorphism of first order structures, the structure $\langle \mathbb{Q}^{\geq 0}, <, s, 0 \rangle$, with $s : q \mapsto q+1$. Hence the name *standard frame*.

In Case 2., we have that the sequence $(F^n(0))_{n \in \mathbb{N}}$ necessarily converges to an irrational number q. For, if the limit q is rational then, by axiom $4'$, there exists $r \in \mathbb{Q}^{\geq 0}$ such that $F(r) = q$. Hence a contradiction since $r < F^n(0)$ holds for some $n \in \mathbb{N}$ and so $q < F^{n+1}(0)$ (by the axioms on F), contradicting the property of q of being the supremum of the sequence.

Therefore, up to isomorphism of first order structures, we can regard $\mathbf{A}_F$ as $\langle A, \triangleleft, F, (0, 0) \rangle$, where:

a. $A = (\mathbb{Q}^{\geq 0} \times \{0\}) \cup (\mathbb{Q} \times \{1\})$;

b. $\triangleleft$ is the order defined in Remark 2.1;

c. $F : (q, 0) \mapsto (q+1, 0)$ for $q \in \mathbb{Q}^{\geq 0}$ and F restricted to $\mathbb{Q} \times \{1\}$ is a bijective order preserving mapping satisfying the condition $\forall x (x < F(x))$.

Consequently, in Case 2, infinitely many frames are possible (depending on the behavior of F on $\mathbb{Q} \times \{1\}$). Hence the name *nonstandard frames*.

A *labelled sentence* is a labelled formula $t : \alpha$, where t is a closed term.

We say that two structures are *temporally elementarily equivalent* (briefly: *tee*) if they satisfy the same labelled sentences.

We say that two frames F_1 and F_2 are *tee* if:

(1) for each structure $\mathbf{A}_1$ based on F_1 (i.e. for each choice of σ_1 and τ_1 on F_1) there exists $\mathbf{A}_2$ based on F_2 such that $\mathbf{A}_1$ and $\mathbf{A}_2$ are *tee*;

(2) property (1) holds by interchanging F_1 and F_2.

It is easy to see that no nonstandard frame is isomorphic, as a first order structure, to the standard frame. At least we would like two such frames being *tee*. Unfortunately this is not the case: each structure based on the standard frame satisfies the consequent of previously defined sequent S_0, but we have shown in Remark 2.1 that there are structures based on a Case 2 frame satisfying the labelled sentence $0 : \neg(p \wedge \Box_{[0,\infty[}(p \to \Box_{[0,1]}p) \to \Box_{[0,\infty[}p)$.

After the previous discussion, we can reformulate Theorem 3.1 as follows:

Theorem 3.2. (Completeness–strong form) *For each sequent $\Gamma \vdash \Delta$ one of the following holds:*

1. *there exists a proof of $\Gamma \vdash \Delta$ in MTL' whose cuts are r-cuts on relational axioms or l-cuts on atomic l-formulas only;*

2. *there exists a structure $\mathbf{A}$ based on the standard frame or on a nonstandard frame (as in Case 2 above) such that $\mathbf{A} \models \gamma$ and $\mathbf{A} \not\models \delta$ for all $\gamma \in \Gamma$ and all $\delta \in \Delta$.*

Therefore, after removing the infinitary part of MTL_∞, apparently we cannot anymore formulate the completeness theorem relative to just one frame.

Question 1 Are any two nonstandard frames *tee*?

Notice that a positive answer to the question above allows to reduce the nonstandard frames to the frame of the structure $\mathbf{A}$ defined in Remark 2.1 when dealing with *closed sequents* (i.e. those containing labelled sentences) only.

Unfortunately the answer is negative. In order to show it, we make the following:

Remark 3.1. notice that we can describe the nonstandard frames $\mathbf{A}_F$ occurring in Case 2 in a more convenient way: up to isomorphism of first order structures we can regard the universe A of any such frame as a disjoint union of the nonnegative rationals with countably many copies of the rationals. Moreover, up to isomorphism, we can assume that $F :\ q \mapsto q + 1$ on A.

As usual, we use $\diamond$ as an abbreviation for $\neg\Box\neg$.

Let p be a propositional letter and let

$$\begin{array}{rl} \alpha \equiv & p \wedge \Box_{[0,\infty[}(p \to \Box_{[0,1[}p) \\ \beta \equiv & (\diamond_{[0,\infty[}\neg p) \wedge \Box_{[0,\infty[}(\neg p \to \Box_{[0,1[}\neg p) \\ \gamma \equiv & \diamond_{[0,\infty[}\Box_{[0,\infty[}p \end{array}$$

Let δ be the labelled sentence $0 : \alpha \wedge \beta \wedge \gamma$. It is easy to check that δ is false in all models whose frame is made by the nonnegative rationals followed by a single copy of the rationals. On the contrary, δ is true in some models whose frame is made by the nonnegative rationals followed by two (or more) copies of the rationals (for instance by letting p true on the nonnegative rationals and on the second copy of the rationals, false elsewhere).

Notice that all the counterexamples to *tee* provided so far do involve at least one structure where some instance of the induction schema fails. We may wonder what happens if we focus on structures that satisfy all the closed instances of the induction schema.

It is an easy remark that there are structures based on a nonstandard frame that satisfy the induction schema. A straightforward way of proving it goes through the translation of labelled formulas into first order formulas described in [BM04]: let $\mathbf{A}$ be any structure based on the standard frame and let $Th(\mathbf{A})^*$ be the set made of the first order translations of all labelled sentences that are true in $\mathbf{A}$ and of all relational axioms. By a standard compactness argument in first order logic (expand the language with a new constant symbol c and add to $Th(\mathbf{A})^*$ the sentences $f^n(0) < c$, $n \in \mathbb{N}$), and by the downward Löwenheim-Skolem theorem, one gets a first order structure that - regarded as a temporal structure - is *tee* to $\mathbf{A}$ and, up to isomorphism, is based on a nonstandard frame of the form described in Remark 3.1.

Of course, application of compactness theorem does not explicitly provide a structure with the required properties. For such a reason, in what follows we describe a structure based on a nonstandard frame that satisfies the induction schema. In addition to that we want such a structure not being *tee* to any structure based on the standard frame.

For simplicity we identify symbols with their interpretations when no confusion arises. We let

$$\mathbf{M} = \langle (\mathbb{Q}^{\geq 0} \times \{0\}) \cup (\mathbb{Q} \times \{1\}) \cup (\mathbb{Q} \times \{2\}), <, f, (0,0), \sigma, \tau \rangle,$$

where $<$ is defined similarly to $\triangleleft$ in Remark 2.1, $f : (q,i) \mapsto (q+1,i)$ for all $(q,i) \in M$ and, for all propositional letters p,

$$p \in \tau(q,i) \;\Leftrightarrow\; i = 0 \text{ or } (i = 1 \text{ and } q \leq 0).$$

For what follows there is no need to specify σ. We let $A = \mathbb{Q}^{\geq 0}$; $B = \mathbb{Q} \times \{1\}$ and $C = \mathbb{Q} \times \{2\}$, so that $M = A \cup B \cup C$.

Remark 3.2. Let $r \in \mathbb{Q}^{\geq 0}$ and let $\mathbf{M}_r = \langle \{s \in M : s \geq (r,0)\}, <, f, (r,0), \sigma, \tau \rangle$. (We do not bother to use a different name for the restriction of $<$ to M_r. Similarly for the restrictions of σ and τ.)

The map defined by $(q,0) \mapsto (q+r,0)$ on A and as identity elsewhere on M is an *isomorphism* of temporal structures, by this meaning that, for all temporal formulas α,

$$\mathbf{M} \models_{(q,0)} \alpha \;\Leftrightarrow\; \mathbf{M}_r \models_{(q+r,0)} \alpha \text{ for all } q \in \mathbb{Q}^{\geq 0}$$

and

$$\mathbf{M} \models_s \alpha \;\Leftrightarrow\; \mathbf{M}_r \models_s \alpha \text{ for all } s \in B \cup C.$$

The proof of the first equivalence is by easy induction on α. The proof of the second one follows from the the fact that truth of a temporal formula depends only on the future.

Notice also that $\mathbf{M}_r \models_{(q+r,0)} \alpha \Leftrightarrow \mathbf{M} \models_{(q+r,0)} \alpha$ for all $q, r \in \mathbb{Q}^{\geq 0}$.

Hence $\mathbf{M} \models_{(q,0)} \alpha \;\Leftrightarrow\; \mathbf{M} \models_{(q+r,0)} \alpha$, for all $q, r \in \mathbb{Q}^{\geq 0}$ and all temporal formulas α.

Moreover, by definition of $\mathbf{M}$, we have

$$\mathbf{M} \models_r \alpha \;\Leftrightarrow\; \mathbf{M} \models_s \alpha$$

for all $r, s \in C$ and all α. (Again we use the fact that truth of a formula depend only on the future and that $r, s \in C$ both "see" the same structure in front of them.)

We call *isomorphism* the properties just described.

We prove a preliminary result.

Proposition 3.3. *Let α be a temporal formula such that $\mathbf{M} \models_r \alpha$ for all (equivalently: for some) $r \in A$. Then there exists $s \in B$ such that $\mathbf{M} \models_t \alpha$ for all $t \leq s$.*

Proof. By induction on α we prove:

1. if $\mathbf{M} \models_r \alpha$ for all $r \in A$ then there exists $s \in B$ such that $\mathbf{M} \models_t \alpha$ for all $t \leq s$;
2. if $\mathbf{M} \models_r \neg\alpha$ for all $r \in A$ then there exists $s \in B$ such that $\mathbf{M} \models_t \neg\alpha$ for all $t \leq s$;

The cases when α is atomic or α is of the form $\neg\beta$ for some temporal formula β follow easily by construction of $\mathbf{M}$ and by induction hypothesis applied to β, respectively. We examine only a few more cases.

(a) Let α be of the form $\beta \wedge \gamma$. For 1., let $s = \min(s_\beta, s_\gamma)$, where s_β and s_γ are given by inductive hypothesis 1. relative to β and γ respectively.

For 2., notice that if $\mathbf{M} \models_r \neg\alpha$ for all $r \in A$, then, without loss of generality, there exists $t \in A$ such that if $\mathbf{M} \models_t \neg\beta$ for some $t \in A$. By *isomorphism* (see Remark 3.2) we get $\mathbf{M} \models_r \neg\beta$ for all $r \in A$. We apply inductive hypothesis 2. relative to β to get the conclusion.

(b) Let α be of the form $\Box_{[m,n]}\beta$. For 1., a straigthforward application of *isomorphism* and of inductive hypothesis 1. relative to β yields $s \in B$ such that $\mathbf{M} \models_t \beta$ for all $t \leq s$. Let $s = (q, 1)$, for some $q \in \mathbb{Q}$. Any $r \in B$ smaller or equal to $(q - n, 1)$ has the required property.

For 2. proceed similarly to 1.

(c) Let α be of the form $\Box_{[m,\infty[}\beta$. Since 1. is trivial, we deal with 2. only. Suppose $\mathbf{M} \models_r \neg\alpha$ for all $r \in A$. Then there exists $r \leq s$ such that $\mathbf{M} \models_s \neg\beta$. If $s \in B$ the conclusion follows easily. If $s \in A$ we first apply *isomorphism* and induction hypothesis relative to $\neg\beta$ to finish (see (b) above).

□

We call *overspill* the property stated in the previous proposition.

Not surprisingly, the definition of $\mathbf{M}$ implies that an *underspill* property holds as well:

Proposition 3.4. *Let α be a temporal formula such that $\mathbf{M} \models_r \alpha$ for all (equivalently: for some) $r \in C$. Then there exists $s \in B$ such that $\mathbf{M} \models_t \alpha$ for all $s \leq t$.*

Proof. Similar to the proof of Proposition 3.3. □

We can now prove our claim.

Proposition 3.5. *The structure $\mathbf{M}$ previously described satisfies all closed instances of the schema of induction.*

Proof. Let α be a temporal formula. We assume

$$\mathbf{M} \models_0 \alpha \wedge \Box_{[0,\infty[}(\alpha \rightarrow \Box_{[0,1]}\alpha)$$

and we prove $\mathbf{M} \models_0 \Box_{[0,\infty[}\alpha$. By *isomorphism* and *overspill*, from $\mathbf{M} \models_0 \alpha$ we get $r \in B$ such that α is true up to r. We use $\mathbf{M} \models_0 \Box_{[0,\infty[}(\alpha \rightarrow \Box_{[0,1]}\alpha)$ to get that α is true in $A \cup B$. It follows from *isomorphism* and from the *underspill* property that α is true also in C.

Finally, we get that all instances of the schema of induction are true in A by *isomorphism*. In particular, in $\mathbf{M}$ all closed instances of the induction schema are true. □

It remains to check that $\mathbf{M}$ is not *tee* to any structure based on the standard frame. This is straigthforward since, for an arbitrary propositional letter p, $\mathbf{M} \models_0 \Diamond_{[0,\infty[}\neg p$ and $\mathbf{M} \models_0 \Box_{[0,n]}p$ for all $n \in \mathbb{N}$.

We finish with the following:

Question 2 Does there exist a temporal sentence that is true in some structure based on a nonstandard frame that satisfies all closed instances of the schema of induction and is false in all structures based on the standard frame?

It is known that in the case of *Linear Time Logic* completeness with respect to the standard frame of natural numbers holds (see [Em90]). Partly motivated by such a result we conjecture that Question 2 has a negative answer. If so we may dispose of nonstandard frames in the statement of Theorem 3.2 when restricting the class of structures to those that satisfy all the instances of the induction schema.

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A The system MTL$'$: syntax and semantics

relational formulas the relational formulas (r-formulas) are the predicate first order formulas in a language with equality and with a countable set Var of variables whose extralogical symbols are:

1. the constant symbol 0;
2. the unary function symbol f;
3. the binary predicate symbol $<$.

temporal formulas the temporal formulas (briefly: formulas) are the formulas in a propositional temporal language whose symbols are:

1. a countably infinite set At of proposition symbols;
2. the propositional connectives;
3. for all $m < n$ in $\mathbf{N}$ the temporal operators

$$\Box_{[m,n]} \quad \Box_{]m,n]} \quad \Box_{[m,n[} \quad \Box_{]m,n[} \quad \Box_{[m,\infty[} \quad \Box_{]m,\infty[} \ .$$

labelled formulas the labelled formulas (l-formulas) are the expressions of the form $t : \alpha$ where t is a relational term and α is a temporal formula.

A.1 The sequent calculus

Let x be a variable and let s, t be terms. We denote by $s[x/t]$ the term obtained by substituting all occurrences of x in s with t.

Sequents are objects of the form $\Gamma \vdash \Delta$, where Γ and Δ are finite lists of r- or l-formulas.

Identity rules

$\varphi \vdash \varphi$ when φ is an r- or an l-formula;

$\vdash \rho$ where ρ is one of the following: an axiom for equality; a first order axiom of the theory of dense linear ordering with least element 0 and no greatest element; one of the following relational axioms:

$$\forall x(x < f(x)); \quad \forall x \forall y(x < y \to f(x) < f(y)); \quad \forall x(f(0) < x \to \exists y(f(y) = x)).$$

$$\frac{\Gamma \vdash \rho, \Delta \quad \Gamma, \rho \vdash \Delta}{\Gamma \vdash \Delta} \quad \text{when } \rho \text{ is an } r\text{-formula} \quad (r\text{–}cut)$$

$$\frac{\Gamma \vdash \lambda, \Delta \quad \Gamma, \lambda \vdash \Delta}{\Gamma \vdash \Delta} \quad \text{when } \lambda \text{ is an } l\text{-formula} \quad (l\text{–}cut)$$

$$\frac{\Gamma, u_1 = u_2, t[x/u_1] : \alpha, t[x/u_2] : \alpha \vdash \Delta}{\Gamma, u_1 = u_2, t[x/u_i] : \alpha \vdash \Delta} \quad i = 1, 2$$

$$\frac{\Gamma, u_1 = u_2 \vdash t[x/u_1] : \alpha, t[x/u_2] : \alpha, \Delta}{\Gamma, u_1 = u_2 \vdash t[x/u_i] : \alpha, \Delta} \quad i = 1, 2$$

Structural rules

Weakening, Exchange and *Contraction* (for all kind of formulas).

Rules for *r*-formulas

By these we mean the rules acting on relational formulas. They are the standard rules of classical sequent calculus for finitary connectives and for the quantifiers $\forall$ and $\exists$ (applicable to *all* r-formulas)

Rules for *l*-formulas

The propositional rules for l-formulas closely follow the first order propositional rules. They require the same label appearing in premises and conclusion: Here are two examples:

$$\frac{\Gamma, t : \alpha \vdash t : \beta, \Delta}{\Gamma \vdash t : \alpha \to \beta, \Delta}$$

$$\frac{\Gamma, t : \beta \vdash \Delta \qquad \Gamma \vdash t : \alpha, \Delta}{\Gamma, t : \alpha \to \beta \vdash \Delta}$$

The rules for temporal operators:

$$\frac{\Gamma, f^m(t) \leq x, x \leq f^n(t) \vdash x : \alpha, \Delta}{\Gamma \vdash t : \Box_{[m,n]}\alpha, \Delta} \quad \text{if } x \text{ does not occur free in } \Gamma \cup \Delta \cup \{t\}$$

$$\frac{\Gamma \vdash f^m(t) \leq s \quad \Gamma \vdash s \leq f^n(t) \quad \Gamma, s : \alpha \vdash \Delta}{\Gamma, t : \Box_{[m,n]}\alpha \vdash \Delta}$$

$$\frac{\Gamma, f^m(t) \leq x \vdash x : \alpha, \Delta}{\Gamma \vdash t : \Box_{[m,\infty[}\alpha, \Delta} \quad \text{if } x \text{ does not occur free in } \Gamma \cup \Delta \cup \{t\}$$

$$\frac{\Gamma \vdash f^m(t) \leq s \quad \Gamma, s : \alpha \vdash \Delta}{\Gamma, t : \Box_{[m,\infty[}\alpha \vdash \Delta}$$

The rules for the other temporal operators are similar.

A.2 Semantics

Formulas are assigned a truth value in structures of the form

$$\mathbf{A} = \langle A, <^{\mathbf{A}}, f^{\mathbf{A}} : A \to A, 0^{\mathbf{A}}, \sigma : Var \to A, \tau : A \to 2^{At} \rangle,$$

where $\langle A, <^{\mathbf{A}}, f^{\mathbf{A}}, 0^{\mathbf{A}} \rangle$ satisfies the relational axioms and σ, τ are arbitrary maps. We say that $\mathbf{A}_F = \langle A, <^{\mathbf{A}}, f^{\mathbf{A}}, 0^{\mathbf{A}} \rangle$ is the *frame* of $\mathbf{A}$. We also say that $\mathbf{A}$ is *based* on its frame.

Let $\mathbf{A}$ be a structure as above. As usual, we denote by $t^{\mathbf{A}}$ the interpretation of relational term t in $\mathbf{A}$.

The semantics of relational formulas is the usual first order semantics. It is taken care by the frame of a structure and by the assignment of values to relational variables.

We define $\mathbf{A} \models_a \alpha$, where $a \in A$ and α is an arbitrary formula, by induction on α :

$\mathbf{A} \models_a p$ if $p \in \tau(a)$ for $p \in At$;

$\mathbf{A} \models_a \neg\alpha$ if not $\mathbf{A} \models_a \alpha$;

$\mathbf{A} \models_a \alpha \bullet \beta$ if $\mathbf{A} \models_a \alpha \quad \bullet \mathbf{A} \models_a \beta$, when $\bullet$ is a binary propositional connective;

$\mathbf{A} \models_a \Box_{[m,n]}\alpha$ if $\mathbf{A} \models_b \alpha$ for all $(f^{\mathbf{A}})^m(a) \leq b \leq (f^{\mathbf{A}})^n(a)$;

$\mathbf{A} \models_a \Box_{]m,\infty[}\alpha$ if $\mathbf{A} \models_b \alpha$ for all $(f^{\mathbf{A}})^m(a) < b$.

(The other cases are similar.)

We let

$$\mathbf{A} \models t : \alpha \;\Leftrightarrow\; \mathbf{A} \models_{t^{\mathbf{A}}} \alpha.$$

We say that sequent $\Gamma \vdash \Delta$ is *true* in the structure $\mathbf{A}$ if

$$\mathbf{A} \models \gamma \text{ for all } \gamma \in \Gamma \;\Rightarrow \mathbf{A} \models \delta \text{ for some } \delta \in \Delta.$$

A.3 A first order translation of *l*-formulas

A translation * of r-formulas and l-formulas into first order formulas of the relational language, expanded with a unary predicate symbol p for each $p \in At$, is defined so that * behaves as identity on r-formulas and has the following inductive definition on l-formulas:

1. $(t : p)^* = p(t)$;
2. $*$ commutes with propositional connectives;
3. $(t : \Box_{[m,n]}\alpha)^* = \forall x(f^m(t) \leq x \leq f^n(t) \to (x : \alpha)^*)$, where x is the first in a list of variables not occurring in t. (Similarly for the other temporal operators.)

If Γ is a list of r- or l-formulas, we denote by Γ^* the list of r-formulas obtained by applying * to each formula in Γ.