Dipartimento di Informatica
Università degli Studi di Verona

Rapporto di ricerca
Research report
December 2006

Stochastic Learning in IFSP for Images with Grey Levels

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Abstract

In previous works we have introduced and studied an algorithm, called SAOP, which optimizes probabilities when a Chaos-Game decompresses a fractal image. Here we describe a suitable modification of SAOP, in order to solve the inverse problem for an image with grey levels. We also find out equilibrium conditions for the Markov Chain which represents the evolution of the probabilities.

Keywords: Image Data, Stochastic Algorithms, Inverse Problem, Random System with Complete Connections, Stochastic Learning
1 Introduction

Fascinating fractal images are a common tool in computer graphics. We will firstly recall how a fractal image is in effect generated.

A black and white image can always be represented by a subset of the unit square. Then, we say that it is generated by an Iterated Function System (IFS) if there exists a set of maps on the unit square, contractive with respect to the Euclidean metrics, such that the operator obtained by their “superposition”, called the Iterated Function System operator, has such a subset as fixed point. The set of points, generated by the iterative application of the IFS operator to any subset of the unit square, asymptotically reproduces the image.

Indeed, such a procedure is too heavy from the computational point of view. Therefore in applications, the generation of the image (also called decompression) is performed by the “Chaos-Game” algorithm. In this case a probability is associated to every assigned contraction map and, instead of iteratively applying the IFS operator, at every step only one map is used, chosen according to the corresponding probability. The resulting random walk on the unit square and the IFS system with the assigned probabilities are respectively called Chaos-Game and IFSP system. By an ergodicity property the set of pixels, visited by the random walk, asymptotically reproduces the image.

Observe that an IFSP system can in principle generate an image with grey levels, if one roughly interprets the frequency of visits by the Chaos-Game in the points of the image as their grey level.

The choice of the probabilities is the critical problem. In fact, with a bad choice, the image could be completely generated only after an enormous time interval.

In case of a black and white image the best choice occurs when the frequencies the pixels are visited with, during the Chaos-Game, reproduce, as well as possible, the uniform distribution on the subset representing the image.

Indeed, with such a set of optimal probabilities, a computer simulation
of the Chaos-Game generates the image associated to the IFS system in the fastest way, because the possibility of being visited twice is approximately the same for all the pixels of the image.

The problem of determining a priori optimal probabilities is still open, but it was recently shown that there exists a stochastic algorithm (SAOP), which computes such optimal probabilities for an arbitrary IFS system on the pixels space ([17]...[21]). Notably, it is possible to prove the existence of the optimal set of probabilities in the previously described sense and to give some convergence results ([17] [18]). Moreover, it is possible to prove that SAOP is an interesting example of Random System with Complete Connections ([20]) and, as a consequence, an example of Stochastic Learning Process ([21]).

The basic structure of SAOP is reported in Section 2.

In Section 3 we recall the theory of Random System with Complete Connections which will be used in the following section.

In Section 4, we describe an extension of SAOP to grey level fractal images.

The new algorithm is based on the same principle of “penalizing-rewarding” used in SAOP, in order to obtain the good set of probabilities associated to the image. Nevertheless the idea of mistake is completey different: we will say that a mistake occurs if the visits frequency in a pixel goes out of an “admitted region”, built around the real grey level of the pixel.

This could in principle compress any given grey level image in the sense that, given an a priori set of available contraction maps, the algorithm is expected to give, asymptotically, the best probabilities associated to them. Then the associated IFSP system, applied to any subset of the unit square, will asymptotically reproduce the given image, with an approximation depending on the set of the a priori available maps. Therefore the last algorithm can be seen as the time evolution of the state of a stochastic system out of equilibrium which, during the relaxation, learns the structure of the given image. The inputs are the “image to learn” and a convenient set of available maps. The output is the optimal set of probabilities (most of them could be equal to zero!) which allows to generate the image by means of a Chaos-Game.
We will also study equilibrium conditions for the Markov Chain which describes the evolution of the probabilities.

2 SAOP and the Chaos-Game

Let \((\mathcal{X}, d)\) be the complete metric space representing the “base space”: we will consider the finite pixels space with the Euclidean metrics; let \(w = (w_1, w_2, \ldots, w_N)\) denote the set of suitably discretized contraction maps which describe the image, defined on \(\mathcal{X}\): the pair \((\mathcal{X}, w)\) is called an IFS (Iterated Function System).

We associate to the IFS system a set of probabilities \(p = (p_1, \ldots, p_N)\), \(p_i > 0\), \(\sum_{i=1}^{N} p_i = 1\): \(p_i\) represents the probability for the map \(w_i\) to be chosen. The new system \((\mathcal{X}, w, p)\) is called an IFSP (Iterated Function System with Probability).

Let \(A\) be the attractor of the system, i.e. the image we want to reconstruct, compressed by means of the maps \((w_1, w_2, \ldots, w_N)\): \(A \in K(\mathcal{X})\), where \((K(\mathcal{X}), h)\) is the complete metric space of compact subsets of \(\mathcal{X}\), with the Hausdorff metrics. \(A\) is the fixed point of the IFS operator \(W(\cdot) = \bigcup_{i=1}^{N} w_i(\cdot)\) defined on \(K(\mathcal{X})\)([2], [3]).

In order to decompress it, the algorithm which is typically used is the so-called Chaos-Game, which can be roughly described in the following way: fixed a starting point \(x_1 \in \mathcal{X}\) we choose a map \(w_i\) with probability \(p_i\) and apply it to \(x_1\), obtaining a new point \(x_2 = w_i(x_1) \in \mathcal{X}\); we then choose a new map, independent of the first one, apply it to \(x_2\) and iterate. The set of points \(\{x_1, \ldots, x_n\}\) approximates \(A\), for \(n\) large enough.

Formally, given an appropriate probability space \((\Omega, \mathcal{F}, \mathbb{P})\), for example the canonical trajectory space, we define the Chaos-Game as a Random Walk \((X_n)_{n=1}^{\infty}\) on \(\mathcal{X}\): given \((Z_n)_{n=1}^{\infty}\) i.i.d. random variables, such that \(\mathbb{P}[Z_n = i] = p_i\), \(\forall i = 1 \ldots N\), the process \((X_n)_{n=1}^{\infty}\) is defined by \(X_{n+1} = w_{Z_n}(X_n)\), \(\forall n \geq 1\), where \(X_1\) is uniformly distributed on \(\mathcal{X}\).

\((X_n)_{n=1}^{\infty}\) is of course an homogeneous Markov chain with state space \(\mathcal{X}\) and
its transition matrix $P = (p_{ij})_{i,j}$ is defined by
\[ p_{ij} := \mathbb{P}[X_{n+1} = x_j | X_n = x_i] = \sum_{k:w_k(x_i) = x_j} p_k, \quad \forall x_i, x_j \in \mathcal{X}. \]

By Elton’s theorem ([6]) we know that
\[ \lim_{m \to \infty} \frac{1}{m} \sum_{n=1}^{m} \delta_{X_n, x} = \pi_x, \quad \text{a.s.} \quad \forall x \in \mathcal{X}, \tag{1} \]
where $(\pi_x)_{x \in \mathcal{X}}$ is the only invariant probability measure for the transition matrix $P$.

Indeed, one could easily see that $A$ is a closed recurrent class for $(X_n)_{n=1}^{\infty}$, and that $\mathcal{X} \setminus A$ contains only transient states ([16]). Then (1) follows by standard results of Markov chains theory.

SAOP (Stochastic Algorithm for the Optimization of Probabilities) decompresses the image, optimizing, in the meanwhile, the probabilities associated to the single maps. Its basic idea consists in a modification of the Chaos-Game, where the arbitrary initial set of probabilities is improved whenever an error occurs: chosen an initial distribution $p^0 = (p^0_1, \ldots, p^0_N)$, one starts with the Chaos-Game, stopping when a site already visited is reached for the second time; at this point a new distribution $p^1$ is considered, where the map which made the mistake is penalized (its probability is reduced), and one of the others is randomly rewarded; the Chaos-Game is used again with the new set of probabilities up to the next mistake.

Formally, we have a process on $\mathcal{X}$, $(\tilde{X}_m)_{m=1}^{\infty}$, the Time Dependent Chaos-Game (TDCG), consisting of a sequence of different Chaos-Games.

We now give a mathematical description of SAOP, following [17] and [18] but with slightly different notations.

From now on, we will assume all random variables defined on a suitable fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

**Block 1:**
Let us fix an initial set of probabilities $p^0 = (p^0_1, \ldots, p^0_N)$, for example $p^0_i = \frac{1}{N}$, $\forall i = 1 \ldots N$, and consider $(Z^0_n)_{n=1}^{\infty}$, i.i.d. random variables such that $\mathbb{P}[Z^0_n = i] = p^0_i$, $\forall i = 1 \ldots N$. 
The corresponding Chaos-Game is defined by the process \((X^1_n)_{n=1}^\infty\), where \(X^1_{n+1} = w_{Z^0_n}(X^1_n)\), \(\forall n \geq 1\) and \(X^1_1\) is uniformly distributed on \(\mathcal{X}\).

The length of this block is represented by the stopping time
\[
\lambda_1 = \inf\{h \geq 2 : \exists 1 \leq k < h \text{ s.t. } X^1_k = X^1_h\}.
\]

**Block j+1:**

Let \(\lambda_j = \inf\{h \geq 2 : \exists 1 \leq k < h \text{ s.t. } X^j_k = X^j_h\}\) be the length of Block \(j\).

Let us also define the random variable \(\zeta_j\), denoting the map chosen to be rewarded:
\[
P[\zeta_{M,j} = i] = \begin{cases} 0 & \text{if } Z^j_{\lambda_{j-1}} = i, \\ \frac{1}{N-1} & \text{otherwise.} \end{cases}
\]

We can then consider the new set of probabilities \(p^j = (p^j_0, \ldots, p^j_N)\) where, with fixed \(\Delta \in (0, 1),\)
\[
p^j_i = \begin{cases} p^{j-1}_i - \Delta & \text{if } Z^j_{\lambda_{j-1}} = i, \\ p^{j-1}_i & \text{if } Z^j_{\lambda_{M,j-1}} \neq i, \zeta_j \neq i, \\ p^{j-1}_i + \Delta & \text{if } Z^j_{\lambda_{M,j-1}} \neq i, \zeta_j = i. \end{cases} \tag{2}
\]

If the map which failed is \(w_i\), its probability is reduced of the fixed quantity \(\Delta\), while if another one made the mistake, its probability is increased of \(\Delta\) if \(w_i\) is chosen to be rewarded, unchanged otherwise.

Let us consider also \((Z^j_n)_{n=1}^\infty\) i.i.d. random variables such that \(P[Z^j_n = i] = p^j_i, \forall i = 1 \ldots N\).

The Chaos-Game is now \((X^{j+1}_n)_{n=1}^\infty\), where \(X^{j+1}_{n+1} = w_{Z^j_n}(X^{j+1}_n)\), \(\forall n \geq 1\) and \(X^{j+1}_1\) is uniformly distributed on \(\mathcal{X}\).

The TDCG \((\hat{X}_m)_{m=1}^\infty\) is then defined by:
\[
\hat{X}_m = X^{j}_{m-T_j-1}, \text{ if } m \in \{T_{j-1} + 1, \ldots, T_j\},
\]

where
\[
\begin{cases} T_0 = 0, \\ T_j = \sum_{i=1}^{j} \lambda_{M,i}, \forall j \geq 1. \end{cases}
\]

TDCG is a sort of regenerative time-dependent process and it is a Markov chain (a Random Walk on \(\mathcal{X}\)), for a fixed choice of the sequences \((p^n)_{n=0}^\infty\) and \((\lambda^n)_{n=1}^\infty\).
The process which describes the corresponding evolution of the probabilities is $Y_j = p^j$, $\forall j \geq 0$, where its value at step $j$ is the set of probabilities used in the stochastic interval $[T_j + 1, T_{j+1}]$ (for more details see [17], [18]).

3 Generalized Random Systems with Complete Connections

We recall here the basic notions of the theory of Dependence with Complete Connections, exhaustively surveyed in [11], by Iosifescu and Grigorescu.

The mentioned theory was introduced in 1935, by Onicescu and Mihoc ([22]), and studied afterwards by the Romanian school, with Ciucu, Theodorescu, Iosifescu and others (see, for example, [4], [12]). It is a non-trivial extension of Markovian Dependence theory, and it was also investigated by Doeblin and Fortet ([5]) and by Harris ([10]).

Examples of Random Systems with Complete Connections are stochastic learning models, urn models, partially observed random chains, decision models and others.

**Definition 3.1** An homogeneous Random System with Complete Connections or RSCC is a quadruple $\{(V, \mathcal{V}), (H, \mathcal{H}), u, P\}$ where

(i) $(V, \mathcal{V})$ and $(H, \mathcal{H})$ are arbitrary measurable spaces;

(ii) $u : V \times H \rightarrow V$ is a $(\mathcal{V} \otimes \mathcal{H}, \mathcal{V})$-measurable map;

(iii) $P$ is a transition probability function from $(V, \mathcal{V})$ to $(H, \mathcal{H})$, i.e., a real valued function defined on $V \times \mathcal{H}$, such that $P(v, \cdot)$ is a probability on $(H, \mathcal{H})$ for any $v \in V$, and $P(\cdot, A)$ is a random variable on $(V, \mathcal{V})$ for any $A \in \mathcal{H}$.

A generalization of this definition is due to Le Calvé and Theodorescu ([14]):

**Definition 3.2** An homogeneous Generalized Random System with Complete Connections or GRSCC is a quadruple $\{(V, \mathcal{V}), (H, \mathcal{H}), \Pi, P\}$ where
(i) \((V, \mathcal{V})\) and \((H, \mathcal{H})\) are arbitrary measurable spaces;

(ii) \(\Pi\) is a transition probability function from \((V \times H, \mathcal{V} \otimes \mathcal{H})\) to \((V, \mathcal{V})\);

(iii) \(P\) is a transition probability function from \((V, \mathcal{V})\) to \((H, \mathcal{H})\).

In both cases an existence theorem was proved: we state here only the one concerning the GRSCC.

**Theorem 3.3** For a given homogeneous GRSCC and an arbitrarily fixed \(v_0 \in V\), there exist a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), a sequence \((\xi_n)_{n=1}^{\infty}\) of \(H\)-valued random variables and a sequence \((\eta_n)_{n=0}^{\infty}\) of \(V\)-valued random variables, both defined on \(\Omega\), such that:

(i) for any \(A \in \mathcal{H}, B \in \mathcal{V}\), and \(n \geq 1\) we have,

\[
\mathbb{P}[\eta_0 \in B] = \delta_{v_0}(B),
\]
\[
\mathbb{P}[\xi_1 \in A] = P(v_0, A),
\]
\[
\mathbb{P}[\xi_{n+1} \in A | \xi_1, \ldots, \xi_n, \eta_0, \ldots, \eta_n] = P(\eta_n, A), \quad \mathbb{P} - \text{a.s.},
\]
\[
\mathbb{P}[\eta_{n+1} \in B | \xi_1, \ldots, \xi_{n+1}, \eta_0, \ldots, \eta_n] = \Pi(\eta_n, \xi_{n+1}, B), \quad \mathbb{P} - \text{a.s.};
\]

(ii) the sequence \((\eta_n)_{n=0}^{\infty}\) is an homogeneous Markov Chain, whose transition probability function \(Q\) and infinitesimal generator \(L\) are given by the equations

\[
Q(v, B) = \int_H \Pi(v, x, B)P(v, dx), \quad (3)
\]

for all \(v \in V\) and \(B \in \mathcal{V}\), and

\[
Lf(v) = \int_V f(v')Q(v, dv') = \int_V f(v') \int_H \Pi(v, x, dv')P(v, dx), \quad (4)
\]

for all \(f\) bounded \(\mathcal{V}\)-measurable functions on \(V\).

The sequence \((\xi_n)_{n=1}^{\infty}\) is called the Generalized Chain with Complete Connections (GCC) or Generalized Chain of Infinite Order (this second term was coined by Harris ([10])), while \((\eta_n)_{n=0}^{\infty}\) is the Markov Chain associated to the GRSCC.

Notice that while \(\eta_{n+1}\) depends both on \(\eta_n\) and \(\xi_{n+1}\), the law of \(\xi_{n+1}\) depends only on \(\eta_n\).
4 Inverse problem for IFSP with grey level

Let us now consider an image represented by a grey level function $L$ on the pixels space $\mathcal{X}$. $L : \chi \rightarrow [0, 1]$ where $\forall r \in \chi$ $L(r)$ is the grey level of the pixel $r$: if $L(r) = 1$ the pixel is black, if $L(r) = 0$ it is white.

Suppose a given set $\Xi$ of $N$ contraction maps is available; then solving the inverse problem, for the image represented by $L$ (and the given set $\Xi$), consists in finding a set of probabilities $(p_1, \ldots, p_N)$ (most of them could in general be equal to zero) such that the frequency of the visits in the various pixels by the associated Chaos-Game approximates $L$ as well as possible. This is equivalent to selecting the “suitable” contraction maps (with the associated probabilities) among the available ones. (For the literature about the inverse problem in image compression see, for example, [1], [7], [8], [9], [15], [23], [24].)

For technical reasons we associate to the real image a “target image”, which is equal to the first one except for uniform variations of the color: $\hat{h} : \chi \rightarrow [0, 1]$ and $\forall r \in \chi$

$$\hat{h}(r) = \frac{L(r)}{\sum_{s \in \chi} L(s)}.$$  

Then $\sum_{s \in \chi} \hat{h}(s) = 1$ and $\hat{h}$ can be seen as a discrete density function on $\chi$, to be approximated with the visits frequency of the Chaos-Game.

**Remark:** If we consider, for example a map without grey levels (a black and white image) made by $N$ pixels, then we have $L(r) \in \{0, 1\}$ and $\hat{h}(r) \in \{0, 1/N\}$. In fact $L(r) = 1$ iff $r$ is a black pixel but the same is not true for $\hat{h}$. What happens is that $\hat{h}(r) = 1$ iff $L(r) \neq 0$ and $L(s) = 0 \forall s \neq r$, which means that the image is made by one pixel: we get then the trivial case, whatever the color of the pixel is. For this reason we can suppose, without any restriction, that $\hat{h}(r) < 1$.

In the following, we shall give an idea of how to proceed to solve the inverse problem: we shall use an algorithm, which is in fact a modification of the SAOP defined before.

Here we reward and penalize the maps of a fixed $\Delta$, and not of a quantity
which depends on the color of the pixel in which the mistakes occurs, as suggested in previous works (see [19], [20], [21]).

Nevertheless, it’s the idea of mistake which is changed: as we shall see later, at each visit in a pixel we will compare the grey level with the visits frequency, in order to decide whether stop the process and penalize or reward a map; we fix an an “admitted region”, built around the value of the grey level \( \hat{h}(r) \) and we will say that a mistake occurs if the visits frequency of the pixel is out of this region. We will choose a region which approximates the real value, when the number of steps of the algorithm increases.

We can now formalize the algorithm with GRSCC notations. Let us observe that for each fixed \( M = 1/\Delta \in \mathbb{N} \), we have a GRSCC: from now on we shall explicit the dependence from \( M \) (i.e. from \( \Delta \)).

Here, then

\[
V_M = \left\{ p \in E_M^\otimes N : \sum_{i=1}^{N} p_i = 1 \right\}
\]

with \( E_M = \{0, 1/M, \ldots, (M-1)/M, 1\} \), \( V_M = \mathcal{P}(V_M) \).

\( V_M \) is the state space for the probability process \( Y^M \), which is a Markov Chain.

The process which visits the pixel is a process with values on \( \chi \), defined by “blocks” of Chaos-Game, as in the original SAOP. If we denote by \( X^M_n = (X^M_{n,i})_{i \geq 1} \) the \( n \)th block and we remark that each block is an infinite sequence, then the Chain of infinite order (i.e. the GCCC) is a process \( X^M = (X^M_n)_n \) with state space \( H_M = H = \chi^\mathbb{N} \), \( \mathcal{H}_M = \mathcal{H} = \mathcal{P}(H) \).

In order to complete the construction of a GRSCC we need to establish the transition rules. Differently from the original SAOP, we decide for a symmetric rule: we fix a minimum and a maximum level and then we penalize the map which makes the visit frequency go over the maximum limit (rewarding another map, randomly) and reward the map which makes the visit frequency go under the minimum limit (penalizing another map, randomly).

Formally the GRSCC is then defined by the two transition functions:

\[
P_M(p, A) = \mathbb{P}(X^M_{n+1} \in A | Y^M_n = p)
\]
\[
= \mathbb{P}(X_1^M \in A | Y_0^M = p) = \mathbb{P}_p(X_1^M \in A),
\]

\[
\Pi_M(p, x, q) = \mathbb{P}(Y_{n+1}^M = q | X_{n+1}^M = x, Y_n^M = p) = \mathbb{P}_p(Y_1^M = q | X_1^M = x).
\]

Let us now define the events

\[S^+_m = \text{“the } i^{\text{th}} \text{ map makes a mistake and the visit frequency}
\]
\[\text{goes over the maximum limit”},
\]

\[S^-_m = \text{“the } i^{\text{th}} \text{ map makes a mistake and the visit frequency}
\]
\[\text{goes under the minimum limit”},
\]

\[S_m = S^+_m \cup S^-_m = \text{“the } i^{\text{th}} \text{ map makes a mistake”},
\]

\[S_M = \bigcup_{i=1}^N S_{M,i} = \text{“some map makes a mistake”}.
\]

Fixed \( p(i, j, M) = p - \delta_i / M + \delta_j / M \), which means, \( \forall k = 1...N, \)

\[
p_k(i, j, M) = \begin{cases} p_i - 1/M & \text{if } k = i \\ p_j + 1/M & \text{if } k = j \\ p_k & \text{otherwise,} \end{cases}
\]

we get

\[
\Pi_M(p, x, q) = \begin{cases} 1 - \mathbb{P}_p(S_M | X_1^M = x) & \text{if } q = p(i, j, M), \text{ and } i \neq j \\ \mathbb{P}_p(S_M | X_1^M = x) & \text{if } q = p \\ 0 & \text{otherwise.} \end{cases}
\]

Let us observe that, in the case there is no mistake in a block, we can then suppose that there is no transition.

Using (3) we obtain then

\[
Q_M(p, p) = 1 - \int_H \mathbb{P}_p(S_M | X_1^M = x) \mathbb{P}_p(X_1^M \in dx)
\]
and, if \( i \neq j \),
\[
Q_M(p, p(i, j, M)) = \frac{1}{N - 1} \left[ \mathbb{P}_p(S_{M,i}^+ | X_i^M = x) + \mathbb{P}_p(S_{M,j}^- | X_j^M = x) \right] \mathbb{P}_p(X_i^M \in dx).
\]

Introducing
\[
a_{M,i}^+(p) = \mathbb{P}_p(S_{M,i}^+ | X_i^M = x) \mathbb{P}_p(X_i^M \in dx),
\]
\[
a_{M,i}^-(p) = \mathbb{P}_p(S_{M,i}^- | X_i^M = x) \mathbb{P}_p(X_i^M \in dx),
\]
\[
a_{M,i}(p) = a_{M,i}^+(p) + a_{M,i}^-(p),
\]
\[
e_M(p) = \mathbb{P}_p(S_M) = \sum_{i=1}^N a_{M,i}(p),
\]
the transition for the Markov Chain \( Y^M \) turns out to be
\[
Q_M(p, q) = \begin{cases} 
\frac{1}{N - 1} \left( a_{M,i}^+(p) + a_{M,j}^-(p) \right) & \text{if } q = p(i, j, M), \\
1 - e_M(p) & \text{if } q = p \\
0 & \text{otherwise.}
\end{cases}
\]

How can we explicitly define \( S_{M,i} \)? When can we say that there has been a mistake? We have already given an idea. Let us then describe it formally.

Let \( \nu_k^M(r) \) be the empirical frequency of visit in \( r \in \chi \) after \( k \) steps (we can always consider the first block, because of the homogeneity of the process):
\[
\nu_k^M(r) = \frac{\#\{2 \leq j \leq k : X_j^{M,1} = r\}}{k} = \frac{1}{k} \sum_{i=2}^k \delta_r(X_i^{M,1}),
\]
from which \( 0 \leq \nu_k^M(r) \leq \frac{k-1}{k} < 1 \).

Let us denote with \( T_k^+(r) \) and \( T_k^-(r) \) respectively the maximum and the minimum grey levels admitted after \( k \) steps in the pixel \( r \in \chi \); in particular we can set \( T_k^+(r) = \hat{h}(r) + K(k) \) and \( T_k^-(r) = \hat{h}(r) - K(k) \), where \( K(k) \geq 0 \), it is decreasing and \( \lim_{k \to +\infty} K^+(k) = 0 \).

Let us define now the “admitted region”
\[
B_k(r) = [T_k^-(r), T_k^+(r)] = [\hat{h}(r) - K(k), \hat{h}(r) + K(k)],
\]
which converges to \( \{ \hat{h}(r) \} \) when \( k \to +\infty \), and the new event \( S_{M,k} = \text{“there is a mistake at time } k \” \):

\[
S_{M,k} = [\nu_k^M(X_k^{M,1}) \in B_k^c(X_k^{M,1})]^c.
\]

Then we get

\[
S_{M,i}^{-} = \bigcup_{k \geq 1} \left[ \bigcap_{j < k} S_{M,j}^c \cap \left[ \nu_k^M(X_k^{M,1}) < T_k^-(X_k^{M,1}) \right] \cap [Z_k^0 = i] \right]
\]

and

\[
S_{M,i}^{+} = \bigcup_{k \geq 1} \left[ \bigcap_{j < k} S_{M,j}^c \cap \left[ \nu_k^M(X_k^{M,1}) > T_k^+(X_k^{M,1}) \right] \cap [Z_k^0 = i] \right],
\]

from which

\[
S_{M,i} = \bigcup_{k \geq 1} \left[ \bigcap_{j < k} S_{M,j}^c \cap S_{M,k} \cap [Z_k^0 = i] \right] = \bigcup_{k \geq 1} [S_{M,k} \cap [Z_k^0 = i]]
\]

and

\[
S_{M} = \bigcup_{k \geq 1} \left[ \bigcap_{j < k} S_{M,j}^c \cap S_{M,k} \right] = \bigcup_{k \geq 1} S_{M,k}.
\]

Then the GRSCC is completely defined. Let us now study the equilibrium condition for the probability process (i.e. the Markov Chain associated to the CRSCC).

**4.1 Mean Equilibrium Condition**

The mean equilibrium condition which has to be satisfied by the process \( Y^M \) is

\[
\mathbb{E} \left[ Y_{n+1}^M | Y_n^M = p \right] = p
\]

which is equivalent, because of the homogeneity, to

\[
\mathbb{E}_p \left[ Y_1^M \right] = \mathbb{E} \left[ Y_1^M | Y_0^M = p \right] = p,
\]
and in particular, considering each component, to

\[ \mathbb{E}_p \left[ Y_i^{t,M} \right] = \mathbb{E} \left[ Y_i^{t,M} \mid Y_0^M = p_i \right] = p_i. \]

Let us then check when it is satisfied.

\[
\begin{align*}
\mathbb{E}_p \left[ Y_i^{t,M} \right] &= p_i - \frac{1}{M} \left[ a_{M,i}^+(p) + \frac{1}{N-1} \sum_{j \neq i} a_{M,j}^-(p) \right] + \\
&\quad + \left[ p_i + \frac{1}{M} \right] \left[ a_{M,i}(p) + \frac{1}{N-1} \sum_{j \neq i} a_{M,j}^+(p) \right] + \\
&\quad + p_i \left[ \frac{N-2}{N-1} \sum_{j \neq i} a_{M,j}^-(p) + \frac{N-2}{N-1} \sum_{j \neq i} a_{M,j}^+(p) + (1 - e_M(p)) \right] \\
&= p_i \left[ a_{M,i}^+(p) + \frac{1}{N-1} \sum_{j \neq i} a_{M,j}^-(p) + a_{M,i}^-(p) + \frac{1}{N-1} \sum_{j \neq i} a_{M,j}^+(p) \right] + \\
&\quad + \frac{N-2}{N-1} \sum_{j \neq i} a_{M,j}^-(p) + \frac{N-2}{N-1} \sum_{j \neq i} a_{M,j}^+(p) + (1 - e_M(p)) \right] \\
&\quad - \frac{1}{M} \left[ a_{M,i}(p) + \frac{1}{N-1} \sum_{j \neq i} a_{M,j}^-(p) \right] + \frac{1}{M} \left[ a_{M,i}(p) + \frac{1}{N-1} \sum_{j \neq i} a_{M,j}^+(p) \right] \\
&= p_i \left[ a_{M,i}(p) + \frac{1}{N-1} \sum_{j \neq i} a_{M,j}(p) + \frac{N-2}{N-1} \sum_{j \neq i} a_{M,j}(p) + (1 - e_M(p)) \right] + \\
&\quad + \frac{1}{M} \left[ (a_{M,i}(p) - a_{M,i}^+(p)) + \frac{1}{N-1} \sum_{j \neq i} (a_{M,j}^+(p) - a_{M,j}^-(p)) \right] \\
&= p_i \left[ a_{M,i}(p) + \sum_{j \neq i} a_{M,j}(p) + [1 - e_M(p)] \right] \\
&\quad + \frac{1}{M} \left[ (a_{M,i}(p) - a_{M,i}^+(p)) + \frac{1}{N-1} \sum_{j \neq i} (a_{M,j}^+(p) - a_{M,j}^-(p)) \right] \\
&= p_i + \frac{1}{M} \left[ (a_{M,i}(p) - a_{M,i}^+(p)) + \frac{1}{N-1} \left[ (e_M(p) - a_{M,i}^+(p)) - (e_M(p) + a_{M,i}^+(p)) \right] \right]
\end{align*}
\]

where we have introduced

\[ e_M^+(p) = \sum_{i=1}^N a_{M,i}^+(p) \]
and
\[ e^{-M}(p) = \sum_{i=1}^{N} a^{-M,i}(p). \]

We obtain then the equilibrium if and only if
\[ \frac{N}{N-1} (a^{-M,i}(p) - a^{+M,i}(p)) + \frac{1}{N-1} (e^{+M}(p) - e^{-M}(p)) = 0 \]
i.e.
\[ a^{+M,i}(p) - a^{-M,i}(p) = \frac{1}{N} (e^{+M}(p) - e^{-M}(p)), \forall i = 1...N, \forall M = \frac{1}{\Delta} \text{ fixed.} \]

### 4.2 Macroscopical Equilibrium Condition (Scaling Limit)

**Theorem 4.1** (see [13] Corollary 7.4.2 pag. 355)

Let \( A = (a_{ij})_{i,j=1,...,d} \) be a continuous, symmetric, non negative matrix with values on \( \mathbb{R}^d \) and \( b : \mathbb{R}^d \to \mathbb{R}^d \), continuous, such that, defining the operator on \( C_c^\infty(\mathbb{R}^d) \)
\[ L = \frac{1}{2} \sum_{ij} a_{ij} \partial_i \partial_j + \sum_i b_i \partial_i, \]
the martingale problem for \((L, \nu)\) has a unique solution for any probability measure \( \nu \) on \( \mathbb{R}^d \).

Let \( \mu_M(\mathbf{x}, \Gamma) \) be transitions on \( \mathbb{R}^d \) and let us define
\[ b_M(\mathbf{x}) = M \int_{|y-x|\leq 1} (y-x)\mu_M(\mathbf{x},dy), \]
and
\[ A_M(\mathbf{x}) = M \int_{|y-x|\leq 1} (y-x)(y-x)^t\mu_M(\mathbf{x},dy). \]

Suppose, \( \forall r > 0, \forall \varepsilon > 0 \) that
\[ a) \lim_{M \to \infty} \sup_{|\mathbf{x}| \leq r} |A_M(\mathbf{x}) - A(\mathbf{x})| = 0 \]
\[ b) \lim_{M \to \infty} \sup_{|\mathbf{x}| \leq r} |b_M(\mathbf{x}) - b(\mathbf{x})| = 0 \]
\[ c) \lim_{M \to \infty} \sup_{|\mathbf{x}| \leq r} M \cdot \mu_M(\mathbf{x}, \{y:|y-x| \geq \varepsilon\}) = 0. \]
Let $Y^M$ be the Markov chain with transition $\mu_M$ and $\tilde{Y}^M_t = Y^M_{[M]}$. If $P^{-1}(Y^M_0) \Rightarrow \nu$, then the process $\tilde{Y}^M$ converges to the solution of the martingale problem for $(L, \nu)$, i.e. to a diffusion process having $A$ as diffusion matrix and $b$ as drift.

Remark:

In our case the transitions are not defined on the whole $\mathbb{R}^N$, because $Y^M$ is defined on $V_M \subset [0, 1]^N$ which is bounded. Then, in the definition of $A_M$ and $b_M$, we don’t have to integrate on a sphere. In fact, for each $M > 0$ fixed we are on $V_M$ and the only transition with positive contribution are those from $p$ to $p(i, j, M)$. Moreover, for these points we have

$$|p - p(i, j, M)| = \left| p - p + \frac{\delta_j}{M} - \frac{\delta_i}{M} \right| = \frac{1}{M} |\delta_j - \delta_i| = \frac{\sqrt{2}}{M} < 1 \text{ if } M > \sqrt{2}.$$

Now

$$b_M(p) = M \sum_{q \in V_M} (q - p)Q_M(p, q)$$

$$= M \sum_{i=1}^{N} \sum_{j=1}^{N} (p(i, j, M) - p)Q_M(p, p(i, j, M))$$

$$= M \sum_{i=1}^{N} \sum_{j \neq i} \left[ p - \frac{\delta_i}{M} + \frac{\delta_j}{M} - p \right] \cdot \frac{1}{N-1} (a^+_{M,i}(p) + a^-_{M,j}(p))$$

$$= \frac{1}{N-1} \sum_{i=1}^{N} \sum_{j \neq i} [\delta_j - \delta_i] \cdot (a^+_{M,i}(p) + a^-_{M,j}(p))$$

$$= \frac{1}{N-1} \sum_{i=1}^{N} \sum_{j \neq i} [\delta_j a^+_{M,i}(p) + \delta_j a^-_{M,j}(p) - \delta_i a^+_{M,i}(p) - \delta_i a^-_{M,j}(p)]$$

$$= \frac{1}{N-1} \sum_{i=1}^{N} \sum_{j \neq i} [\delta_i a^+_{M,j}(p) + \delta_i a^-_{M,j}(p) - \delta_i a^+_{M,i}(p) - \delta_i a^-_{M,j}(p)]$$

$$= \frac{1}{N-1} \sum_{i=1}^{N} \delta_i \sum_{j \neq i} [a^+_{M,j}(p) + a^-_{M,j}(p) - a^+_{M,i}(p) - a^-_{M,j}(p)]$$

$$= \frac{1}{N-1} \sum_{i=1}^{N} \delta_i [e^+_{M}(p) + a^+_{M,i}(p)] + (N-1)a^-_{M,i}(p)$$
\[-(N - 1)a_{M,i}^+(p) - [e_M^-(p) - a_{M,i}^-(p)]\]
\[= \frac{1}{N - 1} \sum_{i=1}^{N} \delta_i \left[ e_M^+(p) - Na_{M,i}^+(p) - e_M^+(p) + Na_{M,i}^-(p) \right] \]

Considering the \(i^{th}\) component, we get
\[b_{M,i}(p) = \frac{1}{N - 1} \left[ (e_M^+(p) - e_M^-(p)) - N(a_{M,i}^+(p) - a_{M,i}^-(p)) \right]. \tag{5}\]

Then, under reasonable hypotheses of uniform convergence for \(a_{M,i}^-, a_{M,i}^+, e_M^-, e_M^+, \) respectively to \(a_i^-, a_i^+, e^-, e^+\), and introducing
\[b_i(p) = \frac{1}{N - 1} \left[ (e_i^+(p) - e_i^-(p)) - N(a_i^+(p) - a_i^-(p)) \right], \]

we get condition b).

For condition a) we get, with a similar computation,
\[A_M(x) = M \sum_{q \in V_M} (q - p)(q - p)^t Q_M(p, q) \]
\[= M \sum_{i=1}^{N} \sum_{j=1}^{N} (p(i, j, M) - p) (p(i, j, M) - p)^t Q_M(p, p(i, j, M)) \]
\[= M \sum_{i=1}^{N} \sum_{j \neq i} \left[ p - \frac{\delta_i}{M} + \frac{\delta_j}{M} - p \right] \left[ p - \frac{\delta_i}{M} + \frac{\delta_j}{M} - p \right]^t \cdot \frac{1}{N - 1} (a_{M,i}^+(p) + a_{M,j}^-(p)) \]
\[= \frac{1}{M(N - 1)} \sum_{i=1}^{N} \sum_{j \neq i} [\delta_j - \delta_i] [\delta_j - \delta_i]^t \cdot (a_{M,i}^+(p) + a_{M,j}^-(p)) \]
\[= \frac{1}{M} \cdot G_N(p) \]

with \(G_N(p)\) bounded. Then condition a) is true with \(A = 0\) (there is no diffusion): \(\forall r > 0\)
\[\sup_{|p| \leq r} |A_M(p) - A(p)| = \sup_{|p| \leq r} |A_M(p)| \leq \frac{1}{M} \sup_{|p| \leq r} |G_N(p)| \leq \frac{1}{M} \cdot C_N,\]

that goes to 0 when \(M\) goes to infinity.
We have then only to control condition c). We should remark that \( Q_M \) is defined only on \( V_M \), but we can imagine it as defined on \( V_M \), setting it equal to 0 out of \( V_M \).

Let be \( M \) fixed and \( p \in V_M \). Then

\[
Q_M(p, \{q : |q - p| \geq \varepsilon\}) = \sum_{q \in V_M : |q - p| \geq \varepsilon} Q_M(p, q) = \sum_{i \neq j, |p(i, j, M) - p| \geq \varepsilon} Q_M(p, p(i, j, M)).
\]

Since \( |p(i, j, M) - p| = \frac{\sqrt{2}}{M} \), then if \( \varepsilon > \frac{\sqrt{2}}{M} \) we obtain \( Q_M(p, \{q : |q - p| \geq \varepsilon\}) = 0 \). Otherwise, if \( \varepsilon \leq \frac{\sqrt{2}}{M} \),

\[
Q_M(p, \{q : |q - p| \geq \varepsilon\}) = \sum_{i \neq j} Q_M(p, p(i, j, M)) = (N - 1) a_i^+(p) + a_j^-(p) = 0.
\]

Finally

\[
Q_M(p, \{q : |q - p| \geq \varepsilon\}) = \begin{cases} 0 & \text{if } \varepsilon > \frac{\sqrt{2}}{M}, \\ e_M(p) & \text{if } \varepsilon \leq \frac{\sqrt{2}}{M}. \end{cases}
\]

Fixed \( \varepsilon > 0 \) and \( r > 0 \) we get then condition c) because

\[
\lim_{M \to \infty} \sup_{|p| \leq r} M \cdot Q_M(p, \{q : |q - p| \geq \varepsilon\}) = \lim_{M \to \infty} Me_M(p) \cdot \delta_{M < \frac{\sqrt{2}}{M}} = 0.
\]

(in fact it is definitely equal to 0).

Using Theorem 4.1 above we can prove the convergence of the process to a deterministic limit.

The equilibrium condition is then

\[
b(p) = 0 \quad \text{that is, by (5),}
\]

\[
[(e^+(p) - e^-(p)) - N(a^+_i(p) - a^-_i(p))] = 0.
\]
i.e.

\[ a_i^+(p) - a_i^-(p) = \frac{1}{N}(e_i^+(p) - e_i^-(p)). \]

### 4.3 Open Problems

Let \( p^* \) be the probability which satisfies the equilibrium conditions and \( p^{**} \) the one which generates the target, i.e. the probability associated to the Chaos-Game for which the visits frequency approximates \( \hat{h} : \nu_k^M(r) \rightarrow \hat{h}(r) \), when \( k \) goes to infinity.

1) Does \( p^{**} \) exist? If yes, is it in the library? and is it unique?
2) \( p^* \) and \( p^{**} \) are the same point?

### References


